Curvature Effects in Surface Superconductivity

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Mathematical Many-Body Theory and its Applications

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joint work with E.L. Giacomelli (Roma 1), N. Rougerie (Grenoble)
Introduction [FH]:
- Ginzburg-Landau (GL) theory and response of a superconductor to an applied magnetic field.
- Surface superconductivity: GL asymptotics and Pan’s conjecture.

Main results & future perspectives [CR1 – 3]:
- Energy and density asymptotics between $H_{c2}$ and $H_{c3}$ [CR1 – 2];
- Curvature effects on surface superconductivity [CR3].
- Effects of boundary singularities (corners) [CG].

Main References

Certain materials which behave like metals at room temperature become superconductors (zero resistance) below a certain $T_c > 0$ (ceramic compound YBa$_2$Cu$_3$O$_7$ in fig.).

A (brief) History of Superconductivity

- [1911] Discover by H. Kamerlingh Onnes cooling down Hg.
- [1933] Discovery of Meissner effect in experiments performed by W. Meissner and R. Ochsenfeld.
- [1957] First microscopic description with BCS theory proposed by J. Bardeen, L. Cooper, J. Schrieffer.
Meissner Effect

- When a type-II superconductor is immersed in a magnetic field and cooled down below $T_c$, the magnetic field is expelled from the bulk.

- *Strong* magnetic fields can penetrate the sample (first at isolated defects) and eventually destroy the superconductivity.

- The response of a superconductor to an external magnetic field (Meissner effect) can be completely described *within* the GL theory.
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SC Maglev train running at 500 km/h ➞
**Ginzburg-Landau Theory**

### GL Energy Functional

The energy of a superconductor confined to an infinite cylinder of \((smooth\) and simply connected\) cross section \(\Omega \subset \mathbb{R}^2\) is obtained by minimizing

\[
G_{\kappa}^{GL}[\Psi, A] = \int_{\Omega} \, dr \left\{ |(\nabla + iA) \Psi|^2 - \kappa^2 |\Psi|^2 + \frac{1}{2} \kappa^2 |\Psi|^4 + |\text{curl}A - h_{ex}|^2 \right\}
\]

- \(|\Psi|^2\) density of superconducting electrons (Cooper pairs).
- \(A\) magnetic potential with magnetic field \(h = \text{curl}A\).
- \(\kappa^{-1}\) penetration depth \((\kappa \to \infty = \text{extreme type-II superconductors})\).
- Uniform applied magnetic field \(\perp\) to \(\Omega\) of size \(h_{ex}\).
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GL Minimizers

\[ G_{\kappa}^{GL}[\Psi] = \int_{\Omega} \, dr \left\{ |(\nabla + iA) \Psi|^2 + \frac{1}{2} \kappa^2 (1 - |\Psi|^2)^2 + |\text{curl} A|^2 \right\} \]

**Perfectly superconducting state**

In absence of applied field, the superconducting state \(|\Psi| \equiv 1, A = 0\) (Meissner state) is the unique minimizer of the GL energy.

**Normal state**

If \(h_{ex} \gg 1\) and \(\kappa\) fixed (huge applied field), the normal state \(\Psi \equiv 0\) with \(\text{curl} A = h_{ex}\) is the unique minimizer of the GL energy.

**Mixed state**

For intermediate applied fields, any minimizer (possibly non-unique) is a mixed state satisfying \(0 \leq |\Psi| \leq 1\).
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For intermediate applied fields, any minimizer (possibly non-unique) is a mixed state satisfying \( 0 \leq |\Psi| \leq 1 \).
Superconductivity is first lost at isolated defects (vortices).

For larger magnetic fields the number of vortices increases and eventually vortices arrange in a triangular lattice, which was predicted by Abrikosov in 1957 and later observed by Essmann, Trauble in 1967.

Vortices in Nb crystal [Ling et al ’00].
Phenomenology (Physics)

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Vortices in $Nb$ crystal [Ling et al '00].

Vortices in $Pb$ at 1.1 K [Essmann, Trauble '67].
Before being totally *lost*, superconductivity survives at the boundary (surface superconductivity) as predicted by *Saint-James, de Gennes* in 1963 and observed by *Strongin et al* in 1964.

*Pb* island of superconductor at 4.32 K [Ning et al ’09].

Vortices and surface superconductivity on a *Pb* island [Ning et al ’09].
# Critical Magnetic Fields

As $\kappa \to \infty$, one can identify three bifurcation values (critical fields) for $h_{\text{ex}}$:

## First Critical Field

If $h_{\text{ex}} < H_{c1}(\kappa) \approx C_\Omega \log \kappa$, one has $|\Psi_{\text{GL}}| > 0$, $A_{\text{GL}} \approx 0$. Above $H_{c1}$ isolated defects of $\Psi_{\text{GL}}$ (vortices), where the superconductivity is lost, start to appear [Sandier, Serfaty ’00].

## Second Critical Field

At $H_{c2}(\kappa) \approx \kappa^2$, superconductivity disappears in the bulk and becomes a boundary phenomenon (surface superconductivity).

## Third Critical Field

If $h_{\text{ex}} > H_{c3}(\kappa) \approx \Theta_0^{-1} \kappa^2$ with $\Theta_0 < 1$ a universal constant (actually $\Theta_0^{-1} \approx 1.6946$), the superconductivity is totally lost and $\Psi_{\text{GL}} \equiv 0$ with $h = h_{\text{ex}}$ is the unique minimizer [Fournais, Helffer ’06].
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Introduction

Second Critical Field

**Second Critical Field (Mathematical Definition)**

- No precise mathematical definition of $H_{c2}$, but only the idea of a vague transition from bulk to boundary behavior.
- $H_{c2}(\kappa) = \kappa^2$ (can be taken as a definition).
- Agmon estimates yield an exponential decay of $\Psi^{GL}$ far from $\partial \Omega$, provided $h_{\text{ex}} > H_{c2}$.

