Asymptotic Models for Geophysical Applications

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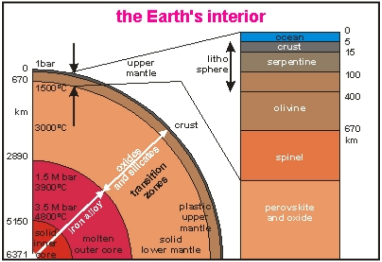
Motivations

HPC-GA Project

- **Numerical Models**
  - Elasto-Acoustic Coupling : to reproduce earthquakes
  - Modeling the effects of the ocean on seismic waves



  - The medium : a Land Area surrounded by a thin Fluid zone

Configurations, Difficulty

- **Configurations**

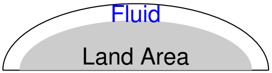


Figure: Configuration A

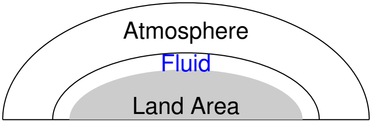


Figure: Configuration B

- **Sub-Configurations**
  1. **Conf. A1** : A thin layer with a **uniform thickness**
  2. **Conf. A2** : A thin layer with a **variable thickness**
- **Difficulty**

  Apply a FEM on a mesh with thin cells in the Fluid and much larger in the Land Area

Motivations

Method

1. Replace the Fluid layer by an Equivalent Boundary Condition

   Approach : an Asymptotic Method

2. Couple this condition with
   - the elastic equation, in **Conf. A**
   - the elastic and acoustic equations, in **Conf. B**

3. Apply a FEM

Figure: Conf. A

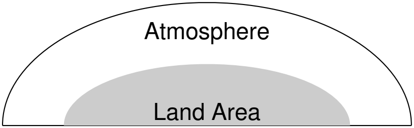


Figure: Conf. B

Motivations

Bibliography and Outline

Bibliography

- Modeling Fluid-Solid Interaction Problems :
  - JONES, 83
  - LUKE-MARTIN, 95
- Derivation of Equivalent Conditions for thin layer Problems :
  - ENGQUIST-NÉDÉLEC, 93
  - BENDALI-LEMRABET, 96
- Derivation of Equivalent Conditions in **Conf. A1**
  - PÉRON, INRIA RR 8163, 2012

Motivations

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Outline

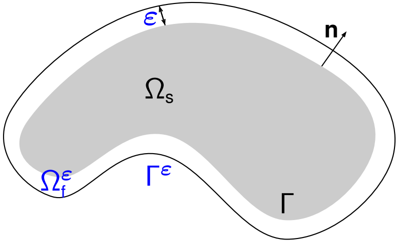
- Asymptotic Analysis for the time-harmonic **Elasto-acoustic** equations in a domain with a thin acoustic layer.

Outline

A Model Problem

The Problem $(\mathbf{P}_\varepsilon)$ set in a smooth domain $\Omega^\varepsilon = \Omega_s \cup \Gamma \cup \Omega_f^\varepsilon$:

$$
\begin{array}{lll}
\nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}) + \omega^2 \rho \mathbf{u} = 0 & \text{in} & \Omega_s \\
\Delta p + \kappa^2 p = 0 & \text{in} & \Omega_f^\varepsilon \\
\partial_{\mathbf{n}} p = \rho_f \omega^2 \mathbf{u} \cdot \mathbf{n} - \partial_{\mathbf{n}} p_i & \text{on} & \Gamma \\
\underline{\underline{\sigma}}(\mathbf{u})\, \mathbf{n} = -p\mathbf{n} - p_i \mathbf{n} & \text{on} & \Gamma \\
p = 0 & \text{on} & \Gamma^\varepsilon
\end{array}
$$



$$\kappa = \frac{\omega}{c_f}$$

Issue : Uniform Estimates for solutions $(\mathbf{u}_\varepsilon, p_\varepsilon)$ of $(\mathbf{P}_\varepsilon)$ as $\varepsilon \to 0$?

