Eigensystem Bootstrap Multiscale Analysis for the Anderson Model

Abel Klein
University of California, Irvine

with C.S. Sidney Tsang
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\[ (\Delta \varphi)(x) := \sum_{y \in \mathbb{Z}^d; \ |y-x|=1} \varphi(y) \quad \text{for} \quad \varphi \in \ell^2(\mathbb{Z}^d). \]
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\[ S_\mu(t) := \sup_{a \in \mathbb{R}} \mu \{ [a, a+t] \} \leq K t^\alpha \quad \text{for} \quad t \in [0, 1]. \]
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Recall \( \sigma(H_{\varepsilon, \omega}) = \Sigma_\varepsilon := [-2d\varepsilon, 2d\varepsilon] + \text{supp} \mu \) with probability one.
Given $\Phi \subset \Theta \subset \mathbb{Z}^d$, we consider $\ell^2(\Phi) \subset \ell^2(\Theta)$ by extending functions on $\Phi$ to functions on $\Theta$ that are identically 0 on $\Theta \setminus \Phi$. 

$\parallel x \parallel = \max_{j=1,2,\ldots,d} |x_j|$ and $|x| = \sqrt{\sum_{j=1}^d x_j^2}$ for $x \in \mathbb{R}^d$. 

We consider $\mathbb{Z}^d \subset \mathbb{R}^d$ and use boxes in $\mathbb{Z}^d$ centered at points in $\mathbb{R}^d$: 

$\Lambda_L(x) = \Lambda_R^L(x) \cap \mathbb{Z}^d$, where $x \in \mathbb{R}^d$ and $\Lambda_R^L(x) = \{ y \in \mathbb{R}^d; \parallel y - x \parallel \leq L \}$. 

Note that $(L - 2)^d < |\Lambda_L(x)| \leq (L + 1)^d$ for $L \geq 2$. 

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Eigenpairs and eigensystems

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- We call \((\varphi, \lambda)\) an eigenpair for \( H_\Theta \) if \( \lambda \) is an eigenvalue for \( H_\Theta \) and \( \varphi \) is a corresponding normalized eigenfunction, that is,

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H_\Theta \varphi = \lambda \varphi, \quad \text{where} \quad \lambda \in \mathbb{R} \quad \text{and} \quad \varphi \in \ell^2(\Theta) \quad \text{with} \quad \| \varphi \| = 1.
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- A collection \( \{ (\varphi_j, \lambda_j) \}_{j \in J} \) of eigenpairs for \( H_\Theta \) will be called an eigensystem for \( H_\Theta \) if \( \{ \varphi_j \}_{j \in J} \) is an orthonormal basis for \( \ell^2(\Theta) \).
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- If $\Theta$ is finite and all eigenvalues of $H_\Theta$ are simple, we can rewrite an eigensystem as $\{(\varphi_\lambda, \lambda)\}_{\lambda \in \sigma(H_\Theta)}$. 
Localizing boxes

Definition

Fix $\beta, \tau \in (0,1)$. Let $m > 0$. A box $\Lambda_L$ will be called $m$-localizing for $H = H_{\varepsilon, \omega}$ if

$$\left| \lambda - \lambda' \right| \geq e^{-L \beta} \quad \text{for all} \quad \lambda, \lambda' \in \sigma(H_{\Lambda_L}), \lambda \neq \lambda'.$$

There exists an $m$-localized eigensystem for $H_{\Lambda_L}$: an eigensystem $\{ (\phi_x, \lambda_x) \}_{x \in \Lambda_L}$ for $H_{\Lambda_L}$ such that $\phi_x$ is $m$-localized for all $x \in \Lambda_L$, that is,

$$\left| \phi_x(y) \right| \leq e^{-m \|y - x\|} \quad \text{for all} \quad y \in \Lambda_L \quad \text{with} \quad \|y - x\| \geq L \tau.$$
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   \[ |\varphi_x(y)| \leq e^{-m\|y-x\|} \quad \text{for all} \quad y \in \Lambda_L \quad \text{with} \quad \|y-x\| \geq L^{\tau}. \]

We need to specify \( \beta, \tau \in (0, 1) \) in the definition of an \( m \)-localizing box.
Theorem

Let $H_{\varepsilon,\omega}$ be an Anderson model. There exists $\varepsilon_0 > 0$ with the following property:

Given $\xi \in (0, 1)$, fix $\beta, \tau \in (0, 1)$ such that, for some $\gamma > 1$,

\[ 0 < \xi < \beta < 1 < \frac{\gamma}{2} < 1 < \gamma < \sqrt{\beta \xi} \text{ and } \max\{1+\frac{\gamma \beta}{2}, (\gamma-1)\beta+1, \gamma \} < \tau < 1. \]

Then there exist a scale $\tilde{L}_\xi$ and $m_\xi > 0$, such that for all $0 < \varepsilon \leq \varepsilon_0$ we have

\[ \inf_{x \in \mathbb{R}^d} P\{\Lambda_{L}(x) \text{ is } m_\xi \text{-localizing for } H_{\varepsilon,\omega}\} \geq 1 - e^{-L_\xi} \text{ for all } L \geq \tilde{L}_\xi. \]

This theorem was originally proved by Elgart and Klein for a fixed $\xi \in (0, 1)$, that is, with $\varepsilon_0$ depending on $\xi$. 
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The Green’s function MSA is done either for a fixed energy, or for all energies but with two boxes with an ‘either or’ statement for each energy. The eigensystem MSA treats all energies in a single box, giving directly a complete picture in a fixed box.

The eigensystem MSA implies the conclusions of the Green’s function MSA. Conversely, we can recover the conclusions of the eigensystem MSA from the Green’s function MSA except for the labeling. The labeling can then be established using the argument based on Hall’s Marriage Theorem.
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1. All eigenvalues of $H_{\Theta}$ are simple, i.e., $|\sigma(H_{\Theta})| = |\Theta|$,
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**Lemma (Klein-Molchanov)**

Let $\Theta \subset \mathbb{Z}^d$ be finite. Then, for all $\varepsilon \leq 1$,

$$P\{\Theta \text{ is } \eta\text{-level spacing for } H_{\varepsilon, \omega}\} \geq 1 - Y \mu \eta^{2\alpha - 1} |\Theta|^2.$$

In the special case of a box $\Lambda_L$, we have

$$P\{\Lambda_L \text{ is } \eta\text{-level spacing for } H_{\varepsilon, \omega}\} \geq 1 - Y \mu (L + 1)^2 \eta^{2\alpha - 1}.$$

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P\{\Theta \text{ is } \eta\text{-level spacing for } H_{\varepsilon,\omega}\} \geq 1 - Y_\mu \eta^{2\alpha-1} |\Theta|^2.
\]
Probability estimates for level spacing sets

The eigensystem MSA does not use a Wegner estimate; it uses instead a probability estimate for level spacing sets derived from Minami’s estimate.

Definition
Let $\eta > 0$. A finite set $\Theta \subset \mathbb{Z}^d$ will be called $\eta$-level spacing for $H$ if

1. all eigenvalues of $H_{\Theta}$ are simple, i.e., $|\sigma(H_{\Theta})| = |\Theta|$, 
2. $|\lambda - \lambda'| \geq \eta$ for all $\lambda, \lambda' \in \sigma(H_{\Theta}), \lambda \neq \lambda'$.

Lemma (Klein-Molchanov)

Let $\Theta \subset \mathbb{Z}^d$ be finite. Then, for all $\varepsilon \leq 1$,

$$\mathbb{P}\{\Theta \text{ is } \eta\text{-level spacing for } H_{\varepsilon, \omega}\} \geq 1 - Y_{\mu} \eta^{2\alpha - 1} |\Theta|^2.$$  
In the special case of a box $\Lambda_L$, we have

$$\mathbb{P}\{\Lambda_L \text{ is } \eta\text{-level spacing for } H_{\varepsilon, \omega}\} \geq 1 - Y_{\mu} (L + 1)^{2d} \eta^{2\alpha - 1}.$$
The eigensystem MSA does not use a Wegner estimate; it uses instead a probability estimate for level spacing sets derived from Minami’s estimate.

Definition
Let \( \eta > 0 \). A finite set \( \Theta \subset \mathbb{Z}^d \) will be called \( \eta \)-level spacing for \( H \) if
1. all eigenvalues of \( H_\Theta \) are simple, i.e., \( |\sigma(H_\Theta)| = |\Theta| \),
2. \( |\lambda - \lambda'| \geq \eta \) for all \( \lambda, \lambda' \in \sigma(H_\Theta), \lambda \neq \lambda' \).

Lemma (Klein-Molchanov)
Let \( \Theta \subset \mathbb{Z}^d \) be finite. Then, for all \( \varepsilon \leq 1 \),
\[
P\{ \Theta \text{ is } \eta \text{-level spacing for } H_{\varepsilon,\omega} \} \geq 1 - Y_\mu \eta^{2\alpha - 1} |\Theta|^2.
\]
In the special case of a box \( \Lambda_L \), we have
\[
P\{ \Lambda_L \text{ is } \eta \text{-level spacing for } H_{\varepsilon,\omega} \} \geq 1 - Y_\mu (L + 1)^{2d} \eta^{2\alpha - 1}.
\]
(We will take \( \eta = e^{-L^\beta} \) and \( \eta = L^{-q} \).)
Level spacing and localizing eigensystems

Definition Fix $H = H_{\varepsilon, \omega}$, $q > 0$ and $\beta, \tau \in (0, 1)$.  

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Level spacing and localizing eigensystems

Definition  Fix $H = H_{\epsilon, \omega}$, $q > 0$ and $\beta, \tau \in (0, 1)$. Let $\Lambda_L$ be a box, $\theta, m > 0$, $s \in (0, 1)$. 
Level spacing and localizing eigensystems

Definition  Fix $H = H_{\epsilon, \omega}$, $q > 0$ and $\beta, \tau \in (0, 1)$. Let $\Lambda_L$ be a box, $\theta, m > 0$, $s \in (0, 1)$.

1. $\Lambda_L$ is polynomially level spacing (PLS) if it is $L^{-q}$-level spacing.

2. $\Lambda_L$ is level spacing (LS) if it is $e^{-L^{\beta}}$-level spacing.

3. A $\theta$-polynomially localized eigensystem (PLE) for $H_{\Lambda_L}$ is an eigensystem $\{ (\phi_x, \lambda_x) \}_{x \in \Lambda_L}$ for $H_{\Lambda_L}$ such that for all $x \in \Lambda_L$ we have $|\phi_x(y)| \leq L^{-\theta}$ for all $y \in \Lambda_L$ with $\|y - x\| \geq L^{1/2}$.

4. A $s$-subexponentially localized eigensystem (SLE) for $H_{\Lambda_L}$ is an eigensystem $\{ (\phi_x, \lambda_x) \}_{x \in \Lambda_L}$ for $H_{\Lambda_L}$ such that for all $x \in \Lambda_L$ we have $|\phi_x(y)| \leq e^{-L^{s}}$ for all $y \in \Lambda_L$ with $\|y - x\| \geq L^{1/2}$.

5. An $m$-localized eigensystem (LE) for $H_{\Lambda_L}$ is an eigensystem $\{ (\phi_x, \lambda_x) \}_{x \in \Lambda_L}$ for $H_{\Lambda_L}$ such that for all $x \in \Lambda_L$ we have $|\phi_x(y)| \leq e^{-m\|y - x\|}$ for all $y \in \Lambda_L$ with $\|y - x\| \geq L^{1/2}$.
Level spacing and localizing eigensystems

Definition  Fix $H = H_{\varepsilon, \omega}$, $q > 0$ and $\beta, \tau \in (0, 1)$.
Let $\Lambda_L$ be a box, $\theta, m > 0$, $s \in (0, 1)$.

1. $\Lambda_L$ is polynomially level spacing (PLS) if it is $L^{-q}$-level spacing.
2. $\Lambda_L$ is level spacing (LS) if it is $e^{-L^\beta}$-level spacing.
Level spacing and localizing eigensystems

Definition  Fix $H = H_{\varepsilon, \omega}$, $q > 0$ and $\beta, \tau \in (0, 1)$. Let $\Lambda_L$ be a box, $\theta, m > 0$, $s \in (0, 1)$.