**Proposition (Agmon Estimates [Helffer, Morame ’01])**

If $h_{\text{ex}} = b\kappa^2$ for some $b > 1$ and $\kappa$ large enough, $\exists A > 0$ such that

$$\int_{\Omega} |\Psi^{GL}(\mathbf{r})|^2 = O(\kappa^{-1}),$$

$$|\Psi^{GL}(\mathbf{r})| = O(\kappa^{-\infty}),$$

for $\text{dist}(\mathbf{r}, \partial \Omega) \gg \kappa^{-1}$. 
**Between $H_{c2}$ and $H_{c3}$**

### Change of units

$$
\mathcal{G}_{\kappa}^{\text{GL}}[\Psi, A] = \int_{\Omega} \, \text{d}r \left\{ |(\nabla + i h_{\text{ex}} A) \Psi|^2 - \kappa^2 |\Psi|^2 + \frac{1}{2} \kappa^2 |\Psi|^4 
+ h_{\text{ex}}^2 |\text{curl} A - 1|^2 \right\}
$$

- We are interested in the regime $H_{c2} < h_{\text{ex}} < H_{c3}$, i.e.,
  $$
h_{\text{ex}} = b \kappa^2, \quad 1 < b < \Theta^{-1}_0
$$
- Change of units to $(\varepsilon, b)$ with $\varepsilon \ll 1$ and rescaling of $A \rightarrow h_{\text{ex}} A$:
  $$
\varepsilon = (b \kappa^2)^{-1/2}
$$
- $E^{\text{GL}}_{\varepsilon} = \min_{(\Psi,A) \in H^1 \times H^1} \mathcal{E}^{\text{GL}}_{\varepsilon}[\Psi, A]$ and $(\Psi^{\text{GL}}, A^{\text{GL}})$ any minimizing pair.
**Introduction**

**Between** \( H_{c2} \) **and** \( H_{c3} \)

### Change of units

\[
E_{\varepsilon}^{GL}[\Psi, A] = \int_{\Omega} \text{d}r \left\{ \left| \left( \nabla + i \frac{A}{\varepsilon^2} \right) \Psi \right|^2 - \frac{1}{2b\varepsilon^2} \left( 2|\Psi|^2 - |\Psi|^4 \right) + \frac{b}{\varepsilon^4} |\text{curl} A - 1|^2 \right\}
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  \[
  \varepsilon = \left( b\kappa^2 \right)^{-1/2}
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- \(E_{\varepsilon}^{GL} = \min_{(\Psi, A) \in H^1 \times H^1} E_{\varepsilon}^{GL}[\Psi, A] \) and \((\Psi^{GL}, A^{GL})\) any minimizing pair.
Heuristics (around $H_{c3}$)

- Suppose we decrease $h_{ex}$ from huge values: above $H_{c3}$ the normal state is the unique minimizer and $\text{curl} \mathbf{A}^{GL} \equiv 1$, with $\mathcal{E}_\varepsilon^{GL} = 0$.
- When $h_{ex}$ is lowered below $H_{c3}$, in first approximation $\text{curl} \mathbf{A}^{GL} = 1$ and $\Psi^{GL}$ is small, so that the energy to minimize is \textit{linear}

$$\int_{\Omega} d\mathbf{r} \left\{ | (\nabla + i \frac{A}{\varepsilon^2}) \Psi |^2 - \frac{1}{b\varepsilon^2} |\Psi|^2 \right\} = \left\langle \Psi | H_\varepsilon - \frac{1}{b\varepsilon^2} | \Psi \right\rangle$$

- When $\lambda_0(\varepsilon) - \frac{1}{b\varepsilon^2} < 0$, with $\lambda_0(\varepsilon)$ the ground state of $H_\varepsilon$?
- The ground state $\psi_\varepsilon$ of $H_\varepsilon$ is \textbf{localized} on a scale $\varepsilon$ and blowing up on that scale one finds 2 alternative effective problems...

### Magnetic Laplacian on the Plane/Half-Plane

- $H_\varepsilon$ on $\mathbb{R}^{2,+}$ with Neumann b.c., $\lambda_0(\varepsilon) = \Theta_0 \varepsilon^{-2}$ and $\psi_\varepsilon$ lives on $\partial \Omega$.
- $H_\varepsilon$ on $\mathbb{R}^2$ (or on $\mathbb{R}^{2,+}$ with Dirichlet b.c.), $\lambda_0(\varepsilon) = \varepsilon^{-2}$ and $\psi_\varepsilon$ lives in the interior of $\Omega$. 
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- Suppose we decrease $h_{ex}$ from huge values: above $H_{c3}$ the normal state is the unique minimizer and $\text{curl} A^{GL} = 1$, with $E^{GL} = 0$.
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Heuristics (between $H_{c2}$ and $H_{c3}$)

- Restriction to a neighborhood of $\partial \Omega$ & tubular coordinates there: $(s, \varepsilon t)$ tangential and normal coordinates (rescaled).