Framework

Assumption

(i) *Hooke's law :* $\underline{\underline{\sigma}}(\mathbf{u}) = \underline{\underline{C}}\ \underline{\underline{\epsilon}}(\mathbf{u})$

(ii) $\underline{\underline{C}}(x) = (C_{ijkl}(x))$ *is symmetric :* $C_{ijkl} = C_{jikl} = C_{klij}$

(iii) $C_{ijkl}(x)$ *are real valued smooth functions, up to* Γ

(iv) *The tensor* $\underline{\underline{C}}(x)$ *is positive*

Framework

Assumption

(i) *Hooke's law :* $\underline{\underline{\sigma}}(\mathbf{u}) = \underline{\underline{C}}\ \underline{\underline{\epsilon}}(\mathbf{u})$
(ii) $\underline{\underline{C}}(x) = (C_{ijkl}(x))$ *is symmetric :* $C_{ijkl} = C_{jikl} = C_{klij}$
(iii) $C_{ijkl}(x)$ *are real valued smooth functions, up to* Γ
(iv) *The tensor* $\underline{\underline{C}}(x)$ *is positive*

Assumption (SA)

The angular frequency ω *is not an eigenfrequency of the problem*

$$\begin{cases} \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}) + \omega^2 \rho \mathbf{u} = 0 & \textit{in} \quad \Omega_s \\ \underline{\underline{\sigma}}(\mathbf{u})\,\mathbf{n} = 0 & \textit{on} \quad \Gamma \end{cases}$$

Uniform Estimates

Proposition

There exists $\varepsilon_0 > 0$ *s.t. for all* $\varepsilon \in (0, \varepsilon_0)$*, the problem* $(\mathbf{P}_\varepsilon)$ *has a unique solution* $(\mathbf{u}_\varepsilon, \mathrm{p}_\varepsilon) \in \mathrm{H}^1(\Omega_\mathrm{s}) \times \mathrm{H}^1_{0,\Gamma^\varepsilon}(\Omega_\mathrm{f}^\varepsilon)$*, and*

$$\|\mathbf{u}_\varepsilon\|_{1,\Omega_\mathrm{s}} + \|\mathrm{p}_\varepsilon\|_{1,\Omega_\mathrm{f}^\varepsilon} \leqslant C\|(\mathrm{p}_i, \partial_\mathbf{n}\mathrm{p}_i)\|_{\frac{1}{2},-\frac{1}{2},\Gamma}$$

Uniform Estimates

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There exists $\varepsilon_0 > 0$ *s.t. for all* $\varepsilon \in (0, \varepsilon_0)$, *the problem* $(\mathbf{P}_\varepsilon)$ *has a unique solution* $(\mathbf{u}_\varepsilon, \mathrm{p}_\varepsilon) \in \mathrm{H}^1(\Omega_s) \times \mathrm{H}^1_{0,\Gamma^\varepsilon}(\Omega_f^\varepsilon)$, *and*

$$\|\mathbf{u}_\varepsilon\|_{1,\Omega_s} + \|\mathrm{p}_\varepsilon\|_{1,\Omega_f^\varepsilon} \leqslant C\|(\mathrm{p}_i, \partial_{\mathbf{n}}\mathrm{p}_i)\|_{\frac{1}{2},-\frac{1}{2},\Gamma}$$

Key for the proof: Introduce a ”Scaled Problem” by means of a Scaling

$$S = \frac{s}{\varepsilon} \in (0,1) \quad \text{when} \quad s \in (0, \varepsilon) \quad \text{is a } \textit{normal coordinate} \quad .$$

Application : Convergence of an asymptotic expansion for $(\mathbf{u}_\varepsilon, \mathrm{p}_\varepsilon)$ as $\varepsilon \to 0$

Outline

Methodology

- Step 1 : Derive an Asymptotic Expansion for $(\mathbf{u}_\varepsilon, \mathrm{p}_\varepsilon)$ when $\varepsilon \to 0$

$$\mathbf{u}_\varepsilon(x) = \mathbf{u}_0(x) + \varepsilon \mathbf{u}_1(x) + \varepsilon^2 \mathbf{u}_2(x) + \cdots$$
$$\mathrm{p}_\varepsilon(x) = \mathfrak{p}_0(y_\alpha, \frac{s}{\varepsilon}) + \varepsilon \mathfrak{p}_1(y_\alpha, \frac{s}{\varepsilon}) + \varepsilon^2 \mathfrak{p}_2(y_\alpha, \frac{s}{\varepsilon}) + \cdots$$

- Step 2 : Equivalent Conditions of order $k \in \mathbb{N}$.
  Identify a simpler problem satisfied by

$$\mathbf{u}_{k,\varepsilon} := \mathbf{u}_0 + \varepsilon \mathbf{u}_1 + \varepsilon^2 \mathbf{u}_2 + \cdots + \varepsilon^k \mathbf{u}_k \quad \text{up to} \quad \mathcal{O}(\varepsilon^{k+1})$$

- Step 3 : Prove Stability & Convergence results for Equivalent models

Step 1 : Multiscale Expansion

First terms

- $$\mathfrak{p}_0 = 0$$
  $$\begin{cases} \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}_0) + \omega^2 \rho \mathbf{u}_0 = 0 & \text{in} \quad \Omega_s \\ \mathbf{T}(\mathbf{u}_0) = -\mathrm{p}_i \mathbf{n} & \text{on} \quad \Gamma \end{cases}$$