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2. $\Lambda_L$ is level spacing (LS) if it is $e^{-L^\beta}$-level spacing.
3. A $\theta$-polynomially localized eigensystem (PLE) for $H_{\Lambda_L}$ is an eigensystem $\{(\phi_x, \lambda_x)\}_{x \in \Lambda_L}$ for $H_{\Lambda_L}$ such that for all $x \in \Lambda_L$ we have

$$|\phi_x(y)| \leq L^{-\theta} \quad \text{for all} \quad y \in \Lambda_L \quad \text{with} \quad \|y - x\| \geq \frac{L}{20}.$$
Level spacing and localizing eigensystems

Definition  Fix $H = H_{ε,ω}$, $q > 0$ and $β, τ ∈ (0,1)$. Let $Λ_L$ be a box, $θ, m > 0$, $s ∈ (0,1)$.

1. $Λ_L$ is polynomially level spacing (PLS) if it is $L^{-q}$-level spacing.
2. $Λ_L$ is level spacing (LS) if it is $e^{-L^β}$-level spacing.
3. A $θ$-polynomially localized eigensystem (PLE) for $H_{Λ_L}$ is an eigensystem $\{(φ_x, λ_x)\}_{x ∈ Λ_L}$ for $H_{Λ_L}$ such that for all $x ∈ Λ_L$ we have

$$|φ_x(y)| ≤ L^{-θ} \text{ for all } y ∈ Λ_L \text{ with } \|y - x\| ≥ \frac{L}{20}.$$ 

4. A $s$-subexponentially localized eigensystem (SLE) for $H_{Λ_L}$ is an eigensystem $\{(φ_x, λ_x)\}_{x ∈ Λ_L}$ for $H_{Λ_L}$ such that for all $x ∈ Λ_L$ we have

$$|φ_x(y)| ≤ e^{-L^s} \text{ for all } y ∈ Λ_L \text{ with } \|y - x\| ≥ \frac{L}{20}.$$
Level spacing and localizing eigensystems

Definition
Fix $H = H_{\varepsilon, \omega}$, $q > 0$ and $\beta, \tau \in (0, 1)$.

Let $\Lambda_L$ be a box, $\theta, m > 0$, $s \in (0, 1)$.

1. $\Lambda_L$ is polynomially level spacing (PLS) if it is $L^{-q}$-level spacing.
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   $$|\varphi_x(y)| \leq L^{-\theta} \quad \text{for all } y \in \Lambda_L \quad \text{with} \quad \|y - x\| \geq \frac{L}{20}.$$

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   $$|\varphi_x(y)| \leq e^{-L^s} \quad \text{for all } y \in \Lambda_L \quad \text{with} \quad \|y - x\| \geq \frac{L}{20}.$$

5. An $m$-localized eigensystem (LE) for $H_{\Lambda_L}$ is an eigensystem $\{(\varphi_x, \lambda_x)\}_{x \in \Lambda_L}$ for $H_{\Lambda_L}$ such that for all $x \in \Lambda_L$ we have
   $$|\varphi_x(y)| \leq e^{-m\|y - x\|} \quad \text{for all } y \in \Lambda_L \quad \text{with} \quad \|y - x\| \geq L^\tau.$$
Definition Fix $H = H_{\varepsilon, \omega}$, $q > 0$ and $\beta, \tau \in (0, 1)$. Let $\Lambda_L$ be a box, and consider $\theta > 0$, $m > 0$, and $s \in (0, 1)$. 
Hierarchy of localizing boxes for the bootstrap MSA

Definition Fix $H = H_{\varepsilon, \omega}$, $q > 0$ and $\beta, \tau \in (0,1)$. Let $\Lambda_L$ be a box, and consider $\theta > 0$, $m > 0$, and $s \in (0,1)$.

1. $\Lambda_L$ is $\theta$-polynomially localizing (PL) if $\Lambda_L$ is PLS and there is a $\theta$-PLE for $H_{\Lambda_L}$.
Hierarchy of localizing boxes for the bootstrap MSA

**Definition** Fix $H = H_{\varepsilon, \omega}$, $q > 0$ and $\beta, \tau \in (0, 1)$. Let $\Lambda_L$ be a box, and consider $\theta > 0$, $m > 0$, and $s \in (0, 1)$.

1. $\Lambda_L$ is $\theta$-polynomially localizing (PL) if $\Lambda_L$ is PLS and there is a $\theta$-PLE for $H_{\Lambda_L}$.
2. $\Lambda_L$ is $m$-mix localizing (ML) if $\Lambda_L$ is PLS and there is an $m$-LE for $H_{\Lambda_L}$.
Hierarchy of localizing boxes for the bootstrap MSA

Definition

Fix $H = H_{\varepsilon, \omega}$, $q > 0$ and $\beta, \tau \in (0, 1)$. Let $\Lambda_L$ be a box, and consider $\theta > 0$, $m > 0$, and $s \in (0, 1)$.

1. $\Lambda_L$ is $\theta$-polynomially localizing (PL) if $\Lambda_L$ is PLS and there is a $\theta$-PLE for $H_{\Lambda_L}$.

2. $\Lambda_L$ is $m$-mix localizing (ML) if $\Lambda_L$ is PLS and there is an $m$-LE for $H_{\Lambda_L}$.

3. $\Lambda_L$ is $s$-subexponentially localizing (SEL) if $\Lambda_L$ is LS and there is a $s$-SLE for $H_{\Lambda_L}$. 
The bootstrap multiscale analysis

Hierarchy of localizing boxes for the bootstrap MSA

Definition  Fix $H = H_{\varepsilon, \omega}$, $q > 0$ and $\beta, \tau \in (0, 1)$. Let $\Lambda_L$ be a box, and consider $\theta > 0$, $m > 0$, and $s \in (0, 1)$.

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3. $\Lambda_L$ is $s$-subexponentially localizing (SEL) if $\Lambda_L$ is LS and there is an $s$-SLE for $H_{\Lambda_L}$.

4. $\Lambda_L$ is $m$-localizing (LOC) if $\Lambda_L$ is LS and there is an $m$-LE for $H_{\Lambda_L}$. 

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The eigensystem bootstrap multiscale analysis

Theorem

Let $H_{\varepsilon, \omega}$ be an Anderson model, and consider $\theta > \left( \frac{6}{2\alpha - 1} + \frac{9}{2} \right) d$. 

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The eigensystem bootstrap multiscale analysis

Theorem

Let $H_{\epsilon,\omega}$ be an Anderson model, and consider $\theta > \left( \frac{6}{2\alpha - 1} + \frac{9}{2} \right) d$. There exists a finite scale $\mathcal{L}(\theta)$ with the following property:
The eigensystem bootstrap multiscale analysis

Theorem

Let $H_{\varepsilon, \omega}$ be an Anderson model, and consider $\theta > \left(\frac{6}{2\alpha - 1} + \frac{9}{2}\right) d$. There exists a finite scale $\mathcal{L}(\theta)$ with the following property: Suppose for some $\varepsilon \in (0, 1]$, $L_0 \geq \mathcal{L}(\theta)$, and $0 \leq P_0 < \frac{1}{2(800)^2d}$, we have

$$\inf_{x \in \mathbb{R}^d} \mathbb{P}\{\Lambda_{L_0}(x) \text{ is } \theta\text{-polynomially localizing for } H_{\varepsilon, \omega}\} \geq 1 - P_0.$$
The bootstrap multiscale analysis

The eigensystem bootstrap multiscale analysis

Theorem

Let $H_{\epsilon, \omega}$ be an Anderson model, and consider $\theta > \left( \frac{6}{2\alpha - 1} + \frac{9}{2} \right) d$.

There exists a finite scale $L(\theta)$ with the following property:

Suppose for some $\epsilon \in (0, 1]$, $L_0 \geq L(\theta)$, and $0 \leq P_0 < \frac{1}{2(800)^{2\alpha}}$, we have

$$\inf_{x \in \mathbb{R}^d} \mathbb{P}\{ \Lambda_{L_0}(x) \text{ is } \theta\text{-polynomially localizing for } H_{\epsilon, \omega}\} \geq 1 - P_0.$$  

Then, given $0 < \xi < 1$, we can find a finite scale $\tilde{L} = \tilde{L}(\theta, \xi, L_0)$ and $m_\xi = m(\xi, \tilde{L}) > 0$ such that

$$\inf_{x \in \mathbb{R}^d} \mathbb{P}\{ \Lambda_L(x) \text{ is } m_\xi\text{-localizing for } H_{\epsilon, \omega}\} \geq 1 - e^{-L^\xi} \quad \text{for } L \geq \tilde{L}.$$
The initial step for the BMSA

Proposition

Given \( q > \frac{2d}{\alpha} \) and \( \varepsilon \in (0, 1] \), let \( \theta_{\varepsilon, L} = \frac{L}{\log L} \log \left( 1 + \frac{L^{-q}}{2d\varepsilon} \right) \).
The initial step for the BMSA

Proposition

Given $q > \frac{2d}{\alpha}$ and $\varepsilon \in (0, 1]$, let $\theta_{\varepsilon, L} = \frac{L}{\log L} \log \left(1 + \frac{L^{-q}}{2d\varepsilon}\right)$.

Then

$$\inf_{x \in \mathbb{R}^d} \mathbb{P}\{\Lambda_L(x) \text{ is } \theta_{\varepsilon, L}-polynomially localizing for } H_{\varepsilon, \omega}\} \geq 1 - \frac{1}{2} K(L + 1)^{2d} (8d\varepsilon + 2L^{-q})^\alpha.$$
The initial step for the BMSA

Proposition

Given $q > \frac{2d}{\alpha}$ and $\varepsilon \in (0, 1]$, let

$$\theta_{\varepsilon, L} = \frac{L}{20} \log \log \left(1 + \frac{L^{-q}}{2d\varepsilon}\right).$$

Then

$$\inf_{x \in \mathbb{R}^d} \mathbb{P}\{\Lambda_L(x)\text{ is } \theta_{\varepsilon, L}\text{-polynomially localizing for } H_{\varepsilon, \omega}\} \geq 1 - \frac{1}{2} K(L + 1)^{2d} \left(8d\varepsilon + 2L^{-q}\right)^{\alpha}.$$ 

In particular, given $\theta > 0$ and $P_0 > 0$, there exists a finite scale $L(q, \theta, P_0)$ such that for all $L \geq L(q, \theta, P_0)$ and all $0 < \varepsilon \leq \frac{1}{4d} L^{-q}$ we have

$$\inf_{x \in \mathbb{R}^d} \mathbb{P}\{\Lambda_L(x)\text{ is } \theta\text{-polynomially localizing for } H_{\varepsilon, \omega}\} \geq 1 - P_0.$$
Comments on the proof of the BMSA

We fix \( \theta > \left( \frac{6}{2\alpha - 1} + \frac{9}{2} \right) d \), \( 0 < \xi < 1 \), and \( p > 0 \).
Comments on the proof of the BMSA

We fix $\theta > \left( \frac{6}{2\alpha - 1} + \frac{9}{2} \right) d$, $0 < \xi < 1$, and $p > 0$. We introduce the following parameters:
We fix $\theta > \left( \frac{6}{2\alpha - 1} + \frac{9}{2} \right) d$, $0 < \xi < 1$, and $p > 0$.

We introduce the following parameters:

- We fix $q, \gamma_1$ such that
  \[ \frac{3d}{2\alpha - 1} < q < \frac{1}{2} \left( \theta - \frac{9}{2} d \right), \]
  \[ 0 < p < (2\alpha - 1)q - 3d, \quad \text{and} \quad 1 < \gamma_1 < \min \left\{ 1 + \frac{p}{p+2d}, \frac{2\theta - 4d}{5d+4q} \right\}. \]
Comments on the proof of the BMSA

We fix $\theta > \left( \frac{6}{2\alpha - 1} + \frac{9}{2} \right) d$, $0 < \xi < 1$, and $p > 0$. We introduce the following parameters:

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- We fix $\zeta, \beta, \gamma, \tau$ such that $0 < \xi < \zeta < \beta < \frac{1}{\gamma} < 1 < \gamma < \sqrt{\frac{\zeta}{\xi}}$
  and $\max \left\{ \frac{1 + \gamma_1}{2\gamma_1}, \frac{1 + \gamma \beta}{2}, \frac{(\gamma - 1)\beta + 1}{\gamma} \right\} < \tau < 1$. 
The bootstrap multiscale analysis

Comments on the proof of the BMSA

We fix \( \theta > \left( \frac{6}{2\alpha-1} + \frac{9}{2} \right) d \), \( 0 < \xi < 1 \), and \( p > 0 \).

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  and \( \max \left\{ \frac{1+\gamma_1}{2\gamma_1}, \frac{1+\gamma\beta}{2}, \frac{(\gamma-1)\beta+1}{\gamma} \right\} < \tau < 1 \).