- **Gauge choice** [Fournaise, Helffer ’10]

$$\Psi^{\text{GL}}(r) = e^{i\phi_{\varepsilon}(s,t)}\psi(s,t), \quad A^{\text{GL}}(r) \rightarrow (-t + O(\varepsilon|\log \varepsilon|)) \tau(s)$$

where $\tau(s)$ is the unit vector tangential to $\partial \Omega$.

- In the regime $1 < b < \Theta_{0}^{-1}$, $|\Psi^{\text{GL}}|$ is approx. constant in the tangential direction, i.e.,

$$\psi(s,t) \simeq f(t) e^{-i\frac{\alpha}{\varepsilon}s}$$

- The GL energy becomes up to $O(\varepsilon|\log \varepsilon|\infty)$ error terms

$$\frac{1}{\varepsilon} \int_{0}^{\partial \Omega} ds \int_{0}^{C|\log \varepsilon|} dt \left\{|\partial_t \psi|^2 + |(\varepsilon \partial_s - it) \psi|^2 + \frac{1}{b} |\psi|^4 - \frac{2}{b} |\psi|^2\right\}$$
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- The GL energy becomes up to $O(\varepsilon \log \varepsilon)$ error terms

\[
\frac{|\partial \Omega|}{\varepsilon} \int_0^{+\infty} dt \left\{ |\partial_t f|^2 + (t + \alpha)^2 f^2 - \frac{1}{2b} \left(2f^2 - f^4\right) \right\}
\]
**Effective 1D Functional**

\[
\mathcal{E}_{0,\alpha}^{1D}[f] := \int_0^{+\infty} dt \left\{ |\partial_t f|^2 + (t + \alpha)^2 f^2 - \frac{1}{2b} (2f^2 - f^4) \right\}
\]

- \( \exists! \) minimizer \( f_{0,\alpha} \geq 0 \) satisfying Neumann boundary conditions at 0.
- \( f_{0,\alpha} \) is non-trivial and thus \( E_{0,\alpha}^{1D} < 0 \) iff \( b^{-1} > \mu_0(\alpha) \)

while \( f_{0,\alpha} \equiv 0 \) (normal state) for \( b > \Theta_0^{-1} \).

- \( \mu_0(\alpha) \) is the ground state energy of \( H_\alpha = -\partial_t^2 + (t + \alpha)^2 \) in \( L^2(\mathbb{R}^+, dt) \) with Neumann boundary conditions and \( \Theta_0 = \min_{\alpha \in \mathbb{R}} \mu_0(\alpha) \).

- For any \( 1 \leq b < \Theta_0^{-1} \), \( \exists \) a phase \( \alpha_0 < 0 \) minimizing \( E_{0,\alpha}^{1D} \) and a density \( f_0 = f_{0,\alpha_0} \neq 0 \), i.e.,

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  \[ E_{0}^{1D} = \inf_{\alpha \in \mathbb{R}} E_{0,\alpha}^{1D} = E_{0,\alpha_0}^{1D} \]
**Introduction**

**Past Results**

**GL energy asymptotics**

- **[Pan ’02]** \( E_{\varepsilon}^{GL} = \frac{|\partial \Omega| E_b}{\varepsilon} + o(\varepsilon^{-1}) \) for \( 1 < b < \Theta_0^{-1} \) and \( E_b < 0 \).
- **[Almog, Helffer ’07; Fournais, Helffer, Persson ’11]**: \[
E_{\varepsilon}^{GL} = \frac{|\partial \Omega| E_0^{1D}}{\varepsilon} + O(1) \]
  for \( 1.25 \leq b < \Theta_0^{-1} \) with \( E_0^{1D} = \inf_{\alpha \in \mathbb{R}} E_{0,\alpha}^{1D} \) (by perturbative methods).

**Order parameter asymptotics**

- **[Fournais, Helffer, Persson ’11]** If \( 1.25 \leq b < \Theta_0^{-1} \) the density \( |\Psi^{GL}|^2 \) is close to \( f_0^2 \), i.e., \( (\tau = \text{dist}(r, \partial \Omega), \tau = \varepsilon t) \)
  \[
  \| |\Psi^{GL}|^2 - |f_0(t)|^2 \|_{L^2(\Omega)} \ll \| f_0^2(t) \|_{L^2(\Omega)} \propto \varepsilon^{1/2}
  \]
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  \left\| |\Psi^{\text{GL}}|^2 - |f_0(t)|^2 \right\|_{L^2(\Omega)} \ll \left\| f_0^2(t) \right\|_{L^2(\Omega)} \propto \varepsilon^{1/2}
  \]
Open Problems

- Extend the GL energy asymptotics to the whole surface superconductivity regime, i.e., for $1 < b < \Theta_0^{-1}$.

Pan’s conjecture

[Pan ’02] The density $|\Psi_{GL}|^2$ is close to $f_0^2(0)$ in $L^\infty(\partial\Omega)$, i.e.,

$$\| |\Psi_{GL}|(r) - f_0(0) \|_{L^\infty(\partial\Omega)} = o(1)$$

- A stronger version of Pan’s conjecture is $\| |\Psi_{GL}| - f_0 \|_{L^\infty(A_\varepsilon)} = o(1)$ in any boundary layer $A_\varepsilon$ containing the bulk of superconductivity.
- Since $f_0 > 0$, Pan’s conjecture would imply no vortices in $A_\varepsilon$.
- In [Almog, Helffer ’07] argument against Pan’s conjecture.
Main Results: Energy Asymptotics

Energy and Density Asymptotics

Theorem (GL asymptotics [MC, Rougerie ’13])