Step 1 : Multiscale Expansion

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- $$\mathfrak{p}_1(., S) = (S-1)\left(\rho_f \omega^2 \mathbf{u}_0 \cdot \mathbf{n} - \partial_{\mathbf{n}} \mathrm{p}_i\right) \ , \quad S \in (0,1)$$
  $$\begin{cases} \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}_1) + \omega^2 \rho \mathbf{u}_1 = 0 & \text{in} \quad \Omega_s \\ \mathbf{T}(\mathbf{u}_1) = \rho_f \omega^2 \mathbf{u}_0 \cdot \mathbf{n}\, \mathbf{n} - \partial_{\mathbf{n}} \mathrm{p}_i \mathbf{n} & \text{on} \quad \Gamma \end{cases}$$

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- $$\mathfrak{p}_2(., S) = (S^2-1)\mathcal{H}[\rho_f \omega^2 \mathbf{u}_0 \cdot \mathbf{n} - \partial_{\mathbf{n}} \mathrm{p}_i] + (S-1)\rho_f \omega^2 \mathbf{u}_1 \cdot \mathbf{n}$$

$$\begin{cases} \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}_2) + \omega^2 \rho \mathbf{u}_2 = 0 & \text{in} \quad \Omega_s \\ \mathbf{T}(\mathbf{u}_2) = \rho_f \omega^2 \mathbf{u}_1 \cdot \mathbf{n}\, \mathbf{n} + \mathcal{H}[\rho_f \omega^2 \mathbf{u}_0 \cdot \mathbf{n} - \partial_{\mathbf{n}} \mathrm{p}_i]\mathbf{n} & \text{on} \quad \Gamma \end{cases}$$

Step 2 : Equivalent Conditions on Γ

We identify a simpler problem

$$(\mathbf{P}_{\varepsilon}^{k}) \quad \begin{cases} \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}_{\varepsilon}^{k}) + \omega^2 \rho \, \mathbf{u}_{\varepsilon}^{k} = 0 & \text{in} \quad \Omega_s \\ \mathbf{T}(\mathbf{u}_{\varepsilon}^{k}) + \mathrm{B}_{k,\varepsilon}(\mathbf{u}_{\varepsilon}^{k} \cdot \mathbf{n}) \, \mathbf{n} = \mathbf{h}_{k,\varepsilon} & \text{on} \quad \Gamma \end{cases}$$

with $\mathrm{B}_{k,\varepsilon}$ a surfacic differential operator.

Step 2 : Equivalent Conditions on Γ

We identify a simpler problem

$$(\mathbf{P}_{\varepsilon}^{k}) \quad \begin{cases} \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}_{\varepsilon}^{k}) + \omega^2 \rho \, \mathbf{u}_{\varepsilon}^{k} = 0 & \text{in} \quad \Omega_s \\ \mathbf{T}(\mathbf{u}_{\varepsilon}^{k}) + \mathrm{B}_{k,\varepsilon}(\mathbf{u}_{\varepsilon}^{k} \cdot \mathbf{n}) \, \mathbf{n} = \mathbf{h}_{k,\varepsilon} & \text{on} \quad \Gamma \end{cases}$$

with $\mathrm{B}_{k,\varepsilon}$ a surfacic differential operator.

- ECs of order $k \in \{0, 1, 2, 3\}$ for a Dirichlet b.c. on Γ^{ε}
- ECs of order $k \in \{0, 1\}$ for a Fourier-Robin b.c. on Γ^{ε}

Step 2 : Equivalent Conditions

Dirichlet b.c.

- $k = 0$:

$$\mathbf{T}(\mathbf{u}_0) = -\mathrm{p}_i \mathbf{n}$$

- $k = 1$:

$$\mathbf{T}(\mathbf{u}^1_\varepsilon) - \varepsilon\omega^2\rho_f \mathbf{u}^1_\varepsilon \cdot \mathbf{n}\,\mathbf{n} = \mathbf{h}_{1,\varepsilon}$$

- $k = 2$:

$$\mathbf{T}(\mathbf{u}^2_\varepsilon) - \varepsilon\omega^2\rho_f \left(1 + \varepsilon\mathcal{H}\right) \mathbf{u}^2_\varepsilon \cdot \mathbf{n}\,\mathbf{n} = \mathbf{h}_{2,\varepsilon}$$

- $k = 3$:

$$\mathbf{T}(\mathbf{u}^3_\varepsilon) - \varepsilon\omega^2\rho_f \left(1 + \varepsilon\mathcal{H} + \frac{\varepsilon^2}{6}\left[2(\Delta_\Gamma + \kappa^2\mathbb{I}) + 8\mathcal{H}^2 - \mathcal{K}\right]\right) \mathbf{u}^3_\varepsilon \cdot \mathbf{n}\,\mathbf{n} = \mathbf{h}_{3,\varepsilon}$$

Step 2 : Equivalent Conditions

Fourier-Robin b.c.