- We fix \( s \) such that \( \max \left\{ \gamma\beta, 1 - 2\gamma \left( \tau - \frac{1+\gamma\beta}{2} \right) \right\} < s < 1 \).
Comments on the proof of the BMSA

We fix $\theta > \left(\frac{6}{2\alpha - 1} + \frac{9}{2}\right) d$, $0 < \xi < 1$, and $p > 0$.

We introduce the following parameters:

- We fix $q, \gamma_1$ such that $\frac{3d}{2\alpha - 1} < q < \frac{1}{2} \left(\theta - \frac{9}{2} d\right)$, $0 < p < (2\alpha - 1)q - 3d$, and $1 < \gamma_1 < \min \left\{1 + \frac{p}{p + 2d}, \frac{2\theta - 4d}{5d + 4q}\right\}$.

- We fix $\zeta, \beta, \gamma, \tau$ such that $0 < \xi < \zeta < \beta < \frac{1}{\gamma} < 1 < \gamma < \sqrt{\frac{\zeta}{\xi}}$ and $\max\left\{\frac{1 + \gamma_1}{2\gamma_1}, \frac{1 + \gamma \beta}{2}, \frac{(\gamma - 1)\beta + 1}{\gamma}\right\} < \tau < 1$.

- We fix $s$ such that $\max\left\{\gamma \beta, 1 - 2\gamma \left(\tau - \frac{1 + \gamma \beta}{2}\right)\right\} < s < 1$.

These parameters $q, \beta, \tau$, etc. will be omitted from the dependence of the constants.
We fix \( \theta > \left( \frac{6}{2\alpha - 1} + \frac{9}{2} \right) d \), \( 0 < \xi < 1 \), and \( p > 0 \).

We introduce the following parameters:

- We fix \( q, \gamma_1 \) such that \( \frac{3d}{2\alpha - 1} < q < \frac{1}{2} \left( \theta - \frac{9}{2} d \right) \),
  \( 0 < p < (2\alpha - 1)q - 3d \), and \( 1 < \gamma_1 < \min \left\{ 1 + \frac{p}{p + 2d}, \frac{2\theta - 4d}{5d + 4q} \right\} \).

- We fix \( \zeta, \beta, \gamma, \tau \) such that \( 0 < \xi < \zeta < \beta < \frac{1}{\gamma} < 1 < \gamma < \sqrt{\frac{\zeta}{\xi}} \)
  and \( \max \left\{ \frac{1 + \gamma_1}{2\gamma_1}, \frac{1 + \gamma \beta}{2}, \frac{(\gamma - 1)\beta + 1}{\gamma} \right\} < \tau < 1 \).

- We fix \( s \) such that \( \max \left\{ \gamma \beta, 1 - 2\gamma \left( \tau - \frac{1 + \gamma \beta}{2} \right) \right\} < s < 1 \).

These parameters \( q, \beta, \tau, \) etc. will be omitted from the dependence of the constants.

The proof of the theorem proceeds by 4 multiscale analysis plus 2 intermediate steps.
The first multiscale analysis

Proposition

Fix \( Y \geq 400 \) and \( P_0 < \frac{1}{2} (2Y)^{-2d} \). There exists a finite scale \( L(Y) \) with the following property:

Suppose for some scale \( L_0 \geq L(Y) \) and \( \epsilon \in (0,1] \) we have

\[
\inf_{x \in \mathbb{R}^d} P\left\{ \Lambda_{L_0}(x) \text{ is } \theta \text{-polynomially localizing for } H_{\epsilon, \omega} \right\} \geq 1 - P_0.
\]

Then, setting \( L_k + 1 = Y L_k \) for \( k = 0, 1, \ldots \), there exists \( K_0 = K_0(Y, L_0, P_0) \in \mathbb{N} \) such that

\[
\inf_{x \in \mathbb{R}^d} P\left\{ \Lambda_{L_k}(x) \text{ is } \theta \text{-polynomially localizing for } H_{\epsilon, \omega} \right\} \geq 1 - L^{-p_k} \text{ for } k \geq K_0.
\]
Proposition

Fix $Y \geq 400$ and $P_0 < \frac{1}{2}(2Y)^{-2d}$. There exists a finite scale $\mathcal{L}(Y)$ with the following property: Suppose for some scale $L_0 \geq \mathcal{L}(Y)$ and $\varepsilon \in (0,1]$ we have

$$\inf_{x \in \mathbb{R}^d} \mathbb{P}\{\Lambda_{L_0}(x) \text{ is } \theta\text{-polynomially localizing for } H_{\varepsilon,\omega}\} \geq 1 - P_0.$$
The first multiscale analysis

Proposition

Fix $Y \geq 400$ and $P_0 < \frac{1}{2} (2Y)^{-2d}$. There exists a finite scale $\mathcal{L}(Y)$ with the following property: Suppose for some scale $L_0 \geq \mathcal{L}(Y)$ and $\varepsilon \in (0, 1]$ we have

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Then, setting $L_{k+1} = YL_k$ for $k = 0, 1, \ldots$, there exists $K_0 = K_0(Y, L_0, P_0) \in \mathbb{N}$ such that

$$\inf_{x \in \mathbb{R}^d} \mathbb{P}\{\Lambda_{L_k}(x) \text{ is } \theta\text{-polynomially localizing for } H_{\varepsilon, \omega}\} \geq 1 - L_k^{-p} \text{ for } k \geq K_0.$$
The first intermediate step

Proposition

Suppose for some scale \( \ell \) and \( \varepsilon \in (0, 1] \) we have

\[
\inf_{x \in \mathbb{R}^d} \mathbb{P}\{ \Lambda_{\ell}(x) \text{ is } \theta\text{-polynomially localizing for } H_{\varepsilon, \omega} \} \geq 1 - \ell^{-p}.
\]
The first intermediate step

**Proposition**

Suppose for some scale $\ell$ and $\varepsilon \in (0, 1]$ we have

$$\inf_{x \in \mathbb{R}^d} \mathbb{P}\{\Lambda_\ell(x) \text{ is } \theta\text{-polynomially localizing for } H_{\varepsilon, \omega}\} \geq 1 - \ell^{-p}.$$ 

Let $L = \ell^{\gamma_1}$. If $\ell$ is sufficiently large, we have

$$\inf_{x \in \mathbb{R}^d} \mathbb{P}\{\Lambda_L(x) \text{ is } m_0^\ast\text{-mix localizing for } H_{\varepsilon, \omega}\} \geq 1 - L^{-p},$$

where

$$m_0^\ast \geq \frac{1}{8} \left( \frac{5d}{2} + q \right) L^{-(1-\tau+\frac{1}{\gamma_1})} \log L.$$
Proposition

*There exists a finite scale $L$ with the following property:*
The second multiscale analysis

Proposition

There exists a finite scale $L$ with the following property: Suppose for some scale $L_0 \geq L$, $\varepsilon \in (0, 1)$, and $m^*_0 \geq L_0^{-\kappa}$, where $0 < \kappa < \tau$, we have

$$\inf_{x \in \mathbb{R}^d} \mathbb{P}\{ \Lambda_{L_0}(x) \text{ is } m^*_0\text{-mix localizing for } H_{\varepsilon, \omega} \} \geq 1 - L_0^{-p}.$$
The second multiscale analysis

Proposition

There exists a finite scale $\mathcal{L}$ with the following property: Suppose for some scale $L_0 \geq \mathcal{L}$, $\epsilon \in (0, 1]$, and $m_0^* \geq L_0^{-\kappa}$, where $0 < \kappa < \tau$, we have

$$\inf_{x \in \mathbb{R}^d} \mathbb{P}\{\Lambda_{L_0}(x) \text{ is } m_0^*-\text{mix localizing for } H_{\epsilon, \omega}\} \geq 1 - L_0^{-p}.$$ 

Then, setting $L_{k+1} = L_k^{\gamma_k}$ for $k = 0, 1, \ldots$, we have

$$\inf_{x \in \mathbb{R}^d} \mathbb{P}\{\Lambda_{L_k}(x) \text{ is } m_0^*-\frac{2}{2}\text{-mix localizing for } H_{\epsilon, \omega}\} \geq 1 - L_k^{-p} \text{ for } k = 0, 1, \ldots.$$
The second multiscale analysis

Proposition

There exists a finite scale $L$ with the following property: Suppose for some scale $L_0 \geq L$, $\epsilon \in (0,1)$, and $m^* \geq L_0^{-\kappa}$, where $0 < \kappa < \tau$, we have

$$\inf_{x \in \mathbb{R}^d} \mathbb{P}\{\Lambda_{L_0}(x) \text{ is } m^*-\text{mix localizing for } H_{\epsilon, \omega}\} \geq 1 - L_0^{-p}.$$

Then, setting $L_{k+1} = L_k^{\gamma_k}$ for $k = 0, 1, \ldots$, we have

$$\inf_{x \in \mathbb{R}^d} \mathbb{P}\{\Lambda_{L_k}(x) \text{ is } \frac{m^*}{2}\text{-mix localizing for } H_{\epsilon, \omega}\} \geq 1 - L_k^{-p} \text{ for } k = 0, 1, \ldots.$$

$\Lambda_L$ is $m^*$-mix localizing $\implies \Lambda_L$ is $\left(1 - \frac{\log \frac{40}{m^*}}{\log L}\right)$-SEL $\implies \Lambda_L$ is $s$-SEL

for sufficiently large $L$ and $m^* < 40$. 

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The third multiscale analysis

Proposition

Fix $Y \geq 400 \frac{1}{1-s}$ and $\tilde{P}_0 < (2(2Y)(\lfloor Y^s \rfloor + 1)^d)^{-\frac{1}{[Y^s]}}$. There exists a finite scale $\mathcal{L}(Y)$ with the following property:
The bootstrap multiscale analysis

The third multiscale analysis

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Fix $Y \geq 400^{\frac{1}{1-s}}$ and $\widetilde{P}_0 < (2(2Y)(\lceil Y^s \rceil + 1)^d)^{-\frac{1}{\lceil Y^s \rceil}}$.

There exists a finite scale $L(Y)$ with the following property:
Suppose for some scale $L_0 \geq L(Y)$ and $\varepsilon \in (0, 1]$ we have

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The third multiscale analysis

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Then, setting $L_{k+1} = YL_k$ for $k = 0, 1, \ldots$, there exists $K_0 = K_0(Y, L_0, \tilde{P}_0) \in \mathbb{N}$ such that

$$\inf_{x \in \mathbb{R}^d} \mathbb{P}\{\Lambda_{L_k}(x) \text{ is } s\text{-SEL for } H_{\epsilon, \omega}\} \geq 1 - e^{-L_k^\zeta} \quad \text{for } k \geq K_0.$$
The second intermediate step

Proposition

Suppose for some scale $\ell$ and $\varepsilon \in (0, 1]$ we have

$$\inf_{x \in \mathbb{R}^d} \mathbb{P}\{\Lambda_\ell(x) \text{ is } s\text{-SEL for } H_{\varepsilon, \omega}\} \geq 1 - e^{-\ell \zeta}.$$
The bootstrap multiscale analysis

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$$\inf_{x \in \mathbb{R}^d} \mathbb{P}\{\Lambda_{\ell}(x) \text{ is } s\text{-SEL for } H_{\varepsilon, \omega}\} \geq 1 - e^{-\ell \zeta}.$$ 

Let $L = \ell^\gamma$. If $\ell$ is sufficiently large, we have

$$\inf_{x \in \mathbb{R}^d} \mathbb{P}\{\Lambda_{L}(x) \text{ is } m_0\text{-localizing for } H_{\varepsilon, \omega}\} \geq 1 - e^{-L \zeta},$$

where

$$m_0 \geq \frac{1}{8} L^{-\left(1 - \tau + \frac{1-s}{\gamma}\right)}.$$
The fourth multiscale analysis

Proposition

There exists a finite scale $L$ with the following property:

$\inf_{x \in \mathbb{R}^d} P\{\Lambda_{L^0}(x) \text{ is } m_0\text{-localizing for } H_\varepsilon, \omega\} \geq 1 - e^{-L} \zeta_0$. 

Then, setting $L_{k+1}^\gamma = L_k$ for $k = 0, 1, ...$, we have

$\inf_{x \in \mathbb{R}^d} P\{\Lambda_{L_k}(x) \text{ is } m_0^2\text{-localizing for } H_\varepsilon, \omega\} \geq 1 - e^{-L} \zeta_k$ for $k = 0, 1, ...$.