Let $\Omega \subset \mathbb{R}^2$ be any smooth simply connected domain. For any fixed $1 \leq b < \Theta_0^{-1}$ in the limit $\varepsilon \to 0$, one has

\[
E_{\varepsilon}^{\text{GL}} = \left| \partial \Omega \right| \frac{E_{0}^{\text{1D}}}{\varepsilon} + \mathcal{O}(1),
\]

\[
\| \Psi_{\varepsilon}^{\text{GL}} \|_{L^2(\Omega)}^2 - f_0^2(t) \|_{L^2(\Omega)} = \mathcal{O}(\varepsilon)
\]

- For $1 \leq b < \Theta_0^{-1}$, $f_0 > 0$ and $\| f_0^2(t) \|_{L^2(A_{\varepsilon})} \propto \varepsilon^{1/2}$.
- The above result is still compatible with vortices in $A_{\varepsilon}$, as it is the error $\mathcal{O}(1)$ in the energy asymptotics.
- $\nabla \Psi_{\varepsilon}^{\text{GL}} = \mathcal{O}(\varepsilon^{-1})$ and a variation of the density on a ball of radius $\mathcal{O}(\varepsilon)$ to produce a zero (vortex) would have an energy cost of order $\varepsilon^{-2} \mathcal{O}(\varepsilon^2) = \mathcal{O}(1)$ (the density terms come multiplied by $\varepsilon^{-2}$).
Main Results: Energy Asymptotics

**Energy and Density Asymptotics**

**Theorem (GL asymptotics [MC, Rougerie ’13])**

Let \( \Omega \subset \mathbb{R}^2 \) be any smooth simply connected domain. For any fixed \( 1 \leq b < \Theta_0^{-1} \) in the limit \( \varepsilon \to 0 \), one has

\[
E_{\varepsilon}^{GL} = \frac{|\partial \Omega|E_{0}^{1D}}{\varepsilon} + \mathcal{O}(1),
\]

\[
\| |\Psi_{GL}^2 - f_0^2(t)\|_{L^2(\Omega)} = \mathcal{O}(\varepsilon)
\]

- For \( 1 \leq b < \Theta_0^{-1} \), \( f_0 > 0 \) and \( \| f_0^2(t)\|_{L^2(A_\varepsilon)} \propto \varepsilon^{1/2} \).
- The above result is still compatible with vortices in \( A_\varepsilon \), as it is the error \( \mathcal{O}(1) \) in the energy asymptotics.
- \( \nabla \Psi_{GL} = \mathcal{O}(\varepsilon^{-1}) \) and a variation of the density on a ball of radius \( \mathcal{O}(\varepsilon) \) to produce a zero (vortex) would have an energy cost of order \( \varepsilon^{-2}\mathcal{O}(\varepsilon^2) = \mathcal{O}(1) \) (the density terms come multiplied by \( \varepsilon^{-2} \)).
To prove a stronger estimate of $\Psi^{\text{GL}}$, one has to refine the error term $O(1) \implies$ the leading $\varepsilon-$dependent terms must be retained.

After the replacement of $A^{\text{GL}}$ and the change to boundary coordinates ($t$ rescaled) with $k(s)$ the boundary curvature, the energy reads

$$\frac{1}{\varepsilon} \int_0^{t_\varepsilon} \int_0^{|\partial \Omega|} ds \int_0^{|\partial \Omega|} dt \left(1 - \varepsilon k(s)t\right) \left\{ |\partial_t \psi|^2 + \frac{1}{(1 - \varepsilon k(s)t)^2} |(\varepsilon \partial_s + ia_\varepsilon) \psi|^2 \right.$$  

$$\left. - \frac{1}{2b} \left[2|\psi|^2 - |\psi|^4\right]\right\}$$

where $t_\varepsilon = c_0 |\log \varepsilon|$ and $a_\varepsilon \simeq -t + \frac{1}{2} \varepsilon k(s)t^2$.

Consider the disc case, i.e., $k(s) \equiv k$ constant and plug the ansatz $\psi(s,t) \approx f(t) e^{-i \frac{\alpha}{\varepsilon} s}$ into the energy: one gets in units $\frac{|\partial \Omega|}{\varepsilon}$

$$\mathcal{E}_{1D}^{k,\alpha}[f] = \int_0^{c_0 |\log \varepsilon|} dt \left(1 - \varepsilon kt\right) \left\{ |\partial_t f|^2 + V_{\varepsilon,\alpha} f^2 - \frac{1}{2b} (2f^2 - f^4)\right\}$$
Refined Boundary Behavior

To prove a stronger estimate of $\Psi^{GL}$, one has to refine the error term $O(1) \implies$ the leading $\varepsilon-$dependent terms must be retained.

After the replacement of $A^{GL}$ and the change to boundary coordinates ($t$ rescaled) with $k(s)$ the boundary curvature, the energy reads

$$
\frac{1}{\varepsilon} \int_0^{\varepsilon} |\partial \Omega| ds \int_0^{t_\varepsilon} dt \left(1 - \varepsilon k(s)t\right) \left\{ |\partial_t \psi|^2 + \frac{1}{(1-\varepsilon k(s)t)^2} |(\varepsilon \partial_s + ia_\varepsilon) \psi|^2 - \frac{1}{2b} [2|\psi|^2 - |\psi|^4] \right\}
$$

where $t_\varepsilon = c_0 |\log \varepsilon|$ and $a_\varepsilon \simeq -t + \frac{1}{2} \varepsilon k(s)t^2$.

