Construction of Ground State Eigenvalues and Resonances

Miguel Ballesteros
miguel.ballesteros@iimas.unam.mx
IIMAS, Universidad Nacional Autonoma de Mexico (UNAM)

Based on joint works with Volker Bach, Jérémy Faupin, Jürg Fröhlich, Martin Könenberg, Lars Menrath, Alessandro Pizzo and Baptiste Schubnel.
Matter and Light
(the presentation of the problem)

The study of eigenvalues (and resonances) embedded in the continuum.

Typical Situation:
A boson Hamiltonian $H_{bo}(\nu)$ and an atom Hamiltonian $H_{at}(\nu)$, defined on the Hilbert spaces $\mathcal{H}_{bo}$ and $\mathcal{H}_{at}$, respectively. Here we take $\nu \geq 0$.

$$\sigma(H_{bo}(\nu)) = e^{-i\nu}[0, \infty), \quad 0 \text{ is the only eigenvalue of } H_{bo}(\nu).$$

$$\sigma(H_{at}(\nu)) = \left\{ e_m \right\}_{m=0}^{M} \bigcup B(\nu),$$

$$e_0 < e_1 < e_2 < \cdots, \quad e_m \leq \Re\left( B(\nu) \right), \quad \forall m.$$
The composite (atom + boson) system is defined in the Hilbert space $\mathcal{H} := \mathcal{H}_{at} \otimes \mathcal{H}_{bo}$. The non-interacting (composite) Hamiltonian is

$$H_0(\nu) := H_{at}(\nu) \otimes 1_{\mathcal{H}_{bo}} + 1_{\mathcal{H}_{at}} \otimes H_{bo}(\nu) \equiv H_{at}(\nu) + H_{bo}(\nu).$$

Moreover,

$$\sigma(H_0(\nu)) = \sigma(H_{at}(\nu)) + \sigma(H_{bo}(\nu)),\quad \{e_m\}_{m=0}^M$$

are eigenvalues of $H_0(\nu)$. 
Main Problem:
We include an interaction on the picture. We denote it by $W(\nu)$, defined in $\mathcal{H}$. The full Hamiltonian is given by

$$H(\nu) := H_0(\nu) + gW(\nu),$$

where $g > 0$ is a small parameter.

- What is the fate of the eigenvalues $\left\{ e_m \right\}_{m=0}^M$ after turning on the interaction?
- Develop methods for the construction of the resulting (perturbed) eigenvalues.
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Examples:

Pauli-Fierz:
The Hilbert spaces for bosons and the atom are (respectively):

\[ \mathcal{H}^{(pf)}_{\text{bo}} := \bigoplus_{N=0}^{\infty} (L^2(\mathbb{R}^3 \times \mathbb{Z}_2))^\otimes S^N, \quad \mathcal{H}^{(pf)}_{\text{at}} := L^2(\mathbb{R}^3). \]

The boson and atom Hamiltonians are (taking \( \bar{k} := (k, \lambda) \in \mathbb{R}^3 \times \mathbb{Z}_2 \))

\[ H^{(pf)}_{\text{bo}}(\nu) := e^{-i\nu} \int d\bar{k} |\bar{k}| a^*_{(pf)}(\bar{k}) a_{(pf)}(\bar{k}), \]
\[ H^{(pf)}_{\text{at}}(\nu) := -e^{-2i\nu} \Delta - V_\nu(x), \]

where \( a^*_{(pf)} \) and \( a_{(pf)} \) are the creation and annihilation operators.
The interacting Hamiltonian is

\[ H^{(pf)}(\nu) := ( - ie^{-i\nu} \nabla_x - A_\nu(x))^2 + V_\nu(x) + H^{(pf)}_{bo}(\nu), \]

where

\[ A_\nu := \frac{\alpha^{3/2}}{4(\pi)^{3/2}} \left[ a^*_{(pf)}(G(\nu)e^{-i\alpha k \cdot x} \bar{\varepsilon}(\bar{k}))+a_{(pf)}(G(-\nu)e^{-i\alpha k \cdot x} \bar{\varepsilon}(\bar{k})) \right] \]

and

\[ G_\mu(\nu)(\bar{k}) \equiv G_\mu(\nu)(k) \equiv G(\nu)(k) := \frac{-e^{-i\nu - i\mu \nu} \exp(e^{-2i\nu}|k|^2)}{|k|^{1/2-\mu}}. \]

Here \( \bar{\varepsilon}(\bar{k}) \) is the polarization vector and \( \alpha \) is the fine structure constant. \( \mu \geq 0 \) is a regularization parameter. \( \mu = 0 \) is the critical case.
Spin-Boson: (two-level atom coupled to a boson field)

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\[ \mathcal{H}_{\text{bo}} := \bigoplus_{N=0}^{\infty} (L^2(\mathbb{R}^3))^\otimes s^N, \quad \mathcal{H}_{\text{at}} := \mathbb{C}^2. \]

The boson and atom Hamiltonians are

\[ H_{\text{bo}}(\nu) := e^{-i\nu} \int dk \, |k| \, a^*(k)a(k), \quad H_{\text{at}}(\nu) := \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \]

where “\( a^* \)” and “\( a \)” are the creation and annihilation operators.