Moreover, we have

$\inf_{x \in \mathbb{R}^d} P\{\Lambda_{L_k}(x) \text{ is } m_0^4\text{-localizing for } H_\varepsilon, \omega\} \geq 1 - e^{-L} \xi$ for all $L \geq L_0^\gamma$. 

Abel Klein
The fourth multiscale analysis

Proposition

*There exists a finite scale $L$ with the following property: Suppose for some scale $L_0 \geq L$, $\epsilon \in (0, 1]$, and $m_0 \geq L_0^{-\kappa}$, where $0 < \kappa < \tau - \gamma \beta$, we have*

$$\inf_{x \in \mathbb{R}^d} P\{L_0(x) \text{ is } m_0\text{-localizing for } H_{\epsilon, \omega}\} \geq 1 - e^{-L_0^\xi}.$$
The fourth multiscale analysis

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Then, setting $L_{k+1} = L_k^{\gamma}$ for $k = 0, 1, \ldots$, we have

$$\inf_{x \in \mathbb{R}^d} \mathbb{P}\{\Lambda_{L_k}(x) \text{ is } m_0/2\text{-localizing for } H_{\varepsilon, \omega}\} \geq 1 - e^{-L_k \xi} \text{ for } k = 0, 1, \ldots.$$
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Moreover, we have

$$\inf_{x \in \mathbb{R}^d} \mathbb{P}\{\Lambda_{L_k}(x) \text{ is } \frac{m_0}{4}\text{-localizing for } H_{\varepsilon, \omega}\} \geq 1 - e^{-L_0\xi} \text{ for all } L \geq L_0^\gamma.$$
Lemmas about eigenpairs

Let $\Psi \subset \Theta \subset \mathbb{Z}^d$. Given $t \geq 1$, we set
Lemmas about eigenpairs

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\partial_{\text{ex}}^\Theta \Psi = \{ v \in (\Theta \setminus \Psi); |v - u| = 1 \text{ for some } u \in \Psi \} \\
\partial_{\text{in}}^\Theta \Psi = \{ u \in \Psi; |v - u| = 1 \text{ for some } v \in \Theta \setminus \Psi \} \\
\Psi_{\Theta,t} = \{ y \in \Psi; \text{dist}(y, \Theta \setminus \Psi) > t \}, \\
\partial_{\Theta,t}^\Psi = \partial_{\text{ex}}^\Theta \Psi \cup \left( \Psi \setminus \Psi_{\Theta,t} \right).
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Lemma

Let $\Phi \subset \Theta \subset \mathbb{Z}^d$ and suppose $(\varphi, \lambda)$ is an eigenpair for $H_\Phi$. Then

$$
\text{dist}(\lambda, \sigma(H_\Theta)) \leq \| (H_\Theta - \lambda) \varphi \| \leq (2d - 1) \varepsilon \left| \partial_{\text{ex}}^\Theta \Phi \right|^{\frac{1}{2}} \sup_{y \in \partial_{\text{in}}^\Theta \Phi} |\varphi(y)|.
$$
Lemma about localized eigenpairs

Lemma

Consider a box $\Lambda_L \subset \Theta \subset \mathbb{Z}^d$, and suppose $(\varphi, \lambda)$ is an eigenpair for $H_{\Lambda_L}$.
Lemma about localized eigenpairs

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Consider a box $\Lambda_L \subset \Theta \subset \mathbb{Z}^d$, and suppose $(\varphi, \lambda)$ is an eigenpair for $H_{\Lambda_L}$.

1. If $\varphi$ is $(x, \theta)$-polynomially localized for some $x \in \Lambda_L^{\Theta, \frac{L}{20}}$, we have

$$\text{dist}(\lambda, \sigma(H_\Theta)) \leq CL^{-\left(\theta - \frac{d-1}{2}\right)}.$$
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2. If $\varphi$ is $(x, s)$-subexponentially localized for some $x \in \Lambda_L^{\Theta, L/20}$, we have
   \[ \text{dist}(\lambda, \sigma(H_{\Theta})) \leq e^{-c_1 L^s}, \text{ where } c_1 \geq 1 - \frac{\log L}{L^s}. \]
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   \]

3. If $\varphi$ is $(x, m)$ localized for some $x \in \Lambda_L^{\Theta, L^\tau}$, we have
   \[
   \text{dist}(\lambda, \sigma(H_{\Theta})) \leq e^{-m_1 L^\tau}, \quad \text{where} \quad m_1 \geq m - C \frac{\log L}{L^\tau}.
   \]
Lemma

Let $\Theta \subset \mathbb{Z}^d$ and $0 < 4\delta < \eta$. Suppose:

- $\mu$ is a simple eigenvalue of $H_{\Theta}$ with normalized eigenfunction $\psi_{\mu}$, with $\text{dist}(\mu, \sigma(H_{\Theta})\{\mu\}) \geq \eta$.
- $\| (H_{\Theta} - \lambda) \varphi \| \leq \delta$, where $\varphi \in \ell^2(\Theta)$ with $\| \varphi \| = 1$ and $\lambda \in \mathbb{R}$ with $|\lambda - \mu| \leq \delta$.

Define $\varphi_\perp$ by $\varphi_\perp = \langle \psi_{\mu}, \varphi \rangle \psi_{\mu} + \varphi_\perp$. Then we have $|\langle \psi_{\mu}, \varphi \rangle|^2 \geq 1 - 2\delta^2 \eta^2$ and $\| \varphi_\perp \| \leq \sqrt{2} \delta \eta$.

Moreover, choosing $\varphi$ so $\langle \psi_{\mu}, \varphi \rangle > 0$, we have $\| \varphi - \psi_{\mu} \| \leq 3\delta^2 \eta$. 
Lemma about approximate eigenpairs

Lemma

Let $\Theta \subset \mathbb{Z}^d$ and $0 < 4\delta < \eta$. Suppose:

1. $\mu$ is a simple eigenvalue of $H_\Theta$ with normalized eigenfunction $\psi_\mu$, with $\text{dist}(\mu, \sigma(H_\Theta) \setminus \{\mu\}) \geq \eta$.

\[ \| (H_\Theta - \lambda) \varphi \| \leq \delta, \quad \| \varphi \| = 1, \quad \lambda \in \mathbb{R} \quad \text{with} \quad |\lambda - \mu| \leq \delta. \]

Define $\varphi_\perp$ by $\varphi_\perp = \langle \psi_\mu, \varphi \rangle \psi_\mu + \varphi_\perp$. Then we have

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$$\left| \langle \psi_\mu, \varphi \rangle \right|^2 \geq 1 - \frac{2\delta^2}{\eta^2} \quad \text{and} \quad \|\varphi^\perp\| \leq \frac{\sqrt{2}\delta}{\eta}.$$
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Moreover, choosing $\varphi$ so $\langle \psi_\mu, \varphi \rangle > 0$, we have

$$\|\varphi - \psi_\mu\| \leq \frac{3\delta}{2\eta}.$$
Localizing boxes

We will show applications of the lemmas on eigenpairs to $m$-localizing boxes. Similar results hold for $\theta$-polynomially localizing, $m$-mix localizing, and $s$-subexponentially localizing boxes.
Localizing boxes

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Lemma

Let $(\psi, \lambda)$ be a generalized eigenpair for $H_\Theta$ and $\Lambda_\ell \subset \Theta$ be an $m$-localizing box with an $m$-localized eigensystem $\{\varphi_x, \lambda_x\}_{x \in \Lambda_\ell}$, and suppose

$$|\lambda - \lambda_x| \geq \frac{1}{2} e^{-L\beta} \quad \text{for all} \quad x \in \Lambda_\ell^{\Theta, \ell\tau}.$$
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Then the following holds for sufficiently large \( L \):
Localizing boxes

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Then the following holds for sufficiently large $L$:

1. If $y \in \Lambda_{\ell}^{\Theta, 2\ell^\tau}$ we have $|\psi(y)| \leq e^{-m_2\ell^\tau} |\psi(y_1)|$ for some $y_1 \in \partial^{\Theta, 2\ell^\tau} \Lambda_\ell$. 
Localizing boxes

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Let \((\psi, \lambda)\) be a generalized eigenpair for \( H_\Theta \) and \( \Lambda_\ell \subset \Theta \) be an \( m \)-localizing box with an \( m \)-localized eigensystem \( \{\varphi_x, \lambda_x\}_{x \in \Lambda_\ell} \), and suppose

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|\lambda - \lambda_x| \geq \frac{1}{2} e^{-L^\beta} \quad \text{for all} \quad x \in \Lambda_{\Theta, \ell^\tau}.
\]

Then the following holds for sufficiently large \( L \):

1. If \( y \in \Lambda_{\ell}^{\Theta, 2\ell^\tau} \) we have \( |\psi(y)| \leq e^{-m_2\ell^\tau} |\psi(y_1)| \) for some \( y_1 \in \partial^{\Theta, 2\ell^\tau} \Lambda_\ell \).
2. If \( y \in \Lambda_{\ell}^{\Theta, 2\ell\tilde{\tau}} \), \( \tilde{\tau} = \frac{1 + \tau}{2} \), we have \( |\psi(y)| \leq e^{-m_3\|y_2 - y\|} |\psi(y_2)| \) for some \( y_2 \in \partial^{\Theta, \ell\tilde{\tau}} \Lambda_\ell \). In particular, \( \|y_2 - y\| > \ell\tilde{\tau} \).
The induction step for the 4th multiscale analysis

**Lemma**

Suppose for some scale $\ell$, $\epsilon \in (0, 1]$, and $m \geq m_- > 0$ we have

$$\inf_{x \in \mathbb{R}^d} \mathbb{P}\{\Lambda_{\ell}(x) \text{ is } m\text{-localizing for } H_{\epsilon, \omega}\} \geq 1 - e^{-\ell \xi}.$$
The induction step for the 4th multiscale analysis

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$$\inf_{x \in \mathbb{R}^d} \mathbb{P}\{ \Lambda_{\ell}(x) \text{ is } m\text{-localizing for } H_{\varepsilon, \omega} \} \geq 1 - e^{-\ell \zeta}.$$ 

Let $L = \ell^\gamma$. If $\ell$ is sufficiently large, we have

$$\inf_{x \in \mathbb{R}^d} \mathbb{P}\{ \Lambda_{L}(x) \text{ is } M\text{-localizing for } H_{\varepsilon, \omega} \} \geq 1 - e^{-L \zeta},$$

where $M \geq m \left(1 - C_{d, m_- , \varepsilon_0} \ell^{-\min\left\{ \frac{1-\tau}{2} , \gamma \tau - (\gamma-1) \tilde{\zeta} - 1 \right\}} \right)$.
Starting the proof of the induction step

♦ We cover the box $\Lambda_L = \Lambda(x_0), \, x_0 \in \mathbb{R}^d$, by boxes $\Lambda_\ell$:

$$\Lambda_L = \bigcup_{a \in \Xi_{L,\ell}} \Lambda_\ell(a), \text{ where } \Xi_{L,\ell} := \left\{x_0 + \rho \ell \mathbb{Z}^d\right\} \cap \Lambda_L^\mathbb{R}(x_0) \text{ with } \frac{3}{5} \leq \rho \leq \frac{4}{5}.$$
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♦ Let $B_N$ denote the event that there exist at most $N$ disjoint boxes $\Lambda_\ell$ in the cover that are not $m$-localizing for $H_{\varepsilon, \omega}$. We take

$$N = N_\ell = \left\lfloor \ell (\gamma^{-1}) \tilde{\zeta} \right\rfloor \quad \implies \quad \mathbb{P}\left\{ B_{N_\ell}^c \right\} \leq \frac{1}{2} e^{-L \xi}. $$
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N = N_\ell = \left\lfloor \ell (\gamma^{-1}) \tilde{\zeta} \right\rfloor \implies \mathbb{P} \left\{ \mathcal{B}_N^c \right\} \leq \frac{1}{2} e^{-L \tilde{\zeta}}.
$$

♦ Fix $\omega \in \mathcal{B}_{N_\ell}$. There exist $a_1, a_2, \ldots, a_R \in \Xi_{L,\ell}$, with $R \leq N_\ell$, such that $|a_i - a_j| \geq \ell$ for $i \neq j$ (the boxes $\{\Lambda_\ell(a_r)\}_{r=1}^R$ are disjoint, possibly non-localizing), and

$$
a \in \Xi_{L,\ell} \text{ with } \min_{r=1}^R |a - a_r| \geq \ell \implies \Lambda_\ell(a) \text{ is } m\text{-localizing}.
$$
Lemma
Let $\Lambda_\ell(a) \subset \Theta \subset \mathbb{Z}^d$, where the box $\Lambda_\ell(a)$ is $m$-localizing with an $m$-localized eigensystem $\{(\varphi_x^{(a)}, \lambda_x^{(a)})\}_{x \in \Lambda_\ell(a)}$, $\Theta$ is $e^{-L^\beta}$-level spacing, and $\{\psi_{\lambda}, \lambda\}_{\lambda \in \sigma(H_\Theta)}$ is an eigensystem for $H_\Theta$. 