Consider the disc case, i.e., $k(s) \equiv k$ constant and plug the ansatz $\psi(s,t) \simeq f(t) e^{-i \frac{\alpha}{\varepsilon} s}$ into the energy: one gets in units $|\partial \Omega| / \varepsilon$

$$
\mathcal{E}^{1D}_{k,\alpha}[f] = \int_0^{c_0 |\log \varepsilon|} dt \left(1 - \varepsilon kt\right) \left\{ |\partial_t f|^2 + V_{\varepsilon,\alpha} f^2 - \frac{1}{2b} (2f^2 - f^4) \right\}
$$
The potential is approximately the translated harmonic potential:

\[ V_{\varepsilon, \alpha}(t) = \frac{(t + \alpha - \frac{1}{2}\varepsilon k t^2)^2}{(1 - \varepsilon k t)^2} = (t + \alpha)^2 + \mathcal{O}(\varepsilon |\log \varepsilon|). \]

Minimization of \( E_{k,\alpha}^{1D}[f] \) w.r.t to \( f \) yields an energy \( E_{k,\alpha}^{1D} \) and a minimizer \( f_{k,\alpha} \), which is non-trivial if \( 1 < b < \Theta_0^{-1} \).

Minimizing \( E_{k,\alpha}^{1D} \) w.r.t \( \alpha \in \mathbb{R} \), we get the energy

\[ E_*(k) = \min_{\alpha \in \mathbb{R}} E_{k,\alpha}^{1D} \]

an optimal phase \( \alpha(k) \) and a density \( f_k = f_{k,\alpha(k)} \).
The potential is approximately the translated harmonic potential:

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an optimal phase \( \alpha(k) \) and a density \( f_k = f_{k, \alpha(k)} \).
Main Results: Density Asymptotics

**Refined Energy Asymptotics**

**Theorem (Energy asymptotics [MC, Rougerie ’14])**

Let \( \Omega \subset \mathbb{R}^2 \) be any smooth simply connected domain with boundary curvature \( k(s) \). For any fixed \( 1 < b < \Theta_0^{-1} \) in the limit \( \varepsilon \to 0 \), one has

\[
E_{\varepsilon}^{\text{GL}} = \frac{1}{\varepsilon} \int_0^{\partial \Omega} ds \, E_\ast(k(s)) + O(\varepsilon |\log \varepsilon|^\infty)
\]

Expanding further \( E_\ast(k(s)) \), one gets [MC, Rougerie ‘15]

\[
E_{\varepsilon}^{\text{GL}} = \frac{|\partial \Omega|}{\varepsilon} E_{0}^{1D} + \mathcal{E}_{\text{corr}}[f_0] + o(1)
\]

\[
\mathcal{E}_{\text{corr}}[f_0] = \int_0^\infty dt \, t \left\{ (f'_0)^2 + (-\alpha(t + \alpha_0) - \frac{1}{b} + \frac{1}{2b} f_0^2) f_0^2 \right\}.
\]
**Proof of Pan’s Conjecture**

**Theorem (Density asymptotics [MC, Rougerie ’14])**

Let $\Omega \subset \mathbb{R}^2$ be any smooth simply connected domain. For any fixed $1 < b < \Theta_0^{-1}$ in the limit $\varepsilon \to 0$, one has

$$\|\Psi^{GL} - f_0(0)\|_{L^\infty(\partial \Omega)} = O(\varepsilon^{1/4}|\log \varepsilon|)$$

- Stronger result $\|\Psi^{GL} - f_0(\varepsilon t)\|_{L^\infty(A_{bl})} = o(1)$ in any suitable boundary layer $A_{bl} \subset \{\text{dist}(\mathbf{r}, \partial \Omega) \leq C\varepsilon \sqrt{|\log \varepsilon|}\}$.
- The estimate is stated in terms of $f_0$ because $\|f_0 - f_k\|_\infty = O(\sqrt{\varepsilon})$, but only $f_k(s)$ yields the refined energy estimate.
- The result proves the original form of Pan’s conjecture and therefore we can conclude that minimizers $\Psi^{GL}$ can not be continuous in $\varepsilon, b$. 
Main Results: Density Asymptotics

Winding Number

- Since $f_0(0) > 0$ for $1 < b < \Theta_0^{-1}$, $\Psi^{GL}$ does not vanish at the boundary and therefore its phase circulation is well defined, i.e.,
  
  $$2\pi \deg (\Psi^{GL}, \partial \Omega) = -i \int_{\partial \Omega} \frac{d\Psi}{\Psi} \frac{\partial s}{|\Psi|} \left( \frac{\Psi}{|\Psi|} \right).$$

- The estimate is stated in terms of $\alpha_0$ because $|\alpha_0 - \alpha(k)| = O(\sqrt{\varepsilon})$, but only $\alpha(k(s))$ yields the refined energy estimate.

Theorem (Degree asymptotics [MC, Rougerie ’14])

Let $\Omega \subset \mathbb{R}^2$ be any smooth simply connected domain. For any fixed $1 < b < \Theta_0^{-1}$ in the limit $\varepsilon \to 0$, one has

$$\deg (\Psi^{GL}, \partial \Omega) = \frac{|\Omega|}{\varepsilon^2} - \frac{\alpha_0}{\varepsilon} + O(\varepsilon^{-3/4}|\log \varepsilon|\infty).$$
Main Results: Curvature Corrections

Curvature Corrections

To leading order $|\Psi^{GL}| \simeq f_0(\varepsilon t)$ and superconductivity is uniformly distributed in the boundary layer. Any effect of the curvature?