Fourier-Robin b.c. : $\partial_{\mathbf{n}} p_{\varepsilon} - i\kappa p_{\varepsilon} = 0$ on Γ^{ε}

- $k = 0$:

$$\mathbf{T}(\mathbf{u}_0) - i\omega c_f \rho_f \mathbf{u}_0 \cdot \mathbf{n}\,\mathbf{n} = \mathbf{g}_0 \quad \text{on} \quad \Gamma$$

- $k = 1$:

$$\mathbf{T}(\mathbf{u}^1_{\varepsilon}) - i\omega c_f \rho_f \left(1 + \varepsilon \left(2\mathcal{H} + i\kappa^{-1}\Delta_{\Gamma}\right)\right)\left(\mathbf{u}^1_{\varepsilon} \cdot \mathbf{n}\right)\mathbf{n} = \mathbf{g}_{1,\varepsilon}$$

Step 3 : Stability and Convergence results

Proposition

There exists $\varepsilon_0 > 0$ *s.t. for all* $\varepsilon \in (0, \varepsilon_0)$, *the problem* $(\mathbf{P}_{\varepsilon}^{k})$ *with data* $\mathrm{h}_k \in \mathrm{L}^2(\Gamma)$ *has a unique solution* $\mathbf{u}_{\varepsilon}^{k} \in \mathrm{H}^1(\Omega_{\mathrm{s}})$

- *Stability*

$$\|\mathbf{u}_{\varepsilon}^{k}\|_{1,\Omega_{\mathrm{s}}} \leqslant C\|\mathrm{h}_k\|_{0,\Gamma}$$

- *Convergence*

$$\|\mathbf{u}_{\varepsilon} - \mathbf{u}_{\varepsilon}^{k}\|_{1,\Omega_{\mathrm{s}}} \leqslant C\varepsilon^{k+1}$$

Outline

Finite Element Method

We use a Discontinuous Galerkin Method (IPDGM).

- Computational domain : $\Omega_s = \mathrm{D}(0; 0.01)$
- $\omega = 1.5 \times 10^6$
- Source : $p_i(\mathbf{x}) = \exp(i\omega\mathbf{x} \cdot \mathbf{d})$ with $\mathbf{d} = (1, 0)$
- $\mathbb{P}_3$-finite elements (Lagrange) in the Library Hou10ni
- Navier equation :

$$\mu\Delta\mathbf{u} + (\lambda + \mu)\nabla \operatorname{div} \mathbf{u} + \omega^2\rho\mathbf{u} = 0 \quad \text{in} \quad \Omega_s$$

with Lamé coefficients : $\mu = 26.32 \times 10^9$ and $\lambda = 51.08 \times 10^9$.

- $c = 1500$ m.s^{-1}, $\rho_f = 1000$ kg.m^{-3} , $\rho = 2700$ kg.m^{-3}.

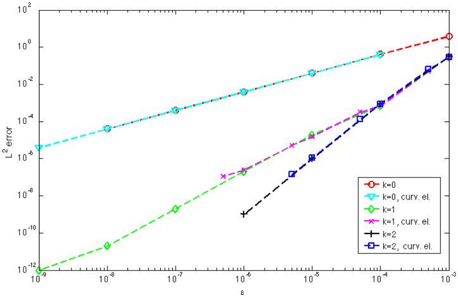
Convergence of the models

Dirichlet b.c.

For $k \in \{0, 1, 2\}$, we plot the L^2-error $\|\mathbf{u}_\varepsilon - \mathbf{u}_\varepsilon^k\|_{0,\Omega_s}$ w.r.t. ε

$\mathbf{u}_\varepsilon$: analytical solution

$\mathbf{u}_\varepsilon^k$: analytical solution / FE solutions with Curved elements



The convergence rate coincide with $\|\mathbf{u}_\varepsilon - \mathbf{u}_\varepsilon^k\|_{0,\Omega_s} = \mathcal{O}(\varepsilon^{k+1})$

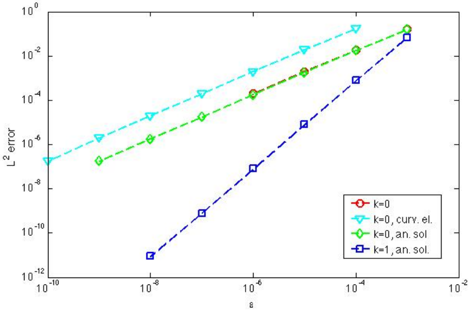
Convergence of the models

Fourier-Robin b.c.