1. There exists an injection $x \in \Lambda_\Theta$, $\ell$ $\tau_\ell(a) \xrightarrow{\sim} \tilde{\lambda}_x(a) x \in \sigma(H_\Theta)$, such that $|\tilde{\lambda}_x(a) x - \lambda_x(a) x| \leq e^{-m_1 \tau_\ell(a)}$ for all $x \in \Lambda_\Theta$, $\ell$ $\tau_\ell(a)$, and, multiplying each $\varphi_x^{(a)} x$ by a suitable phase factor, $\|\psi_{\tilde{\lambda}_x(a)} x - \varphi_x^{(a)} x\| \leq e^{-m'_1 \ell \tau_\ell(a)}$.

2. Let $\sigma^\{a\} (H_\Theta) := \{\tilde{\lambda}_x(a) x, x \in \Lambda_\Theta, \ell \tau_\ell(a)\}$. Then for $\lambda \in \sigma^\{a\} (H_\Theta)$ we have $|\psi_{\lambda}(y)| \leq e^{-m'_1 \ell \tau_\ell(a)}$ for all $y \in \Theta \setminus \Lambda_\ell(a)$.

3. If $\lambda \in \sigma(H_\Theta) \setminus \sigma^\{a\} (H_\Theta)$, we have $|\lambda - \lambda_x(a) x| \geq \frac{1}{2} e^{-L^\beta}$ for $x \in \Lambda_\Theta, \ell$ $\tau_\ell(a)$.

4. If $\Lambda_\ell(a), \Lambda_\ell(b) \subset \Theta$ are both as above, and $x \in \Lambda_\ell(a), y \in \Lambda_\ell(b)$, $\tilde{\lambda}_x(a) x = \tilde{\lambda}_x(b) y \Rightarrow \|x - y\| < 2 \ell \tau_\ell(a)$. 

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Lemma

Let $\Lambda_\ell(a) \subset \Theta \subset \mathbb{Z}^d$, where the box $\Lambda_\ell(a)$ is $m$-localizing with an $m$-localized eigensystem $\{(\phi_x^{(a)}, \lambda_x^{(a)})\}_{x \in \Lambda_\ell(a)}$, $\Theta$ is $e^{-L^\beta}$-level spacing, and $\{(\psi_\lambda, \lambda)\}_{\lambda \in \sigma(H_\Theta)}$ is an eigensystem for $H_\Theta$.

1. There exists an injection $x \in \Lambda_\ell^{\Theta,\ell^\tau}(a) \mapsto \tilde{\lambda}_x^{(a)} \in \sigma(H_\Theta)$, such that $|\tilde{\lambda}_x^{(a)} - \lambda_x^{(a)}| \leq e^{-m_1\ell^\tau}$ for all $x \in \Lambda_\ell^{\Theta,\ell^\tau}(a)$, and, multiplying each $\phi_x^{(a)}$ by a suitable phase factor, $\|\psi_{\tilde{\lambda}_x^{(a)}} - \phi_x^{(a)}\| \leq e^{-m_1'\ell^\tau}$.
Lemma

Let \( \Lambda_\ell(a) \subset \Theta \subset \mathbb{Z}^d \), where the box \( \Lambda_\ell(a) \) is \( m \)-localizing with an \( m \)-localized eigensystem \( \{ (\varphi^{(a)}_x, \lambda^{(a)}_x) \}_{x \in \Lambda_\ell(a)}, \Theta \) is \( e^{-L\beta} \)-level spacing, and \( \{ (\psi_\lambda, \lambda) \}_{\lambda \in \sigma(H_\Theta)} \) is an eigensystem for \( H_\Theta \).

1. There exists an injection \( x \in \Lambda^{\Theta, \ell^\tau}(a) \mapsto \tilde{\lambda}^{(a)}_x \in \sigma(H_\Theta) \), such that \( |\tilde{\lambda}^{(a)}_x - \lambda^{(a)}_x| \leq e^{-m_1 \ell^\tau} \) for all \( x \in \Lambda^{\Theta, \ell^\tau}(a) \), and, multiplying each \( \varphi^{(a)}_x \) by a suitable phase factor, \( \| \psi^{(a)}_{\tilde{\lambda}^{(a)}_x} - \varphi^{(a)}_x \| \leq e^{-m'_1 \ell^\tau} \).

2. Let \( \sigma\{a\}(H_\Theta) := \{ \tilde{\lambda}^{(a)}_x, x \in \Lambda^{\Theta, \ell^\tau}(a) \} \). Then for \( \lambda \in \sigma\{a\}(H_\Theta) \) we have \( |\psi_\lambda(y)| \leq e^{-m'_1 \ell^\tau} \) for all \( y \in \Theta \setminus \Lambda_\ell(a) \).
Lemma

Let $\Lambda_\ell(a) \subset \Theta \subset \mathbb{Z}^d$, where the box $\Lambda_\ell(a)$ is $m$-localizing with an $m$-localized eigensystem $\{((\varphi_x^{(a)}, \lambda_x^{(a)}))_{x \in \Lambda_\ell(a)}, \Theta \text{ is } e^{-L\beta} \text{-level spacing, and} \{((\psi_\lambda, \lambda))_{\lambda \in \sigma(H_\Theta)} \}$ is an eigensystem for $H_\Theta$.

1. There exists an injection $x \in \Lambda_\ell^{\Theta,\ell^\tau}(a) \mapsto \tilde{\lambda}_x^{(a)} \in \sigma(H_\Theta)$, such that $|\tilde{\lambda}_x^{(a)} - \lambda_x^{(a)}| \leq e^{-m_1\ell^\tau}$ for all $x \in \Lambda_\ell^{\Theta,\ell^\tau}(a)$, and, multiplying each $\varphi_x^{(a)}$ by a suitable phase factor, $\|\psi_{\tilde{\lambda}_x^{(a)}} - \varphi_x^{(a)}\| \leq e^{-m'_1\ell^\tau}$.

2. Let $\sigma\{a\}(H_\Theta) := \{\tilde{\lambda}_x^{(a)}, x \in \Lambda_\ell^{\Theta,\ell^\tau}(a)\}$. Then for $\lambda \in \sigma\{a\}(H_\Theta)$ we have $|\psi_\lambda(y)| \leq e^{-m'_1\ell^\tau}$ for all $y \in \Theta \setminus \Lambda_\ell(a)$.

3. If $\lambda \in \sigma(H_\Theta) \setminus \sigma\{a\}(H_\Theta)$, we have $|\lambda - \lambda_x^{(a)}| \geq \frac{1}{2}e^{-L\beta}$ for $x \in \Lambda_\ell^{\Theta,\ell^\tau}(a)$.
Lemma

Let $\Lambda_\ell(a) \subset \Theta \subset \mathbb{Z}^d$, where the box $\Lambda_\ell(a)$ is $m$-localizing with an $m$-localized eigensystem $\{(\varphi_x^{(a)}, \lambda_x^{(a)})\}_{x \in \Lambda_\ell(a)}$, $\Theta$ is $e^{-L\beta}$-level spacing, and $\{(\psi_\lambda, \lambda)\}_{\lambda \in \sigma(H_\Theta)}$ is an eigensystem for $H_\Theta$.

1. There exists an injection $x \in \Lambda^{\Theta,\ell^\tau}(a) \mapsto \tilde{\lambda}_x^{(a)} \in \sigma(H_\Theta)$, such that $|\tilde{\lambda}_x^{(a)} - \lambda_x^{(a)}| \leq e^{-m_1\ell^\tau}$ for all $x \in \Lambda^{\Theta,\ell^\tau}(a)$, and, multiplying each $\varphi_x^{(a)}$ by a suitable phase factor, $\|\psi_{\tilde{\lambda}_x^{(a)}}(a) - \varphi_x^{(a)}\| \leq e^{-m_1'\ell^\tau}$.

2. Let $\sigma_{\{a\}}(H_\Theta) := \{\tilde{\lambda}_x^{(a)}, \ x \in \Lambda^{\Theta,\ell^\tau}(a)\}$. Then for $\lambda \in \sigma_{\{a\}}(H_\Theta)$ we have $|\psi_\lambda(y)| \leq e^{-m_1'\ell^\tau}$ for all $y \in \Theta \setminus \Lambda_\ell(a)$.

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4. If $\Lambda_\ell(a), \Lambda_\ell(b) \subset \Theta$ are both as above, and $x \in \Lambda_\ell(a)$, $y \in \Lambda_\ell(b)$, $\tilde{\lambda}_x^{(a)} = \tilde{\lambda}_y^{(b)} \implies \|x - y\| < 2\ell^\tau$. 

Abel Klein
Buffered subsets

If boxes $\Lambda_\ell \subset \Lambda_L$ are not $m$-localizing, we surround them with a buffer of $m$-localizing boxes and study eigensystems for the augmented subset.
**Buffered subsets**

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**Definition**

We call $\Upsilon \subset \Lambda_L$ a buffered subset of $\Lambda_L$ if the following holds:
Buffered subsets

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**Definition**

We call $\Upsilon \subset \Lambda_L$ a buffered subset of $\Lambda_L$ if the following holds:

1. $\Upsilon$ is a connected set in $\mathbb{Z}^d$ of the form $\Upsilon = \bigcup_{j=1}^{J} \Lambda_{R_j}(a_j) \cap \Lambda_L$, where $J \in \mathbb{N}$, $a_1, a_2, \ldots, a_J \in \Lambda_L^R$, and $\ell \leq R_j \leq L$ for $j = 1, 2, \ldots, J$. 

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2. $\Upsilon$ is $e^{-L^\beta}$-level spacing for $H$. 
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2. $\Upsilon$ is $e^{-L^\beta}$-level spacing for $H$.

3. There exists $G_\Upsilon \subset \Lambda_L^R$ such that:
   4. For all $a \in G_\Upsilon$ we have $\Lambda_\ell(a) \subset \Lambda_L$, $\Lambda_\ell(a)$ is an $m$-localizing box for $H$. 
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2. $\Upsilon$ is $e^{-L^\beta}$-level spacing for $H$.

3. There exists $\mathcal{G}_\Upsilon \subset \Lambda_L^R$ such that:
   1. For all $a \in \mathcal{G}_\Upsilon$ we have $\Lambda_\ell(a) \subset \Lambda_L$, $\Lambda_\ell(a)$ is an $m$-localizing box for $H$.
   2. For all $y \in \partial_{\text{in}}^{\Lambda_L}\Upsilon$ there exists $a_y \in \mathcal{G}_\Upsilon$ such that $y \in \Lambda_{\ell,2^\tau}(a_y)$. 
Buffered subsets

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2. $\Upsilon$ is $e^{-L^\beta}$-level spacing for $H$.
3. There exists $\mathcal{G}_\Upsilon \subset \Lambda_L^R$ such that:
   1. For all $a \in \mathcal{G}_\Upsilon$ we have $\Lambda_\ell(a) \subset \Lambda_L$, $\Lambda_\ell(a)$ is an $m$-localizing box for $H$.
   2. For all $y \in \partial_{\text{in}}^\Upsilon \Upsilon$ there exists $a_y \in \mathcal{G}_\Upsilon$ such that $y \in \Lambda_{\Upsilon,2\ell^\tau}(a_y)$.