Recalling that $E^{1D}(k) = E_0^{1D} + \mathcal{E}_{\text{corr}}[f_0] + \mathcal{O}(\varepsilon^{3/2}|\log \varepsilon|\infty)$.

The sign of $C_2(b)$ determines whether superconductivity is attracted or repelled by points of large curvature.

Theorem (Curvature corrections [MC, Rougerie ’15])

For any $1 < b < \Theta_0^{-1}$ as $\varepsilon \to 0$ and for any “rectangular” set $D$

$$\int_D |\Psi^{GL}|^4 = \varepsilon C_1(b) |\partial \Omega \cap \partial D| + \varepsilon^2 C_2(b) \int_{\partial D \cap \partial \Omega} ds k(s) + o(\varepsilon^2)$$

with $C_1(b) = -2bE^{1D}_0 \geq 0$ and $C_2(b) = 2b\mathcal{E}_{\text{corr}}[f_0]$. 
Effect of Corners

- So far we have considered only domains with smooth boundary. What happens if the boundary is not smooth but contains corners?

- The presence of corners might affect the boundary distribution of superconductivity.

- The third critical field $H_{c3}$ can also be shifted because of corners.

- From now on we will assume that the boundary of $\Omega$ is a Lipschitz boundary with finitely many corners.

- The normal $n(s)$ as well as tubular coordinates and the curvature $k(s)$ are all defined only a.e., with jumps at corners.
**Main Results: Effect of Corners**

$H_{c3}$ **with Corners**

- Back to the linear problem: if we decrease $h_{ex}$ from huge values:
  \[
  E_{GL}^{\varepsilon}[\Psi] \simeq \int_{\Omega} dr \left\{ \left| \left( \nabla + i \frac{A}{\varepsilon^2} \right) \Psi \right|^2 - \frac{1}{b\varepsilon^2} |\Psi|^2 \right\} = \left\langle \Psi \left| H_\varepsilon - \frac{1}{b\varepsilon^2} \right| \Psi \right\rangle
  \]
  
  The ground state $\psi_\varepsilon$ of $H_\varepsilon$ is still **localized** on a scale $\varepsilon$ and blowing up a new effective problem emerges, i.e., the magnetic Laplacian on a sector with opening angle $\vartheta$.

- The ground state energy $\gamma(\vartheta)/\varepsilon^2$ of $H_\varepsilon$ is mostly unknown, but
  
  - $\gamma(\vartheta) \to 0$ as $\vartheta \to 0$ [Bonnaillie-Noël, Dauge '06];
  
  - $\gamma(\pi) = \Theta_0$ and $\gamma(\vartheta) < \Theta_0$ if $\vartheta < \pi/2$ [Bonnaillie-Noël '05];

**Conjecture (⋆ Bonnaillie-Noël, Dauge ’07)**

Motivated by numerical computations, one expects that

- $\gamma(\vartheta)$ is **increasing** in $\vartheta$;

- $\gamma(\vartheta) < \Theta_0$ for $\vartheta < \pi$;

- $\gamma(\vartheta) = \Theta_0$ for $\vartheta \geq \pi$. 
Main Results: Effect of Corners

$H_{c3}$ with Corners

- Back to the linear problem: if we decrease $h_{ex}$ from huge values:
  $$\mathcal{E}_{\varepsilon}^{GL}[^{[\Psi]}] \sim \int_{\Omega} dr \left\{ \left| (\nabla + i \frac{A}{\varepsilon^2}) \Psi \right|^2 - \frac{1}{b\varepsilon^2} |\Psi|^2 \right\} = \langle \Psi | H_{\varepsilon} - \frac{1}{b\varepsilon^2} | \Psi \rangle$$

- The ground state $\psi_{\varepsilon}$ of $H_{\varepsilon}$ is still localized on a scale $\varepsilon$ and blowing up a new effective problem emerges, i.e., the magnetic Laplacian on a sector with opening angle $\vartheta$.

- The ground state energy $\gamma(\vartheta)/\varepsilon^2$ of $H_{\varepsilon}$ is mostly unknown, but
  - $\gamma(\vartheta) \rightarrow 0$ as $\vartheta \rightarrow 0$ [Bonnaillie-Noël, Dauge ’06];
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Motivated by numerical computations, one expects that

- $\gamma(\vartheta)$ is increasing in $\vartheta$;
- $\gamma(\vartheta) < \Theta_0$ for $\vartheta < \pi$;
- $\gamma(\vartheta) = \Theta_0$ for $\vartheta \geq \pi$. 
A New Critical Field $H_{\text{corner}}$?

$H_{c3}$ with corners (Bonnaillie-Noël, Fournais ’07)

Assuming $(\star)$, in presence of corners of angles $\vartheta_j$ (at least one $\vartheta_j < \pi$)

$$H_{c3} = \lambda_*^{-1} \varepsilon^{-2} + O(1)$$

with $\lambda_* = \min_j \lambda(\vartheta_j)$.

- According to the conjecture $(\star)$, $\lambda_* < \Theta_0$ and therefore $H_{c3}$ is larger in presence of corners.
- Before disappearing, superconductivity gets concentrated near a corner with smallest opening angle and $\Psi_{\text{GL}}$ decays exponentially in the distance from that corner.
- What happens to surface superconductivity? is there another field $H_{c2} < H_{\text{corner}} < H_{c3}$ marking the transition from boundary to corner concentration?
The presence of corners has no effect to leading order.