For $k \in \{0, 1\}$, we plot the L^2-error $\|\mathbf{u}_\varepsilon - \mathbf{u}_\varepsilon^k\|_{0,\Omega_s}$ w.r.t. ε

$\mathbf{u}_\varepsilon$: analytical solution

$\mathbf{u}_\varepsilon^k$: analytical solution / FE solutions



Outline

Framework

- Parameterization of the layer Ω_f^ε

$$\Omega_f^\varepsilon = \left\{ \mathbf{x}(t) + s\, f(t) \mathbf{n}(t) \in \mathbb{R}^2 \mid \quad \mathbf{x}(t) \in \Gamma, \quad s \in (0, \varepsilon) \right\}$$

t : arc-length on Γ

f : smooth and periodic function

$\mathbf{n}(t) = \mathbf{n}\left(\mathbf{x}(t)\right)$: normal vector on Γ

Framework

- Parameterization of the layer Ω_f^ε

$$\Omega_f^\varepsilon = \left\{ \mathbf{x}(t) + s\, f(t)\mathbf{n}(t) \in \mathbb{R}^2 \;\middle|\; \mathbf{x}(t) \in \Gamma, \quad s \in (0, \varepsilon) \right\}$$

t : arc-length on Γ

f : smooth and periodic function

$\mathbf{n}(t) = \mathbf{n}\left(\mathbf{x}(t)\right)$: normal vector on Γ

- Euclidean metric of Ω_f^ε defined in the coordinates (t, s):

$$\begin{pmatrix} \left(1 + s\, f(t)\mathrm{c}(t)\right)^2 + \left(s\, f'(t)\right)^2 & s\, f(t)f'(t) \\ s f(t)f'(t) & f(t)^2 \end{pmatrix}$$

Step 2 : Equivalent Conditions

Dirichlet b.c.

- $k = 0$:

$$\mathbf{T}(\mathbf{u}_0) = -\mathrm{p}_i\mathbf{n}$$

- $k = 1$:

$$\mathbf{T}(\mathbf{u}_\varepsilon^1) - \varepsilon f(t)\omega^2\rho_{\mathrm{f}}\mathbf{u}_\varepsilon^1 \cdot \mathbf{n}\,\mathbf{n} = \mathbf{h}_{1,\varepsilon}$$

- $k = 2$:

$$\mathbf{T}(\mathbf{u}_\varepsilon^2) - \varepsilon f(t)\omega^2\rho_{\mathrm{f}}\left(1 - \frac{\varepsilon}{2}f(t)\mathrm{c}(t)\right)\mathbf{u}_\varepsilon^2 \cdot \mathbf{n}\,\mathbf{n} = \mathbf{h}_{2,\varepsilon}$$

Step 2 : Equivalent Conditions

Fourier-Robin b.c.

- $k = 0$:

$$\mathbf{T}(\mathbf{u}_0) - i\omega c_f \rho_f \mathbf{u}_0 \cdot \mathbf{n}\,\mathbf{n} = \mathbf{g}_0$$

- $k = 1$:

$$\mathbf{T}(\mathbf{u}_\varepsilon^1) - i\omega c_f \rho_f \left(1 + \varepsilon \left(-f(t)\mathrm{c}(t) + i\kappa^{-1}\left[g(t)\partial_t + f(t)\partial_t^2\right]\right)\right) \mathbf{u}_\varepsilon^1 \cdot \mathbf{n}\,\mathbf{n} = \mathbf{g}_{1,\varepsilon}$$

with

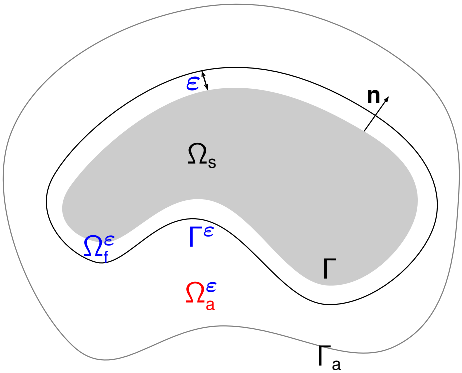
$$g(t) = \left(2 - f^2(t)\right) f'(t)$$

Outline

A Model Problem (Conf. B)