The interaction is

\[ W(\nu) := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \left[ a^*(G_\mu(\nu)) + a(G_\mu(-\nu)) \right], \]

\( \mu = 0 \) is the critical case.
Existing Methods:

- **Spectral Renormalization** [Bach-Fröhlich-Sigal 1995]

Variants we recently worked on:

- Continuous version: [Bach-B-Fröhlich 2015].
- Without spectral reparametrization:
  [B-Faupin-Fröhlich-Schubnel 2015]
  + [Faupin-Fröhlich-Schubnel 2014].

- **Multi-scale Analysis (or Pizzo Method)** [Pizzo 2003] +

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- Resonances (Pauli-Fierz) (critical and non-critical $\mu$):
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We take the spin-boson model for our analysis.

**Spectral Renormalization (Continuous [BBF 2015]) :**

We define

\[ H(z) := H(\nu) + z. \]

The key ingredient is the Feshbach-Schur map, that permits isolating spectral regions of interests (near the eigenvalues, \( e_0 = 0, \ e_1 = 2 \), of the free Hamiltonian). Denote by \( \Pi^{(i)} \) the eigen projections corresponding to \( e_i \) of \( H_{\at} \) and define, for \( \alpha > 0 \),

\[ P_{\alpha}^{(i)} := \Pi^{(i)} \otimes \chi_{\alpha}(H_{\bo}(\nu)), \quad P_{\alpha}^{(i)} := 1 - P_{\alpha}^{(i)}, \]

where \( \chi_{\alpha} \) is the characteristic function of the disc of radius \( e^{-\alpha} \), centered at 0.
The Feshbach-Schur Map:

\[ F_{\alpha}^{(i)}(H(z)) = P_{\alpha}^{(i)} H(z) P_{\alpha}^{(i)} \]

\[ - P_{\alpha}^{(i)} H(z) \frac{1}{P_{\alpha}^{(i)} H(z) P_{\alpha}^{(i)}} H(z) P_{\alpha}^{(i)} \] (2)

We modify the Feshbach-Schur map by re-scaling in the following form:

\[ k \in \mathbb{R}^3 \mapsto e^{-\alpha} k \in \mathbb{R}^3, \]

for \( \alpha > 0 \). We denote the resulting operator by \( \hat{\mathcal{R}}_{\alpha}(H(z)) \).
Isospectrality:

(a) $H(z)$ is (bounded) invertible if and only if $\hat{R}_\alpha(H(z))$ is (bounded) invertible.

(b) $H(z)$ is not injective if and only if $\hat{R}_\alpha(H(z))$ is not injective.
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(b) \( H(z) \) is not injective if and only if \( \hat{\mathcal{R}}_\alpha(H(z)) \) is not injective.
There exist operators $T_\alpha(z)$ and $W_\alpha(z)$ such that

$$\hat{R}_\alpha(H(z)) = T_\alpha(z) + W_\alpha(z),$$  \hspace{1cm} (3)

where $W_\alpha(z)$ decays exponentially in $\alpha$.

- $T_\alpha(z) :=$ renormalized free Hamiltonian
- $W_\alpha(z) :=$ renormalized interaction.
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Theorem (Bach-B-Fröhlich 2015)

There exists a family \( \{E_s\}_{s \geq 0} \) of biholomorphic functions

\[
E_s : D_{\rho/2} \rightarrow E_s(D_{\rho/2}) \subset D_{\rho/2}
\]

such that

\[
\forall \zeta \in E_s(D_{\rho/2}) : H(\zeta) = H(\nu) + \zeta \in \text{dom}(\hat{R}_s).
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The family of functions $\{E_s\}_{s \geq 0}$ is the *renormalization* semi-group (or *renormalization group*) of the spectral parameter $z$. The function $E_s$ is the reparametrization of the spectral parameter $z$.