\( H_{c2} \) is unaffected but, if (⋆) is correct, one would expect that

\[
H_{\text{corner}} = \Theta_0^{-1} \varepsilon^{-2} + O(1).
\]
Derive the first order corrections to the GL energy expansion in presence of corners [MC, GIACOMELLI in progress]:

\[ E^{GL} = \frac{|\partial \Omega|}{\varepsilon} E^{1D}_0 + \mathcal{E}_{corr}[f_0] + \sum_j \mathcal{E}_{corners}(\vartheta_j) + o(1). \]

- Proof of Pan’s conjecture with corners \( \implies \) existence and asymptotic value of \( H_{\text{corner}} \).
- Derive superconductivity distribution with corners.

Thank you for the attention!
Perspectives

- Derive the first order corrections to the GL energy expansion in presence of corners [MC, GIACOMELLI in progress]:

\[ E_{GL} = \left| \partial \Omega \right| \frac{E_{1D}^0}{\varepsilon} + \mathcal{E}_{corr}[f_0] + \sum_j \mathcal{E}_{corners}(\vartheta_j) + o(1). \]

- Proof of Pan’s conjecture with corners \( \Rightarrow \) existence and asymptotic value of \( H_{\text{corner}} \).

- Derive superconductivity distribution with corners.

Thank you for the attention!
Sketch of the Proof

1. Restriction to the boundary layer + magnetic field replacement.
2. Upper bound (trivial): test $\mathcal{E}_{hp}$ on $\psi_{\text{trial}}(\sigma, t) \simeq f_0(t) e^{-i\alpha_0 \sigma}$.
   - Lower bound:
     4. Use of the potential function.
     5. Positivity of the cost function.

Magnetic field replacement [Fournais, Helffer ’10]

- Agmon estimates $\implies$ restriction to the boundary layer (with $\varepsilon t = \text{dist}(r, \partial \Omega)$, $\sigma = \varepsilon s$) $A_\varepsilon = \left\{ 0 \leq \sigma \leq \frac{|\partial \Omega|}{\varepsilon}, 0 \leq t \leq c_0 |\log \varepsilon| \right\}$.
- Gauge choice + elliptic estimates $\implies$ up to error terms of order $O(\varepsilon)$, $E^{GL}$ is given by (with $\psi = e^{-i\phi_\varepsilon} \Psi^{GL}$)

$$\mathcal{E}_{hp}[\psi] = \int_{\mathbb{R} \times \mathbb{R}^+} d\sigma dt \left\{ |(\nabla - ite_\sigma) \psi|^2 - \frac{1}{b} |\psi|^2 + \frac{1}{2b} |\psi|^4 \right\}$$
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  $E_{GL}$ is given by (with $\psi = e^{-i\phi}\Psi^{GL}$)

  $$\mathcal{E}_{hp}[^{\psi}] = \int_{\mathbb{R} \times \mathbb{R}^+} d\sigma dt \left\{ |(\nabla - ite_\sigma)\psi|^2 - \frac{1}{b} |\psi|^2 + \frac{1}{2b} |\psi|^4 \right\}$$
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$$
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$$
Energy Asymptotics: Lower Bound

\[ \mathcal{E}_{hp}[\psi] = \int_{\mathbb{R} \times \mathbb{R}^+} d\sigma dt \left\{ |(\nabla - i\mathbf{e}_\sigma) \psi|^2 - \frac{1}{b} |\psi|^2 + \frac{1}{2b} |\psi|^4 \right\} \]

3 Energy Splitting

- If \( 1 \leq b < \Theta_0^{-1} \), one can set \( \psi(\sigma, t) = f_0(t)e^{-i\alpha_0 \sigma} v(\sigma, t) \).

- Using the variational equation of \( f_0 \) and its boundary conditions

\[ \mathcal{E}_{hp}[\psi] = \left| \frac{\partial \Omega}{\varepsilon} \right| E_0^{1D} + \mathcal{E}[v] \]

with \( j(v) = \frac{i}{2} (v \nabla v^* - v^* \nabla v) \) the superconducting current and

\[ \mathcal{E}[v] = \int d\sigma dt f_0^2 \left\{ |\nabla v|^2 - 2(t + \alpha_0) \mathbf{e}_\sigma \cdot \mathbf{j} + \frac{1}{2b} f_0^2 (1 - |v|^2)^2 \right\} \]

- It remain to bound \( \mathcal{E}[v] \) and we will eventually show that \( \mathcal{E}[v] \geq 0 \).
Energy Asymptotics: Lower Bound

\[ \mathcal{E}[v] = \int_{\mathbb{R} \times \mathbb{R}^+} d\sigma dt \, f_0^2 \left\{ |\nabla v|^2 - 2(t + \alpha_0) e_\sigma \cdot j(v) + \frac{1}{2b} f_0^2 \left( 1 - |v|^2 \right)^2 \right\} \]

4 Use of the Potential Function

- The field \( 2(t + \alpha_0) f_0^2 e_\sigma \) is divergence free so that one can find \( F \) such that \( \nabla \perp F = 2(t + \alpha_0) f_0^2 e_\sigma \), e.g., the potential function

  \[ F_0(t) = 2 \int_0^t d\eta \, (\eta + \alpha_0) f_0^2(\eta). \]

- \( F_0(0) = F_0(\infty) = 0 \) (by optimality of \( \alpha_0 \)), \( F_0'(0) < 0 \) and \( F_0 \) has a unique extreme point \( \Rightarrow F_0 \leq 0 \).