$(\mathbf{u}_\varepsilon, p_\varepsilon, p^a_\varepsilon)$ satisfies

$$
\begin{array}{lll}
\nabla\cdot\underline{\underline{\sigma}}(\mathbf{u}) + \omega^2\rho\mathbf{u} = 0 & \text{in} & \Omega_s \\
\Delta p + \kappa^2 p = 0 & \text{in} & \Omega_f^\varepsilon \\
\Delta p^a + \kappa_a^2 p^a = 0 & \text{in} & \Omega_a^\varepsilon \\
\partial_{\mathbf{n}} p = \rho_f\omega^2\mathbf{u}\cdot\mathbf{n} - \partial_{\mathbf{n}} p_i & \text{on} & \Gamma \\
\underline{\underline{\sigma}}(\mathbf{u})\,\mathbf{n} = -p\mathbf{n} - p_i\mathbf{n} & \text{on} & \Gamma \\
p^a = p & \text{on} & \Gamma^\varepsilon \\
c_a^2\partial_{\mathbf{n}} p^a = c_f^2\partial_{\mathbf{n}} p & \text{on} & \Gamma^\varepsilon \\
\text{ABC} & \text{on} & \Gamma_a
\end{array}
$$



$$\kappa = \frac{\omega}{c_f}, \quad \kappa_a = \frac{\omega}{c_a}$$

Methodology

- Step 1 : Derive an Asymptotic Expansion for $\left(\mathbf{u}_\varepsilon, \mathrm{p}_\varepsilon, \mathrm{p}^a_\varepsilon\right)$ when $\varepsilon \to 0$

$$\mathbf{u}_\varepsilon(x) = \mathbf{u}_0(x) + \varepsilon \mathbf{u}_1(x) + \varepsilon^2 \mathbf{u}_2(x) + \cdots$$
$$\mathrm{p}_\varepsilon(x) = \mathrm{p}_0(x;\varepsilon) + \varepsilon \mathrm{p}_1(x;\varepsilon) + \varepsilon^2 \mathrm{p}_2(x;\varepsilon) + \cdots$$
$$\mathrm{p}^a_\varepsilon(x) = \mathrm{p}^a_0(x) + \varepsilon \mathrm{p}^a_1(x) + \varepsilon^2 \mathrm{p}^a_2(x) + \cdots \quad \text{in} \quad \Omega^\varepsilon_a$$

- Step 2 : Derive Equivalent Conditions of order $k \in \mathbb{N}$

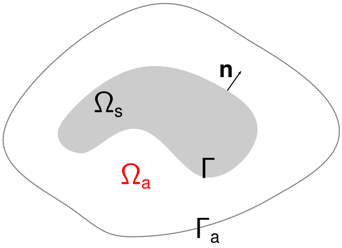
  (i) Extend p^a_ε in $\Omega_a := \Omega \setminus \overline{\Omega_s}$

  (ii) Identify a simpler problem satisfied by

$$\mathbf{u}_{k,\varepsilon} := \mathbf{u}_0 + \varepsilon \mathbf{u}_1 + \varepsilon^2 \mathbf{u}_2 + \cdots + \varepsilon^k \mathbf{u}_k \quad \text{up to} \quad \mathcal{O}(\varepsilon^{k+1}) \;,$$
$$\mathrm{p}^a_{k,\varepsilon} := \mathrm{p}^a_0 + \varepsilon \mathrm{p}^a_1 + \varepsilon^2 \mathrm{p}^a_2 + \cdots + \varepsilon^k \mathrm{p}^a_k \quad \text{up to} \quad \mathcal{O}(\varepsilon^{k+1}) \quad \text{in} \quad \Omega_a$$

Step 2 - Order 0

$$\begin{cases} \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}_0) + \omega^2 \rho \mathbf{u}_0 = 0 & \text{in} \quad \Omega_s \\ \Delta p_0^a + \kappa_a^2 p_0^a = 0 & \text{in} \quad \Omega_a \\ \mathbf{T}(\mathbf{u}_0) = -p_0^a \mathbf{n} - p_i \mathbf{n} & \text{on} \quad \Gamma \\ \partial_{\mathbf{n}} p_0^a = \left(\frac{c_f}{c_a}\right)^2 \rho_f \, \omega^2 \mathbf{u}_0 \cdot \mathbf{n} - \left(\frac{c_f}{c_a}\right)^2 \partial_{\mathbf{n}} p_i & \text{on} \quad \Gamma \\ \text{ABC} & \text{on} \quad \Gamma_a \end{cases}$$



