Entropy production in repeated quantum measurements

Joint work with Tristan Benoist (Toulouse), Vojkan Jakšić (McGill) & Yan Pautrat (Orsay)

Claude-Alain Pillet

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Introduction — Irreversibility in Quantum Mechanics

Irreversibility vs Measurements

- 1927: Heisenberg’s "reduction of the wave packet"
- 1932: von Neumann’s "Mathematical Foundations of QM"
- 1937: Landau-Lifschitz "Statistical Physics"
- ... 
- 1964: Aharonov-Bergmann-Lebowitz "two state vector formalism" and "time-symmetric statistical ensembles"
- ... 
- More recently: "statistical mechanics of repeated measurements" Kümmerer-Maassen’04, Barchielli-Gregoratti’09, Bauer-Benoist-Bernard’11, Benoist-Pellegrini’14, Ballesteros-Fraas-Fröhlich-Schubnel’16,...
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In this talk:
- Emergence of the "arrow of time" in repeated quantum measurement processes
- Relation with the Gallavotti-Cohen "fluctuations relations"
- Thermodynamic formalism for entropy production in classical dynamical systems
Framework

Repeated Quantum Measurements

- Finite dimensional Hilbert space $\mathcal{H}$
- Finite alphabet $\mathcal{A} = \{1, 2, \ldots, \ell\}$
- Quantum instrument $\{\Phi_a\}_{a \in \mathcal{A}}$
  - CP maps $\Phi_a : B(\mathcal{H}) \to B(\mathcal{H})$
  - Unital $\Phi(\mathbb{1}) = \sum_{a \in \mathcal{A}} \Phi_a(\mathbb{1}) = \mathbb{1}$
  - Duality $\text{tr}(\Phi_a^*(X)Y) = \text{tr}(X\Phi_a(Y))$
- Initial state $\rho$
- Probability measures on finite "quantum trajectories"
  \[
  P_T(\omega_1\omega_2\cdots\omega_T) = \text{tr}(\rho \Phi_{\omega_1} \circ \Phi_{\omega_2} \circ \cdots \circ \Phi_{\omega_T}(\mathbb{1}))
  \]
  extend to a probability $P$ on $\Omega = \mathcal{A}^\mathbb{N}$
- Time-reversal
  \[
  \Theta_T(\omega_1\omega_2\cdots\omega_T) = \theta(\omega_T) \cdots \theta(\omega_2)\theta(\omega_1)
  \]
  for some involution $\theta : \mathcal{A} \to \mathcal{A}$

Assumption A

Initial state $\rho$ is faithful and invariant: $\rho > 0$, $\Phi^*(\rho) = \rho$
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\[ P_T(\omega_1\omega_2 \cdots \omega_T) = \text{tr}(\rho \Phi_{\omega_1} \circ \Phi_{\omega_2} \circ \cdots \Phi_{\omega_T}(\mathbb{1})) \]

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Assumption A

Initial state $\rho$ is faithful and invariant: $\rho > 0$, $\Phi^*(\rho) = \rho$
Basic Properties

- $\Phi^*(\rho) = \rho \Rightarrow \mathbb{P}$ is invariant under the left shift $\tau : \Omega \to \Omega$

Quantum instrument $(\{\Phi_a\}_{a \in \mathcal{A}}, \rho) \implies$ classical dynamical system $(\Omega, \tau, \mathbb{P})$

- $\rho > 0 \Rightarrow$ the upper quasi-Bernoulli property holds

$$\mathbb{P}_{T+T'} \leq C \mathbb{P}_T \mathbb{P}_{T'} \circ \tau^{-T}, \quad \hat{\mathbb{P}}_{T+T'} \leq C \hat{\mathbb{P}}_T \hat{\mathbb{P}}_{T'} \circ \tau^{-T}$$

- The probability of time-reversed trajectories $\hat{\mathbb{P}}_T = \mathbb{P}_T \circ \Theta_T$ describes the instrument $\{\hat{\Phi}_a\}_{a \in \mathcal{A}}$ ([Crooks’08])

$$\hat{\Phi}_a(X) = \rho^{-1/2} \Phi_{\theta(a)}(\rho^{1/2} X \rho^{1/2}) \rho^{-1/2}$$

- If 1 is simple eigenvalue of $\Phi$, then $\mathbb{P}$ is ergodic ($\iff \Phi$ irreducible)
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- If 1 is simple eigenvalue of $\Phi$, then $\mathbb{P}$ is ergodic ($\Leftarrow \Phi$ irreducible)

Remark

Special case of ”finitely correlated states” or ”matrix product states” of 1D spin chains [Fannes-Nachtergaele-Werner’92]
Basic Properties

- \( \Phi^*(\rho) = \rho \Rightarrow \mathbb{P} \) is invariant under the left shift \( \tau : \Omega \rightarrow \Omega \)