- Stokes formula yields

  \[ \mathcal{E}[v] = \int_{\mathbb{R} \times \mathbb{R}^+} d\sigma dt \, \left\{ f_0^2(t) |\nabla v|^2 + F_0(t) \mu + \frac{1}{2b} f_0^4(t) \left( 1 - |v|^2 \right)^2 \right\} \]

  with \( \mu = \text{curl}(j(v)) \) the vorticity measure, satisfying \( |\mu| \leq |\nabla v|^2 \).
\[ E[v] = \int_{\mathbb{R} \times \mathbb{R}^+} \text{d}\sigma \text{d}t \, f_0^2 \left\{ |\nabla v|^2 - 2(t + \alpha_0) \mathbf{e}_\sigma \cdot \mathbf{j}(v) + \frac{1}{2b} f_0^2 (1 - |v|^2)^2 \right\} \]

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**Energy Asymptotics: Lower Bound**

\[ E[v] = \int_{\mathbb{R} \times \mathbb{R}^+} d\sigma dt \ f_0^2 \left\{ |\nabla v|^2 - 2(t + \alpha_0) e_\sigma \cdot j(v) + \frac{1}{2b} f_0^2 (1 - |v|^2)^2 \right\} \]

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Energy Asymptotics: Lower Bound

\[ \mathcal{E}[v] = \int_{\mathbb{R} \times \mathbb{R}^+} d\sigma dt \, f_0^2 \left\{ |\nabla v|^2 - 2(t + \alpha_0) e_\sigma \cdot j(v) + \frac{1}{2b} f_0^2 (1 - |v|^2)^2 \right\} \]

Use of the Potential Function

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  with \( \mu = \text{curl}(j(v)) \) the vorticity measure, satisfying \( |\mu| \leq |\nabla v|^2 \).
**Positivity of the cost function**

- We define the vortex cost function as
  \[ K_0(t) = f_0^2(t) + F_0(t) \]

- If \( 1 \leq b < \Theta_0^{-1} \), \( K_0(t) \geq 0 \), for any \( t \in \mathbb{R}^+ \), which allows to conclude that \( \mathcal{E}[u] \geq 0 \) and the lower bound is proven.

- Optimality condition + variational equation for \( f_0 \) imply a remarkable identity for \( F_0(t) \) yielding
  \[ K_0(t) = (1 - \frac{1}{b}) f_0^2(t) + (t + \alpha_0)^2 f_0^2(t) + \frac{1}{2b} f_0^4(t) - f_0'^2(t) \]

- \( K_0(0) > 0 \) and \( K_0(+\infty) = 0 \) \iff \( K < 0 \) somewhere \( \exists t_0 > 0 \) global minimum for \( K_0 \) and \( K'_0(t_0) = 0 \). Since \( K'_0 = 2 f_0 f'_0 + 2 (t + \alpha_0) f_0^2 \) one has \( f'_0(t_0) = -(t_0 + \alpha_0) f_0(t_0) \) and
  \[ K_0(t_0) = (1 - \frac{1}{b}) f_0^2(t_0) + \frac{1}{2b} f_0^4(t_0) \geq 0 \]
5. **Positivity of the cost function**

- We define the vortex cost function as
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- If \( 1 \leq b < \Theta^{-1}_0 \), \( K_0(t) \geq 0 \), for any \( t \in \mathbb{R}^+ \), which allows to conclude that \( \mathcal{E}[u] \geq 0 \) and the lower bound is proven.

- Optimality condition + variational equation for \( f_0 \) imply a remarkable identity for \( F_0(t) \) yielding
  \[ K_0(t) = \left( 1 - \frac{1}{b} \right) f_0^2(t) + (t + \alpha_0)^2 f_0^2(t) + \frac{1}{2b} f_0^4(t) - f_0'^2(t) \]

- \( K_0(0) > 0 \) and \( K_0(\infty) = 0 \) \( \implies \) if \( K < 0 \) somewhere \( \exists t_0 > 0 \) global minimum for \( K_0 \) and \( K_0'(t_0) = 0 \). Since \( K_0' = 2f_0f_0' + 2(t + \alpha_0)f_0^2 \) one has
  \[ f_0'(t_0) = -(t_0 + \alpha_0)f_0(t_0) \]
  and
  \[ K_0(t_0) = \left( 1 - \frac{1}{b} \right) f_0^2(t_0) + \frac{1}{2b} f_0^4(t_0) \geq 0 \]
We define the vortex cost function as

\[ K_0(t) = f_0^2(t) + F_0(t) \]

If \( 1 \leq b < \Theta_0^{-1} \), \( K_0(t) \geq 0 \), for any \( t \in \mathbb{R}^+ \), which allows to conclude that \( \mathcal{E}[u] \geq 0 \) and the lower bound is proven.

Optimality condition + variational equation for \( f_0 \) imply a remarkable identity for \( F_0(t) \) yielding

\[ K_0(t) = (1 - \frac{1}{b}) f_0^2(t) + (t + \alpha_0)^2 f_0^2(t) + \frac{1}{2b} f_0^4(t) - f_0'(t)^2 \]

\( K_0(0) > 0 \) and \( K_0(+\infty) = 0 \) \( \Rightarrow \) if \( K < 0 \) somewhere \( \exists t_0 > 0 \) global minimum for \( K_0 \) and \( K'_0(t_0) = 0 \). Since \( K'_0 = 2f_0f'_0 + 2(t + \alpha_0)f_0^2 \) one has \( f'_0(t_0) = -(t_0 + \alpha_0)f_0(t_0) \) and

\[ K_0(t_0) = (1 - \frac{1}{b}) f_0^2(t_0) + \frac{1}{2b} f_0^4(t_0) \geq 0 \]