Using this Theorem we can define a family (a semi-group) $\{H_s(z)\}_{s \geq 0}$ of operators by

$$\forall z \in D_{\rho/2} : H_s(z) = \hat{R}_s(H(E_s(z))),$$

$$H_0(z) = H(\nu) + z.$$
Theorem (Bach-B-Fröhlich-Sigal 2015)

For every $s \geq 0$, there exist operators $T_s(z)$ and $W_s(z)$ such that

$$H_s(z) = T_s(z) + W_s(z),$$

(7)

where the spectrum of $T_s(z)$ can be computed explicitly and there are constants $\iota > 0$ and $C > 0$ such that

$$\|W_s(z)\| \leq Ce^{-\iota s}.$$  

(8)

In the case that $s = 0$ we take $T_0(z) = H_0(\nu) + z$ and $W_0(z) = W(\nu)$. 
Theorem (Bach-B-Fröhlich 2015)

For a suitable space of operators $H[\mathcal{W}_\xi]^{(0)}$ there is a flow map

$$\Phi : H[\mathcal{W}_\xi]^{(0)} \times [0, \infty) \mapsto H[\mathcal{W}_\xi]^{(0)},$$

such that, for every $H(z) \in H[\mathcal{W}_\xi]^{(0)}$ and every $s \geq 0$,

$$\Phi(H(z), s) = H_s(z).$$

The function $\Phi$ satisfies the flow (or semigroup) property

$$\Phi(H_s(z), t) = \Phi(H(z), s + t), \quad \forall s, t \geq 0, \forall H(z) \in H[\mathcal{W}_\xi]^{(0)}.$$  \hspace{1cm} (9)
Spectral Renormalization
(without Spectral Reparametrization)
[BFFS-2015]+[FFS-2014]

In [BFFS-2015] (and [FFS-2014]) we present a bare-bones simplification of the (spectral renormalization) method, consisting in the following:

- We do not rescale the free energy.
- We do not need the reparametrizations $E_S$. 
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- We do not rescale the free energy.
- We do not need the reparametrizations $E_s$. 
For a convenient sequence of parameters \((\alpha_j)_{j \in \mathbb{N}}\) (converging to \(\infty\)), we construct:

- a nested, shrinking, sequence of balls \(B_j, j \in \mathbb{N}\),
- a sequence of operators (with shrinking domains) \((H_j(z))_{j \in \mathbb{N}}\), for every \(z \in B_j\),

such that

- \(H_j(z)\) is obtained from \(H_{j-1}(z)\), \((z \in B_j)\), by an application of a Feshbach-Schur map (with parameter \(\alpha_j\)).
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Continuous Renormalization Vs. Renormalization without Spectral Reparametrization

Continuous renormalization is the most complicated approach, but the family $H_s(z), s \geq 0,$ has the best structure: It is a semi-group of operators and all operators $H_s(z)$ are defined for every $z \in D_{\rho/2}$. Moreover, it allows a presentation in terms of differential equations.

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Multi-scale Analysis (or Pizzo method)
Spin-Boson: (two-level atom coupled to a boson field)
(recall that, for simplicity, we analyze the spin-boson model)

Idea:
- Construct a sequence of infrared-cutoff Hamiltonians

\[ (H^{(n)}(\nu))_{n \in \mathbb{N}} \]

such that as the parameter \( n \) increases more and more low energies are allowed. In the limit, when \( n \) tends to \( \infty \), all cutoffs are removed.

- For every \( n \): Prove the existence of an eigenvalue \( e_i^{(n)} \) of \( H^{(n)}(\nu) \) near \( e_i \).
- The limit

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Now, we materialize the idea:
We define a sequence of cutoff parameters

\[ \sigma_n := \beta^n \sigma_0, \quad n \in \mathbb{N}, \]

where \( \beta \) and \( \sigma_0 \) are small enough.
We cut low boson energies off defining

\[ \mathcal{H}^{(n)}_{bo} := \bigoplus_{N=0}^{\infty} (L^2(|k| \geq \sigma_n)) \otimes s^N \]
and

\[ H^{(n)}_{\text{bo}}(\nu) := e^{-i\nu} \int_{|k| \geq \sigma_n} |k| a^*(k) a(k). \]

We, furthermore, set

\[ W^{(n)}(\nu) := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \left[ a^*(G^{(n)}(\nu)) + a(G^{(n)}(-\nu)) \right], \]

where \( G^{(n)} \) is the restriction of \( G \) to \( |k| \geq \sigma_n \).
The cutoff free and interacting Hamiltonians are given by

\[ H_0^{(n)}(\nu) := H_{\text{at}} + H_{\text{bo}}^{(n)}(\nu) \]

and

\[ H^{(n)}(\nu) := H_0^{(n)}(\nu) + gW^{(n)}(\nu), \]

defined in

\[ \mathcal{H}^{(n)} := \mathcal{H}_{\text{at}} \otimes \mathcal{H}_{\text{bo}}. \]
The tensor product structure of $\mathcal{H}^{n+1}$ allows us to define $H_0^{(n)}(\nu)$, $H^{(n)}(\nu)$ and $W^{(n)}(\nu)$ in $\mathcal{H}^{(n+1)}$, by cutting all $k$’s with $\sigma_{n+1} \leq |k| < \sigma_n$ off.