Step 2 - Order 1

$$\begin{cases} \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}_1) + \omega^2 \rho \mathbf{u}_1 = 0 & \text{in} \quad \Omega_s \\ \Delta \mathrm{p}_1^a + \kappa_a^2 \mathrm{p}_1^a = 0 & \text{in} \quad \Omega_a \\ \mathbf{T}(\mathbf{u}_1) - \theta_{1,\varepsilon} \mathbf{u}_1 \cdot \mathbf{n}\, \mathbf{n} = -\mathrm{p}_1^a\, \mathbf{n} + \mathbf{h}_1^{\varepsilon} & \text{on} \quad \Gamma \\ \partial_{\mathbf{n}} \mathrm{p}_1^a + \varepsilon \left(\left(\frac{c_f}{c_a}\right)^2 - 1 \right) \Delta_\Gamma \mathrm{p}_1^a = \left(\frac{c_f}{c_a}\right)^2 \theta_{2,\varepsilon} \mathbf{u}_1 \cdot \mathbf{n} + \mathrm{g}_1^{\varepsilon} & \text{on} \quad \Gamma \\ \text{ABC} & \text{on} \quad \Gamma_a \end{cases}$$

Step 2 - Order 1

$$\begin{cases} \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}_1) + \omega^2 \rho \mathbf{u}_1 = 0 & \text{in} \quad \Omega_s \\ \Delta p_1^a + \kappa_a^2 p_1^a = 0 & \text{in} \quad \Omega_a \\ \mathbf{T}(\mathbf{u}_1) - \theta_{1,\varepsilon} \mathbf{u}_1 \cdot \mathbf{n}\, \mathbf{n} = -p_1^a\, \mathbf{n} + \mathbf{h}_1^\varepsilon & \text{on} \quad \Gamma \\ \partial_\mathbf{n} p_1^a + \varepsilon \left(\left(\frac{c_f}{c_a}\right)^2 - 1 \right) \Delta_\Gamma p_1^a = \left(\frac{c_f}{c_a}\right)^2 \theta_{2,\varepsilon} \mathbf{u}_1 \cdot \mathbf{n} + g_1^\varepsilon & \text{on} \quad \Gamma \\ \text{ABC} & \text{on} \quad \Gamma_a \end{cases}$$

$$\theta_{1,\varepsilon} = \varepsilon \rho_f \omega^2 \left[1 - \left(\frac{c_f}{c_a}\right)^2 \right]$$

$$\theta_{2,\varepsilon} = \rho_f \omega^2 + 2\varepsilon (1 - \rho_f \omega^2) \mathcal{H}$$

Order 1 - Formal proof

Transmission Conditions

$(\mathbf{u}_{1,\varepsilon}, \mathrm{p}_{1,\varepsilon})$ satisfies

$$\begin{cases} \mathbf{T}(\mathbf{u}_1) - \varepsilon\rho_f\omega^2\mathbf{u}_1 \cdot \mathbf{n}\,\mathbf{n} = -\mathrm{p}_1^a\mathbf{n} - \varepsilon\partial_{\mathbf{n}}\mathrm{p}_1^a\mathbf{n} + \mathbf{h}_1^\varepsilon \\ \partial_{\mathbf{n}}\mathrm{p}_1^a + \varepsilon(\frac{c_f}{c_a})^2(\kappa^2\mathbb{I} + \Delta_\Gamma)\mathrm{p}_1^a + \varepsilon\partial_{\mathbf{n}}^2\mathrm{p}_1^a = (\frac{c_f}{c_a})^2[\rho_f\omega^2 + 2\varepsilon\mathcal{H}]\mathbf{u}_1 \cdot \mathbf{n} + g_1^\varepsilon \end{cases}$$

up to $\mathcal{O}(\varepsilon^2)$.

Order 1 - Formal proof

Transmission Conditions

$(\mathbf{u}_{1,\varepsilon}, \mathrm{p}_{1,\varepsilon})$ satisfies

$$\begin{cases} \mathbf{T}(\mathbf{u}_1) - \varepsilon\rho_f\omega^2\mathbf{u}_1 \cdot \mathbf{n}\,\mathbf{n} = -\mathrm{p}_1^a\mathbf{n} - \varepsilon\partial_{\mathbf{n}}\mathrm{p}_1^a\mathbf{n} + \mathbf{h}_1^\varepsilon \\ \partial_{\mathbf{n}}\mathrm{p}_1^a + \varepsilon\left(\frac{c_f}{c_a}\right)^2(\kappa^2\mathbb{I} + \Delta_\Gamma)\mathrm{p}_1^a + \varepsilon\partial_{\mathbf{n}}^2\mathrm{p}_1^a = \left(\frac{c_f}{c_a}\right)^2[\rho_f\omega^2 + 2\varepsilon\mathcal{H}]\mathbf{u}_1 \cdot \mathbf{n} + g_1^\varepsilon \end{cases}$$

up to $\mathcal{O}(\varepsilon^2)$.