In this case we set

$$
\widehat{\Upsilon} = \bigcup_{a \in \mathcal{G}_\Upsilon} \Lambda_\ell(a), \quad \widehat{\Upsilon}^\tau = \bigcup_{a \in \mathcal{G}_\Upsilon} \Lambda_{\Upsilon,2\ell^\tau}(a), \quad \hat{\Upsilon} = \Upsilon \setminus \widehat{\Upsilon}, \quad \text{and} \quad \hat{\Upsilon}^\tau = \Upsilon \setminus \widehat{\Upsilon}^\tau.
$$
Lemma

Let $\gamma$ be a buffered subset of $\Lambda_L$, and let $\{(\psi_v, v)\}_{v \in \sigma(H_\gamma)}$ be an eigensystem for $H_\gamma$.
Lemma

Let $\Upsilon$ be a buffered subset of $\Lambda_L$, and let $\{(\psi_v, \nu)\}_{\nu \in \sigma(H_\Upsilon)}$ be an eigensystem for $H_\Upsilon$. Let

$$\sigma_{G_\Upsilon}(H_\Upsilon) = \bigcup_{a \in G_\Upsilon} \sigma_{\{a\}}(H_\Upsilon) \quad \text{and} \quad \sigma_B(H_\Upsilon) = \sigma(H_\Upsilon) \setminus \sigma_{G_\Upsilon}(H_\Upsilon).$$
Lemma

Let \( \Upsilon \) be a buffered subset of \( \Lambda_L \), and let \( \{(\psi_v, \nu)\}_{\nu \in \sigma(H_\Upsilon)} \) be an eigensystem for \( H_\Upsilon \). Let

\[
\sigma_{G_\Upsilon}(H_\Upsilon) = \bigcup_{a \in G_\Upsilon} \sigma_{\{a\}}(H_\Upsilon) \quad \text{and} \quad \sigma_{B}(H_\Upsilon) = \sigma(H_\Upsilon) \setminus \sigma_{G_\Upsilon}(H_\Upsilon).
\]

For all \( \nu \in \sigma_{B}(H_\Upsilon) \) we have

\[
|\psi_\nu(y)| \leq e^{-m_2 \ell^\tau} \text{ for all } y \in \Upsilon^\tau.
\]
Lemma

Let $\Upsilon$ be a buffered subset of $\Lambda_L$, and let $\{(\psi_\nu, \nu)\}_{\nu \in \sigma(H_\Upsilon)}$ be an eigensystem for $H_\Upsilon$. Let

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1. For all $\nu \in \sigma_B(H_\Upsilon)$ we have

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2. Let $\Lambda_L$ be level spacing for $H$, and let $\{(\phi_{\lambda}, \lambda)\}_{\lambda \in \sigma(H_{\Lambda_L})}$ be an eigensystem for $H_{\Lambda_L}$. 

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Lemma

Let $\Upsilon$ be a buffered subset of $\Lambda_L$, and let $\{(\psi_\nu, \nu)\}_{\nu \in \sigma(H_\Upsilon)}$ be an eigensystem for $H_\Upsilon$. Let

$$\sigma_{g_\Upsilon}(H_\Upsilon) = \bigcup_{a \in G_\Upsilon} \sigma_{\{a\}}(H_\Upsilon) \quad \text{and} \quad \sigma_B(H_\Upsilon) = \sigma(H_\Upsilon) \setminus \sigma_{g_\Upsilon}(H_\Upsilon).$$

1. For all $\nu \in \sigma_B(H_\Upsilon)$ we have

$$|\psi_\nu(y)| \leq e^{-m_2 \ell^\tau} \text{ for all } y \in \cdots^\tau.$$

2. Let $\Lambda_L$ be level spacing for $H$, and let $\{(\phi_\lambda, \lambda)\}_{\lambda \in \sigma(H_{\Lambda_L})}$ be an eigensystem for $H_{\Lambda_L}$. There exists an injection

$$\nu \in \sigma_B(H_\Upsilon) \mapsto \tilde{\nu} \in \sigma(H_{\Lambda_L}) \setminus \sigma_{g_\Upsilon}(H_{\Lambda_L}),$$

such that

$$|\tilde{\nu} - \nu| \leq e^{-m_4 \ell^\tau} \text{ for all } \nu \in \sigma_B(H_\Upsilon) \text{ and, multiplying each } \psi_\nu \text{ by a suitable phase factor,} \quad \|\phi_{\tilde{\nu}} - \psi_\nu\| \leq e^{-m_4' \ell^\tau}.$$
Lemma

Let $\Upsilon$ be a buffered subset of $\Lambda_L$, and let $\{(\psi_\nu, \nu)\}_{\nu \in \sigma(H_\Upsilon)}$ be an eigensystem for $H_\Upsilon$. Let

$$\sigma_{G_\Upsilon}(H_\Upsilon) = \bigcup_{a \in G_\Upsilon} \sigma_{\{a\}}(H_\Upsilon) \quad \text{and} \quad \sigma_B(H_\Upsilon) = \sigma(H_\Upsilon) \setminus \sigma_{G_\Upsilon}(H_\Upsilon).$$

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2. Let $\Lambda_L$ be level spacing for $H$, and let $\{(\phi_\lambda, \lambda)\}_{\lambda \in \sigma(H_{\Lambda_L})}$ be an eigensystem for $H_{\Lambda_L}$. There exists an injection

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such that

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We set

$$\sigma_\Upsilon(\Lambda_L) := \{\tilde{\nu}; \nu \in \sigma_B(H_\Upsilon)\} \subset \sigma(H_{\Lambda_L}) \setminus \sigma_{G_\Upsilon}(H_{\Lambda_L}).$$
Back to the proof of the induction step

\[ \Lambda_L = \bigcup_{a \in \Xi_{L,\ell}} \Lambda_\ell(a), \text{ where } \Xi_{L,\ell} := \{ x_0 + \rho \ell \mathbb{Z}^d \} \cap \Lambda^R_L(x_0) \text{ with } \frac{3}{5} \leq \rho \leq \frac{4}{5}. \]
Back to the proof of the induction step

$\Lambda_L = \bigcup_{a \in \Xi_{L,\ell}} \Lambda_\ell(a)$, where $\Xi_{L,\ell} := \{x_0 + \rho \ell \mathbb{Z}^d\} \cap \Lambda_L^{\mathbb{R}}(x_0)$ with $\frac{3}{5} \leq \rho \leq \frac{4}{5}$.

$\mathcal{B}_N$ is the event that there exist at most $N = N_\ell = \left\lfloor \ell(\gamma - 1) \tilde{\zeta} \right\rfloor$ disjoint boxes $\Lambda_\ell$ in the cover that are not $m$-localizing for $H_{\varepsilon, \omega}$. 
Back to the proof of the induction step

♦ $\Lambda_L = \bigcup_{a \in \Xi_{L,\ell}} \Lambda_\ell(a)$, where $\Xi_{L,\ell} := \{x_0 + \rho \ell \mathbb{Z}^d\} \cap \Lambda_L^\mathbb{R}(x_0)$ with $\frac{3}{5} \leq \rho \leq \frac{4}{5}$.

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♦ We have $\mathbb{P}\{\mathcal{B}_N^c\} \leq \frac{1}{2} e^{-L\zeta}$.
Back to the proof of the induction step

♦ $\Lambda_L = \bigcup_{a \in \Xi_{L,\ell}} \Lambda_\ell(a)$, where $\Xi_{L,\ell} := \{x_0 + \rho \ell \mathbb{Z}^d\} \cap \Lambda_L^R(x_0)$ with $\frac{3}{5} \leq \rho \leq \frac{4}{5}$.

♦ $\mathcal{B}_N$ is the event that there exist at most $N = N_\ell = \lfloor \ell(\gamma-1)\tilde{\zeta} \rfloor$ disjoint boxes $\Lambda_\ell$ in the cover that are not $m$-localizing for $H_{\varepsilon,\omega}$.

♦ We have $\mathbb{P}\{\mathcal{B}_N^c\} \leq \frac{1}{2} e^{-L\zeta}$.

♦ Fix $\omega \in \mathcal{B}_N$, and put the $\leq N$ possibly non-localizing boxes inside subsets $\Upsilon_r$, $r = 1, 2, \ldots, R$, which clearly satisfies all the requirements to be a buffered subset of $\Lambda_L$, except that we do not know if each $\Upsilon_r$ is $L$-level spacing for $H_{\varepsilon,\omega}$.
Key ingredients for the proof of the BMSA

Back to the proof of the induction step

♦ $\Lambda_L = \bigcup_{a \in \Xi_L, \ell} \Lambda_{\ell}(a)$, where $\Xi_L, \ell := \{x_0 + \rho \ell \mathbb{Z}^d\} \cap \Lambda_L^R(x_0)$ with $\frac{3}{5} \leq \rho \leq \frac{4}{5}$.

♦ $\mathcal{B}_N$ is the event that there exist at most $N = N_\ell = \left\lfloor \ell(\gamma - 1) \tilde{\zeta} \right\rfloor$ disjoint boxes $\Lambda_{\ell}$ in the cover that are not $m$-localizing for $H_{\varepsilon, \omega}$.

♦ We have $\mathbb{P}\{\mathcal{B}_N^c\} \leq \frac{1}{2} e^{-L\zeta}$.

♦ Fix $\omega \in \mathcal{B}_N$, and put the $\leq N$ possibly non-localizing boxes inside subsets $\Upsilon_r$, $r = 1, 2, \ldots, R$, which clearly satisfies all the requirements to be a buffered subset of $\Lambda_L$, except that we do not know if each $\Upsilon_r$ is $L$-level spacing for $H_{\varepsilon, \omega}$. Letting $\mathcal{S}_N$ be the event that the box $\Lambda_L$ and the possible choices for the subsets $\Upsilon_r$ are all $L$-level spacing for $H_{\varepsilon, \omega}$, we get

$$\mathbb{P}\{\mathcal{S}_N^c\} \leq Y_{\varepsilon_0} \left(1 + (L + 1)^d N_\ell! \left(d 4^d \right)^{N_\ell - 1} \right) (L + 1)^{2d} e^{-(2\alpha - 1)L\beta} < \frac{1}{2} e^{-L\zeta}.$$
Back to the proof of the induction step

\[ \Lambda_L = \bigcup_{a \in \Xi_L, \ell} \Lambda(\ell(a)), \text{ where } \Xi_L, \ell := \{x_0 + \rho \ell \mathbb{Z}^d\} \cap \Lambda_L^R(x_0) \text{ with } \frac{3}{5} \leq \rho \leq \frac{4}{5}. \]

\[ \mathcal{B}_N \text{ is the event that there exist at most } N = N_\ell = \left\lfloor \ell(\gamma - 1)\tilde{\zeta} \right\rfloor \text{ disjoint boxes } \Lambda_\ell \text{ in the cover that are not } m\text{-localizing for } H_{E, \omega}. \]

\[ \mathbb{P}\{\mathcal{B}_N^c\} \leq \frac{1}{2} e^{-L\zeta}. \]

\[ \text{Fix } \omega \in \mathcal{B}_N, \text{ and put the } \leq N \text{ possibly non-localizing boxes inside subsets } \Upsilon_r, r = 1, 2, \ldots, R, \text{ which clearly satisfies all the requirements to be a buffered subset of } \Lambda_L, \text{ except that we do not know if each } \Upsilon_r \text{ is } L\text{-level spacing for } H_{E, \omega}. \text{ Letting } \mathcal{L}_N \text{ be the event that the box } \Lambda_L \text{ and the possible choices for the subsets } \Upsilon_r \text{ are all } L\text{-level spacing for } H_{E, \omega}, \text{ we get} \]

\[ \mathbb{P}\{\mathcal{L}_N^c\} \leq Y_{E_0} \left(1 + (L + 1)^d N_\ell! \left(d4^d\right)^{N_\ell - 1}\right) (L + 1)^{2d} e^{-(2\alpha - 1)L^\beta} < \frac{1}{2} e^{-L\zeta}. \]

\[ \text{We now define the event } \mathcal{E}_N = \mathcal{B}_N \cap \mathcal{L}_N, \text{ so } \mathbb{P}\{\mathcal{E}_N\} > 1 - e^{-L\zeta}. \]
Back to the proof of the induction step

♦ \( \Lambda_L = \bigcup_{a \in \Xi_{L,\ell}} \Lambda(\ell(a)) \), where \( \Xi_{L,\ell} := \{ x_0 + \rho \ell \mathbb{Z}^d \} \cap \Lambda_L^R(x_0) \) with \( \frac{3}{5} \leq \rho \leq \frac{4}{5} \).

♦ \( \mathcal{B}_N \) is the event that there exist at most \( N = N_\ell = \left\lfloor \ell (\gamma-1) \tilde{\zeta} \right\rfloor \) disjoint boxes \( \Lambda_\ell \) in the cover that are not \( m \)-localizing for \( H_{\varepsilon,\omega} \).

♦ We have \( \mathbb{P}\{ \mathcal{B}_N^c \} \leq \frac{1}{2} e^{-L \zeta} \).