We denote

$$\tilde{H}_0^{(n)}(\nu) := H_0^{(n+1)}(\nu) + gW^{(n)}(\nu).$$

Notice that $e_0 = 0$ and $e_1 = 2$ are isolated eigenvalues of $H_0^{(n)}(\nu)$:

$$\text{dist} \left( e_i, \sigma(H_0^{(n)}(\nu)) \ \setminus \ {e_i} \right) \geq \sigma_n.$$
For small enough $g$, we expect to have an eigenvalue $e_i^{(n)}$ of $H^{(n)}(\nu) = H_0^{(n)}(\nu) + gW^{(n)}(\nu)$, near $e_i$. However, constructed in this way, $g$ depends on $n$. Then we need an inductive scheme.

If we take $i = 0$ and $\nu = 0$, then $H^{(n)}(\nu)$ is self-adjoint and the spectral theorem and the min-max principle can be used to prove that the parameter $g$ above can be taken independent of $n$. This is one of the reasons why constructing resonances is more difficult than constricting ground states. However, even for ground states, an inductive argumentation is necessary (for similar reasons).
Inductive Scheme (simplified)

We inductively construct (simple) eigenvalues $e_i^{(n)}$, such that the following key estimate holds true

$$\left\| \left( H^{(n+1)}(\nu) - \tilde{H}^{(n)}(\nu) \right) \frac{1}{\tilde{H}^{(n)}(\nu) - z} \right\| \leq C^n \sigma_n^\mu,$$

for some constant $C$ and every $z$ in a suitable set $\Gamma_n$.

The exponential decay of the term $C^n \sigma_n^\mu$ enables us to complete the induction scheme and it implies that the sequence $(e_i^{(n)})_{n \in \mathbb{N}}$ converges to an eigenvalue of $H(\nu)$ [BBP 2016]:

$$e_i := \lim_{n \to \infty} e_i^{(n)}$$

is an eigenvalue of $H(\nu)$. 
The Critical Case (Spin-boson + Pizzo Method) [BBKM 2016]

In the critical case, [Arai/Hirokawa/Hiroshima, 2008] proved absence of ground states for a variety of operators similar to the spin-boson Hamiltonian. It is, therefore, necessary to impose additional symmetries to the models. For the spin-boson model that we consider, the fact that the interaction is off-diagonal implies the existence of a ground state (see [BBKM 2016]).
The key technical new ingredient in the proof is the equality

\[ P_n \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} P_n = 0, \quad \forall n \in \mathbb{N}, \tag{11} \]

where \( P_n \) is the ground state projection of \( H^{(n)}(0) \). Eq. (11) implies an estimate of the form (10) (which is much more complicated in this case). I do not expect the same method to hold work for resonances, because this method uses a non-inductive construction of the energies \( (e_n)_{n \in \mathbb{N}} \) (that is not available for resonances).
The Critical Case (Pauli-Fierz + Pizzo Method) [BBKM 2014]

Here the required symmetry is encoded by the Pauli-Fierz transformation.

We chose a suitable function $\eta \in C^\infty_0(\mathbb{R}^3; \mathbb{R})$ and define

$$A_{PF} := -\frac{\alpha^3/2}{4(\pi)^{3/2}} \left[ a^*_{(pf)} \left( G(0)\eta(|x||k|) x \cdot \bar{\epsilon}(k) \right) + a_{(pf)} \left( G(0)\eta(|x||k|) x \cdot \bar{\epsilon}(k) \right) \right].$$

We set ($e^{-iA_{PF}}$ is the Pauli-Fierz transformation)

$$H(0) := e^{-iA_{PF}} H(0) e^{iA_{PF}}.$$  \hspace{1cm} (12)
We trade the Hamiltonian $H(0)$ for the unitarily equivalent operator $\mathcal{H}(0)$, and we analytically extend $\mathcal{H}(0)$ to a family of operators $\mathcal{H}(\theta)$, $\theta \in \mathbb{C}$.

We analyze as before the new Hamiltonian $\mathcal{H}(\theta)$, but the key estimate Eq. (10) cannot be directly derived, instead we bound

$$\left\| \left( F(H^{(n+1)}(\nu) - z) - F(\bar{H}^{(n)}(\nu) - z) \right) \frac{1}{F(\bar{H}^{(n)}(\nu) - z)} \right\|, \quad (13)$$

for a suitable Feshbach-Schur map $F$. 

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Thank You
For Your Attention!