We substitute $\partial_{\mathbf{n}}\mathrm{p}_1^a$ into the first TC; we neglect the terms of order ε^2 :

$$\mathbf{T}(\mathbf{u}_1) \cdot \mathbf{n} - \theta_{1,\varepsilon}\mathbf{u}_1 \cdot \mathbf{n} = -\mathrm{p}_1^a + \mathrm{h}_1^\varepsilon \quad \text{on} \quad \Gamma\,.$$

Order 1 - Formal proof

Transmission Conditions

$(\mathbf{u}_{1,\varepsilon}, \mathrm{p}_{1,\varepsilon})$ satisfies

$$\begin{cases} \mathbf{T}(\mathbf{u}_1) - \varepsilon\rho_f\omega^2\mathbf{u}_1\cdot\mathbf{n}\,\mathbf{n} = -\mathrm{p}_1^a\mathbf{n} - \varepsilon\partial_{\mathbf{n}}\mathrm{p}_1^a\mathbf{n} + \mathbf{h}_1^\varepsilon \\ \partial_{\mathbf{n}}\mathrm{p}_1^a + \varepsilon(\frac{c_f}{c_a})^2(\kappa^2\mathbb{I} + \Delta_\Gamma)\mathrm{p}_1^a + \varepsilon\partial_{\mathbf{n}}^2\mathrm{p}_1^a = (\frac{c_f}{c_a})^2[\rho_f\omega^2 + 2\varepsilon\mathcal{H}]\mathbf{u}_1\cdot\mathbf{n} + g_1^\varepsilon \end{cases}$$

up to $\mathcal{O}(\varepsilon^2)$.

We substitute $\partial_{\mathbf{n}}\mathrm{p}_1^a$ into the first TC; we neglect the terms of order ε^2 :

$$\mathbf{T}(\mathbf{u}_1)\cdot\mathbf{n} - \theta_{1,\varepsilon}\mathbf{u}_1\cdot\mathbf{n} = -\mathrm{p}_1^a + \mathrm{h}_1^\varepsilon \quad \text{on} \quad \Gamma\,.$$

The acoustic equation writes on Γ :

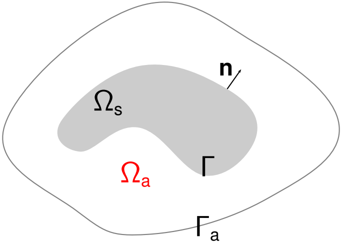
$$\partial_{\mathbf{n}}^2\mathrm{p}_1^a = 2\mathcal{H}\partial_{\mathbf{n}}\mathrm{p}_1^a - \Delta_\Gamma\mathrm{p}_1^a - \kappa_a^2\mathrm{p}_1^a \quad \text{on} \quad \Gamma.$$

We substitute this equation into the second TC

$$\partial_{\mathbf{n}}\mathrm{p}_1^a = \left[(\frac{c_f}{c_a})^2\theta_{2,\varepsilon}\mathbf{u}_1\cdot\mathbf{n} - \varepsilon\left((\frac{c_f}{c_a})^2 - 1\right)\Delta_\Gamma\mathrm{p}_1^a + \mathrm{g}_{1,\varepsilon}\right]$$

Step 2 - Order k

$$\begin{cases} \nabla\cdot\underline{\underline{\sigma}}(\mathbf{u}_\varepsilon^k)+\omega^2\rho\mathbf{u}_\varepsilon^k=0 & \text{in} \quad \Omega_s \\ \Delta \mathrm{p}_\varepsilon^k+\kappa_a^2\mathrm{p}_\varepsilon^k=0 & \text{in} \quad \Omega_a \\ \mathbf{T}(\mathbf{u}_\varepsilon^k)+\mathrm{B}_{k,\varepsilon}(\mathbf{u}_\varepsilon^k\cdot\mathbf{n})\mathbf{n}=-\mathrm{p}_\varepsilon^k\mathbf{n}+\mathbf{h}_{k,\varepsilon} & \text{on} \quad \Gamma \\ \partial_{\mathbf{n}}\mathrm{p}_\varepsilon^k+\mathrm{C}_{k,\varepsilon}(\mathrm{p}_\varepsilon^k)=\mathrm{D}_{k,\varepsilon}\mathbf{u}_\varepsilon^k\cdot\mathbf{n}+\mathrm{g}_{k,\varepsilon} & \text{on} \quad \Gamma \\ \text{ABC} & \text{on} \quad \Gamma_a \end{cases}$$



Thank you for your attention!