♦ Fix \( \omega \in \mathcal{B}_N \), and put the \( \leq N \) possibly non-localizing boxes inside subsets \( \Upsilon_r \), \( r = 1, 2, \ldots, R \), which clearly satisfies all the requirements to be a buffered subset of \( \Lambda_L \), except that we do not know if each \( \Upsilon_r \) is \( L \)-level spacing for \( H_{\varepsilon,\omega} \). Letting \( \mathcal{S}_N \) be the event that the box \( \Lambda_L \) and the possible choices for the subsets \( \Upsilon_r \) are all \( L \)-level spacing for \( H_{\varepsilon,\omega} \), we get

\[
\mathbb{P}\{ \mathcal{S}_N^c \} \leq Y_{\varepsilon_0} \left( 1 + (L + 1)^d N_\ell! \left( d4^d \right)^{N_\ell-1} \right) (L + 1)^{2d} e^{-(2\alpha-1)L^\beta} < \frac{1}{2} e^{-L \zeta}.
\]

♦ We now define the event \( \mathcal{E}_N = \mathcal{B}_N \cap \mathcal{S}_N \), so \( \mathbb{P}\{ \mathcal{E}_N \} > 1 - e^{-L \zeta} \).

♦ To finish the proof we need to show that for all \( \omega \in \mathcal{E}_N \) the box \( \Lambda_L \) is \( M \)-localizing for \( H_{\varepsilon,\omega} \).
Fix \( \omega \in G_N \). We have

\[
\Lambda_L = \left\{ \bigcup_{a \in G} \Lambda_{\ell, \frac{\ell}{10}}(a) \right\} \cup \left\{ \bigcup_{r=1}^{R} \Upsilon_{r, \frac{\ell}{10}} \right\},
\]

where \( G = \{ a \in \Xi_{L, \ell}; \Lambda_{\ell}(a) \text{ is } m\text{-localizing for } H_{\varepsilon, \omega} \} \) and \( \{ \Upsilon_r \}_{r=1}^{R} \) are buffering subsets of \( \Lambda_L \).
♦ Fix $\omega \in \mathcal{E}_N$. We have

$$\Lambda_L = \left\{ \bigcup_{a \in \mathcal{G}} \Lambda_{\ell, \frac{\ell}{10}}(a) \right\} \cup \left\{ \bigcup_{r=1}^{R} \gamma_{r, \frac{\ell}{10}} \right\},$$

where $\mathcal{G} = \{ a \in \Xi_{L, \ell} ; \, \Lambda_\ell(a) \text{ is } m\text{-localizing for } H_{\varepsilon, \omega} \}$ and $\{ \gamma_r \}_{r=1}^{R}$ are buffering subsets of $\Lambda_L$.

♦ We set (we omit $\varepsilon$ and $\omega$ from the notation.)

$$\sigma_{\mathcal{G}}(H_{\Lambda_L}) = \bigcup_{a \in \mathcal{G}} \sigma_{\{a\}}(H_{\Lambda_L}) \quad \text{and} \quad \sigma_{\mathcal{B}}(H_{\Lambda_L}) = \bigcup_{r=1}^{R} \sigma_{\gamma_r}(H_{\Lambda_L}).$$
Fix $\omega \in \mathcal{E}_N$. We have

$$\Lambda_L = \left\{ \bigcup_{a \in \mathcal{G}} \Lambda_{\ell, \frac{\ell}{10}}(a) \right\} \cup \left\{ \bigcup_{r=1}^{R} \Upsilon_{r, \frac{\ell}{10}} \right\},$$

where $\mathcal{G} = \left\{ a \in \Xi_{L, \ell} ; , \; \Lambda_{\ell}(a) \text{ is } m\text{-localizing for } H_{\varepsilon, \omega} \right\}$ and $\{ \Upsilon_r \}_{r=1}^{R}$ are buffering subsets of $\Lambda_L$.

We set (we omit $\varepsilon$ and $\omega$ from the notation.)

$$\sigma_{\mathcal{G}}(H_{\Lambda_L}) = \bigcup_{a \in \mathcal{G}} \sigma_{\{a\}}(H_{\Lambda_L}) \quad \text{and} \quad \sigma_{\mathcal{B}}(H_{\Lambda_L}) = \bigcup_{r=1}^{R} \sigma_{\Upsilon_r}(H_{\Lambda_L}).$$

We prove

$$\sigma(H_{\Lambda_L}) = \sigma_{\mathcal{G}}(H_{\Lambda_L}) \cup \sigma_{\mathcal{B}}(H_{\Lambda_L}).$$
Fix $\omega \in \mathcal{E}_N$. We have

$$\Lambda_L = \left\{ \bigcup_{a \in \mathcal{G}} \Lambda_{\ell}^{\Lambda_L, \ell/10}(a) \right\} \cup \left\{ \bigcup_{r=1}^{R} \gamma_r^{\Lambda_L, \ell/10} \right\},$$

where $\mathcal{G} = \{ a \in \Xi_{L,\ell};, \, \Lambda_{\ell}(a) \text{ is } m\text{-localizing for } H_{\varepsilon,\omega} \}$ and $\{ \gamma_r \}_{r=1}^{R}$ are buffering subsets of $\Lambda_L$.

We set (we omit $\varepsilon$ and $\omega$ from the notation.)

$$\sigma_{\mathcal{G}}(H_{\Lambda_L}) = \bigcup_{a \in \mathcal{G}} \sigma_{\{a\}}(H_{\Lambda_L}) \quad \text{and} \quad \sigma_{\mathcal{B}}(H_{\Lambda_L}) = \bigcup_{r=1}^{R} \sigma_{\gamma_r}(H_{\Lambda_L}).$$

We prove

$$\sigma(H_{\Lambda_L}) = \sigma_{\mathcal{G}}(H_{\Lambda_L}) \cup \sigma_{\mathcal{B}}(H_{\Lambda_L}).$$

We now index the eigenvalues and eigenvectors of $H_{\Lambda_L}$ by sites in $\Lambda_L$ using Hall’s Marriage Theorem, which states a necessary and sufficient condition for the existence of a perfect matching in a bipartite graph.
Hall's Marriage Theorem

Let $G = (A, B; E)$ be a bipartite graph with vertex sets $A$ and $B$ and edge set $E \subset A \times B$ (the bipartite condition).
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Given a vertex $a \in A$, let $\mathcal{N}(a) = \{b \in B; (a, b) \in E\}$, the set of neighbors of $a$. Let $\mathcal{N}(U) = \bigcup_{u \in U} \mathcal{N}(u)$ for $U \subset A$. 
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Hall’s Marriage Theorem

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Hall’s Marriage Theorem

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Let \( G = (A, B; E) \) be a bipartite graph with \( |A| = |B| \). There exists a perfect matching in \( G \) if and only if the graph \( G \) fulfills Hall’s condition.
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Hall’s Marriage Theorem

Let $G = (A, B; E)$ be a bipartite graph with $|A| = |B|$. There exists a perfect matching in $G$ if and only if the graph $G$ fulfills Hall’s condition

$$|U| \leq |N(U)| \quad \text{for all} \quad U \subseteq A.$$
We consider the bipartite graph $\mathcal{G} = (\Lambda_L, \sigma(H_{\Lambda_L}); \mathbb{E})$, where the edge set $\mathbb{E} \subset \Lambda_L \times \sigma(H_{\Lambda_L})$ is defined as follows.

$\bullet$ $N(x)$ was defined to ensure $|\psi_\lambda(x)| \ll 1$ for $\lambda \not\in N(x)$.

$\bullet$ We set $N(\Theta) = \bigcup_{x \in \Theta} N(x)$ for $\Theta \subset \Lambda_L$. Abi Klein
We consider the bipartite graph \( G = (\Lambda_L, \sigma(H_{\Lambda_L}); E) \), where the edge set \( E \subset \Lambda_L \times \sigma(H_{\Lambda_L}) \) is defined as follows. For each \( \lambda \in \sigma(H_{\Lambda_L}) \) we fix \( \lambda^{(a_\lambda)} \) such that \( \lambda = \tilde{\lambda}^{(a_\lambda)} \), and set

\[
N_0(x) = \begin{cases} 
\{ \lambda \in \sigma(H_{\Lambda_L}); \| x_\lambda - x \| < \ell^\tau \} & \text{for } x \in \Lambda_L \setminus \bigcup_{r=1}^R \hat{\Gamma}_r \\
\emptyset & \text{for } x \in \bigcup_{r=1}^R \hat{\Gamma}_r
\end{cases}
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We consider the bipartite graph $G = (\Lambda_L, \sigma(H_{\Lambda_L}); \mathcal{E})$, where the edge set $\mathcal{E} \subset \Lambda_L \times \sigma(H_{\Lambda_L})$ is defined as follows. For each $\lambda \in \sigma_{\mathcal{G}}(H_{\Lambda_L})$ we fix $\lambda^{(a_{\lambda})}_{x_{\lambda}}$ such that $\lambda = \lambda^{(a_{\lambda})}_{x_{\lambda}}$, and set

$$\mathcal{N}_0(x) = \begin{cases} 
\{ \lambda \in \sigma_{\mathcal{G}}(H_{\Lambda_L}); \| x_{\lambda} - x \| < \ell^r \} & \text{for } x \in \Lambda_L \setminus \bigcup_{r=1}^{R} \hat{\gamma}_r \\
\emptyset & \text{for } x \in \bigcup_{r=1}^{R} \hat{\gamma}_r
\end{cases}$$

We define

$$\mathcal{N}(x) = \begin{cases} 
\mathcal{N}_0(x) & \text{for } x \in \Lambda_L \setminus \bigcup_{r=1}^{R} \hat{\gamma}_{r,\tau} \\
\sigma_{\mathcal{G}_r}(H_{\Lambda_L}) & \text{for } x \in \hat{\gamma}_r, \; r = 1, 2, \ldots, R \\
\mathcal{N}_0(x) \cup \sigma_{\mathcal{G}_r}(H_{\Lambda_L}) & \text{for } x \in \hat{\gamma}_{r,\tau} \setminus \hat{\gamma}_r, \; r = 1, 2, \ldots, R
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and set $\mathcal{E} = \{(x, \lambda) \in \Lambda_L \times \sigma(H_{\Lambda_L}); \lambda \in N(x)\}$. 
We consider the bipartite graph $G = (\Lambda_L, \sigma(H\Lambda_L); E)$, where the edge set $E \subset \Lambda_L \times \sigma(H\Lambda_L)$ is defined as follows. For each $\lambda \in \sigma_g(H\Lambda_L)$ we fix $\lambda_{x_\lambda}^{(a_\lambda)}$ such that $\lambda = \tilde{\lambda}_{x_\lambda}^{(a_\lambda)}$, and set

\[
N_0(x) = \begin{cases} 
\{ \lambda \in \sigma_g(H\Lambda_L); \|x_\lambda - x\| < \ell \tau \} & \text{for } x \in \Lambda_L \setminus \bigcup_{r=1}^{R} \hat{\Upsilon}_r \\
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$$N(x) = \begin{cases} N_0(x) & \text{for } x \in \Lambda_L \setminus \bigcup_{r=1}^{R} \hat{\Upsilon}_r, \\ \sigma_{\Upsilon_r}(H_{\Lambda_L}) & \text{for } x \in \hat{\Upsilon}_r, \ r = 1, 2, \ldots, R, \\ N_0(x) \cup \sigma_{\Upsilon_r}(H_{\Lambda_L}) & \text{for } x \in \hat{\Upsilon}_{r, \tau} \setminus \hat{\Upsilon}_r, \ r = 1, 2, \ldots, R \end{cases},$$

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We set $N(\Theta) = \bigcup_{x \in \Theta} N(x)$ for $\Theta \subset \Lambda_L$. 

 Abel Klein
Since $|\Lambda_L| = |\sigma(H_{\Lambda_L})|$, to apply Hall’s Marriage Theorem we verify Hall’s condition:

$$|\Theta| \leq |\mathcal{N}(\Theta)| \quad \text{for all} \quad \Theta \subset \Lambda_L.$$
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We apply Hall’s Marriage Theorem, concluding that there exists a bijection

$$x \in \Lambda_L \mapsto \lambda_x \in \sigma(H_{\Lambda_L}), \quad \text{where} \quad \lambda_x \in \mathcal{N}(x).$$

We set $\psi_x = \psi_{\lambda_x}$ for all $x \in \Lambda_L$. 
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To finish the proof we show that $\{(\psi_x, \lambda_x)\}_{x \in \Lambda_L}$ is an $M$-localized eigensystem for $\Lambda_L$. 
\{(\psi_x, \lambda_x)\}_{x \in \Lambda_L} \text{ is an } M\text{-localized eigensystem for } \Lambda_L

We fix } \psi \in \Lambda_L \text{ and take } y \in \Lambda_L \text{ such that } \|x - y\| \gg N_{\ell \ell} \approx \ell^{(\gamma-1)\zeta+1}, \text{ and consider two cases:} \)
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1. If \( y \in \Lambda^{\Lambda_L, \ell \frac{\ell}{10}} (a) \) for some \( a \in \mathcal{G} \), we must have \( \lambda_x \notin \sigma_{\{a\}} (H_{\Lambda_L}) \), so

\[
|\psi_x(y)| \leq e^{-m_3} \|y_1 - y\| |\psi_x(y_1)| \quad \text{for some} \quad y_1 \in \partial^{\Lambda_L, \ell \tilde{\zeta}} \Lambda_\ell (a).
\]
\[(\psi_x, \lambda_x)\] is an $M$-localized eigensystem for $\Lambda_L$

We fix $x \in \Lambda_L$ and take $y \in \Lambda_L$ such that $\|x - y\| \gg N_\ell \ell \approx \ell(\gamma^{-1})^{\tilde{\zeta}} + 1$, and consider two cases:

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\[
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\]

2. If $y \in \gamma^{\ell, 10}_r$ for some $r \in \{1, 2, \ldots, R\}$, we must have $\lambda_x \notin \sigma_{\mathcal{G}, \gamma_r}(H_{\Lambda_L}) \cup \sigma_{\gamma_r}(H_{\Lambda_L})$, so

\[
|\psi_x(y)| \leq e^{-m_5\ell^\tau} |\psi_x(v)| \quad \text{for some} \quad v \in \partial^{\Lambda_L, 2\ell^\tau}_L \gamma_r.
\]
• Now let let us fix $x \in \Lambda_L$, and take $y \in \Lambda_L$ such that $\|y - x\| \geq L^\tau$. 
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• Take $|\psi_x(y)| > 0$, since otherwise there is nothing to prove.
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• Take \( |\psi_x(y)| > 0 \), since otherwise there is nothing to prove.
• We estimate \( |\psi_x(y)| \) using the two possibilities repeatedly, as appropriate, stopping when we get too close to \( x \). (Note that this must happen since \( |\psi_x(y)| > 0 \).)
Key ingredients for the proof of the BMSA

- Now let us fix $x \in \Lambda_L$, and take $y \in \Lambda_L$ such that $\|y - x\| \geq L^\tau$.
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- We accumulate decay only when we use the first possibility, and just use $e^{-m_5 L^\tau} < 1$ otherwise, getting
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• We accumulate decay only when we use the first possibility, and just use $e^{-m_{5}^{\ell} \tau} < 1$ otherwise, getting

$$|\psi_x(y)| \leq e^{-m_{3}\left(\|y-x\| - \sum_{r=1}^{R} \text{diam } \gamma_{r} - 2\ell\right)} \leq e^{-m_{3}\left(\|y-x\| - 5\ell(y-1)\tilde{\zeta} + 1 - 2\ell\right)} \leq e^{-M\|y-x\|},$$
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• We accumulate decay only when we use the first possibility, and just use $e^{-m_5 \ell^\tau} < 1$ otherwise, getting

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where

$$M \geq m \left(1 - C_{d,m_-,\varepsilon_0} \ell^{-\min\left\{ \frac{1-\tau}{2}, \gamma\tau-(\gamma-1)\tilde{\zeta}-1 \right\}} \right).$$
• Now let let us fix $x \in \Lambda_L$, and take $y \in \Lambda_L$ such that $\|y - x\| \geq L^\tau$.
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where $M \geq m \left(1 - C_{d,m_-,\varepsilon_0} \ell^{-\min\left\{\frac{1-\tau}{2}, \gamma \tau - (\gamma - 1)\tilde{\zeta} - 1\right\}}\right)$.

• We conclude that $\{(\psi_x, \lambda_x)\}_{x \in \Lambda_L}$ is an $M$-localized eigensystem for $\Lambda_L$, so the box is $\Lambda_L$ is $M$-localizing for $H_{\varepsilon, \omega}$. 

Abel Klein
Generalized eigenfunctions and eigenvalues

We fix $\nu > \frac{d}{2}$, and set $\langle x \rangle = \sqrt{1 + \|x\|^2}$. 
Generalized eigenfunctions and eigenvalues

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- $\psi: \Theta \subset \mathbb{Z}^d \to \mathbb{C}$ is a $\nu$-generalized eigenfunction for $H_\Theta$ with generalized eigenvalue $\lambda \in \mathbb{R}$ if

  $0 < \|\langle x \rangle^{-\nu} \psi\| < \infty$ and $(H\phi)(x) = \lambda \phi(x)$ for all $x \in \Theta$. 

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  \[
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  \]

- Given \( \lambda \in \mathbb{R} \) we let \( \mathcal{V}(\lambda) \) denote the collection of \( \nu \)-generalized eigenfunctions for \( H_\Omega \) with generalized eigenvalue \( \lambda \).
Generalized eigenfunctions and eigenvalues

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- Given \( \lambda \in \mathbb{R} \) we let \( \mathcal{V}(\lambda) \) denote the collection of \( \nu \)-generalized eigenfunctions for \( H_\omega \) with generalized eigenvalue \( \lambda \).

- Given \( \lambda \in \mathbb{R} \) and \( a, b \in \mathbb{Z}^d \), we set
  
  \[ W^{(a)}_\lambda(b) := \sup_{\psi \in \mathcal{V}(\lambda)} \frac{|\psi(b)|}{\|\langle x - a \rangle^{-\nu} \psi\|} \quad \text{if} \quad \mathcal{V}(\lambda) \neq \emptyset \quad \text{and} \quad 0 \quad \text{otherwise}. \]
Generalized eigenfunctions and eigenvalues

We fix $\nu > \frac{d}{2}$, and set $\langle x \rangle = \sqrt{1 + \|x\|^2}$.

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- Given $\lambda \in \mathbb{R}$ we let $\mathcal{V}(\lambda)$ denote the collection of $\nu$-generalized eigenfunctions for $H_\omega$ with generalized eigenvalue $\lambda$.

- Given $\lambda \in \mathbb{R}$ and $a, b \in \mathbb{Z}^d$, we set
  
  $$W_\lambda^{(a)}(b) := \sup_{\psi \in \mathcal{V}(\lambda)} \frac{|\psi(b)|}{\|\langle x - a \rangle^{-\nu} \psi\|} \quad \text{if} \quad \mathcal{V}(\lambda) \neq \emptyset \quad \text{and} \quad 0 \quad \text{otherwise}.$$ 

- For all $a, b \in \mathbb{Z}^d$ we have
  
  $$W_\lambda^{(a)}(b) \leq \langle a - b \rangle^\nu, \quad \text{and, in particular,} \quad W_\lambda^{(a)}(a) \leq 1.$$
Theorem encapsulating localization for the Anderson model

**Theorem**

Let \( H_{\varepsilon, \omega} \) be an Anderson model. There exists \( \varepsilon_0 > 0 \) such that, given \( \xi \in (0, 1) \), we can find a scale \( \hat{L}_\xi \) and \( m_\xi > 0 \), such that for all \( 0 < \varepsilon \leq \varepsilon_0 \), \( L \geq \hat{L}_\xi \) with \( L \in 2\mathbb{N} \), and \( a \in \mathbb{Z}^d \) there exists an event \( \mathcal{Y}_{\varepsilon, L,a} \) with the following properties:

1. \( \mathcal{Y}_{\varepsilon, L,a} \) depends only on the random variables \( \{\omega_x\} x \in \Lambda_{5L}(a) \), and \( \mathbb{P}\{\mathcal{Y}_{\varepsilon, L,a}\} \geq 1 - C\varepsilon_0 e^{-L_\xi} \).
2. For all \( \omega \in \mathcal{Y}_{\varepsilon, L,a} \) and \( \lambda \in \mathbb{R} \) we have, with \( W(a)_{\omega, \varepsilon, \lambda} \frac{W(y)}{y} > e^{-1/4} m_\xi L = \Rightarrow W(a)_{\omega, \varepsilon, \lambda} \leq e^{-7/132} m_\xi \|y - a\| \) for all \( y \in A_{L}(a) \).

In particular, \( W(a)_{\omega, \varepsilon, \lambda} \leq e^{-7/132} m_\xi \|y - a\| \) for all \( y \in A_{2\varepsilon, L,a} \).
Theorem encapsulating localization for the Anderson model

**Theorem**

Let $H_{\varepsilon,\omega}$ be an Anderson model. There exists $\varepsilon_0 > 0$ such that, given $\xi \in (0,1)$, we can find a scale $\hat{L}_\xi$ and $m_\xi > 0$, such that for all $0 < \varepsilon \leq \varepsilon_0$, $L \geq \hat{L}_\xi$ with $L \in 2\mathbb{N}$, and $a \in \mathbb{Z}^d$ there exists an event $Y_{\varepsilon,L,a}$ with the following properties:

1. $Y_{\varepsilon,L,a}$ depends only on the random variables $\{\omega_x\}_{x \in \Lambda_{5L}(a)}$, and $\mathbb{P}\{Y_{\varepsilon,L,a}\} \geq 1 - C_\varepsilon e^{-L_\xi}$.
Theorem encapsulating localization for the Anderson model

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1. $\mathcal{Y}_{\varepsilon,L,a}$ depends only on the random variables $\{\omega_x\}_{x \in \Lambda_5 L(a)}$, and
   \[ \mathbb{P} \{ \mathcal{Y}_{\varepsilon,L,a} \} \geq 1 - C_{\varepsilon_0} e^{-L_\xi}. \]

2. For all $\omega \in \mathcal{Y}_{\varepsilon,L,a}$ and $\lambda \in \mathbb{R}$ we have, with
   \[ W_{\omega,\varepsilon,\lambda}^{(a)}(a) > e^{-\frac{1}{4} m_\xi L} \implies W_{\omega,\varepsilon,\lambda}^{(a)}(y) \leq e^{-\frac{7}{132} m_\xi \|y-a\|} \text{ for all } y \in A_L(a), \]
   where $A_L(a) := \left\{ y \in \mathbb{Z}^d; \frac{8}{7} L \leq \|y-a\| \leq \frac{33}{14} L \right\}$. 

Abel Klein
Theorem encapsulating localization for the Anderson model

Let $H_{\varepsilon, \omega}$ be an Anderson model. There exists $\varepsilon_0 > 0$ such that, given $\xi \in (0, 1)$, we can find a scale $\hat{L}_\xi$ and $m_\xi > 0$, such that for all $0 < \varepsilon \leq \varepsilon_0$, $L \geq \hat{L}_\xi$ with $L \in 2\mathbb{N}$, and $a \in \mathbb{Z}^d$ there exists an event $\mathcal{Y}_{\varepsilon, L, a}$ with the following properties:

1. $\mathcal{Y}_{\varepsilon, L, a}$ depends only on the random variables $\{\omega_x\}_{x \in \Lambda_{5L}(a)}$, and
   $$\mathbb{P}\{\mathcal{Y}_{\varepsilon, L, a}\} \geq 1 - C_0 e^{-L_\xi}.$$

2. For all $\omega \in \mathcal{Y}_{\varepsilon, L, a}$ and $\lambda \in \mathbb{R}$ we have, with
   $$W_{\omega, \varepsilon, \lambda}^{(a)}(a) > e^{-\frac{1}{4} m_\xi L} \implies W_{\omega, \varepsilon, \lambda}^{(a)}(y) \leq e^{-\frac{7}{132} m_\xi \|y-a\|}$$ for all $y \in A_L(a)$,
   where
   $$A_L(a) := \left\{ y \in \mathbb{Z}^d; \frac{8}{7} L \leq \|y-a\| \leq \frac{33}{14} L \right\}.$$
   In particular,
   $$W_{\omega, \varepsilon, \lambda}^{(a)}(a) W_{\omega, \varepsilon, \lambda}^{(a)}(y) \leq e^{-\frac{7}{132} m_\xi \|y-a\|}$$ for all $y \in A_L(a)$. 
The theorem encapsulates localization, as shown by Germinet and Klein. It implies Anderson localization, dynamical localization, and more.
Localization for the Anderson model

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- The theorem yields dynamical localization in expectation with any desired subexponential decay.

- Infinite volume localization results for the Anderson model at high disorder are well known.