  Quantum instrument \( \{\Phi_a\}_{a \in A}, \rho \) \( \Rightarrow \) classical dynamical system \( (\Omega, \tau, \mathbb{P}) \)

- \( \rho > 0 \Rightarrow \) the upper quasi-Bernoulli property holds

  \[
  \mathbb{P}_{T+T'} \leq C \mathbb{P}_T \mathbb{P}_{T'} \circ \tau^{-T}, \quad \hat{\mathbb{P}}_{T+T'} \leq C \hat{\mathbb{P}}_T \hat{\mathbb{P}}_{T'} \circ \tau^{-T}
  \]

- The probability of time-reversed trajectories \( \hat{\mathbb{P}}_T = \mathbb{P}_T \circ \Theta_T \) describes the instrument \( \{\hat{\Phi}_a\}_{a \in A} \) ([Crooks'08])

  \[
  \hat{\Phi}_a(X) = \rho^{-1/2} \Phi^{*\theta(a)}(\rho^{1/2}X\rho^{1/2})\rho^{-1/2}
  \]

- If 1 is simple eigenvalue of \( \Phi \), then \( \mathbb{P} \) is ergodic (\( \Leftarrow \) \( \Phi \) irreducible)

Goal

Quantify the emergence of the arrow of time as a "distance" between \( \mathbb{P}_T \) and \( \hat{\mathbb{P}}_T \) in the limit \( T \rightarrow \infty \)
Strategy

- Universal mechanism for entropic fluctuation relations out of equilibrium
- Applies to classical and quantum dynamical systems
- Need to develop a thermodynamic formalism for non-Gibbsian dynamical systems
- Motivated by a body of recent works on subadditive ergodic theory and multifractal analysis of measures [Feng-Käenmäki-Barreira,...]
Strategy

- Universal mechanism for entropic fluctuation relations out of equilibrium
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Assumption B

\( \mathbb{P} \) is ergodic and \( \text{supp} \hat{\mathbb{P}}_T = \text{supp} \mathbb{P}_T \) for all \( T \) (large enough)

- \( \mathbb{P}_T \) and \( \hat{\mathbb{P}}_T \) equivalent for all \( T \) (large enough)
- Entropy production reflects the dichotomy:
  \[
  \mathbb{P} = \hat{\mathbb{P}} \text{ (equilibrium, detailed balance) or } \mathbb{P} \perp \hat{\mathbb{P}} \text{ (nonequilibrium)}
  \]
- Out of equilibrium, the separation between \( \mathbb{P}_T \) and \( \hat{\mathbb{P}}_T \) as \( T \to \infty \) is quantified by relative entropies
  \[
  S(\mathbb{P}_T | \hat{\mathbb{P}}_T) = \mathbb{P}_T(\sigma_T) \geq 0, \quad S_\alpha(\mathbb{P}_T | \hat{\mathbb{P}}_T) = \log \mathbb{P}_T(e^{-\alpha \sigma_T})
  \]
  expectation and cumulant generating function of the entropy production random variable
  \[
  \sigma_T(\omega) = \log \frac{\mathbb{P}_T(\omega)}{\hat{\mathbb{P}}_T(\omega)} = -\sigma_T \circ \Theta_T(\omega)
  \]
Fluctuation relations

\[ P_T(s) = \mathbb{P}\left( \{ \omega \mid \frac{1}{T} \sigma_T(\omega) = s \} \right) \]

- Law of mean entropy production rate on \([0, T]\)
- Assumption B \(\Rightarrow P_T(s) > 0 \iff P_T(-s) > 0\)
- Relative entropies
  \[ S(\mathbb{P}_T | \hat{\mathbb{P}}_T) = \sum sP_T(s) \geq 0, \quad S_\alpha(\mathbb{P}_T | \hat{\mathbb{P}}_T) = \log \sum e^{-\alpha s} P_T(s) \]
- Symmetry of the Rényi entropy \(S_{1-\alpha}(\mathbb{P}_T | \hat{\mathbb{P}}_T) = S_\alpha(\mathbb{P}_T | \hat{\mathbb{P}}_T)\) yields the finite-time fluctuation relation
  \[ \frac{P_T(-s)}{P_T(s)} = e^{-Ts} \]
- More (LDP, CLT, Chernoff & Hoeffding exponents, Gallavotti-Cohen fluctuation relations) if we can control
  \[ \lim_{T \to \infty} \frac{1}{T} S_\alpha(\mathbb{P}_T | \hat{\mathbb{P}}_T) \]
Entropy production

Results from ergodic theory

- Gibbs-Shannon entropy: \( S(\mathbb{P}_T) = -\sum_{\omega \in \mathcal{A}} \mathbb{P}(\omega) \log \mathbb{P}(\omega) \)
- Kolmogorov-Sinai entropy: \( S(\mathbb{P}) = \lim_{T \to \infty} T^{-1} S(\mathbb{P}_T) \in [0, \log \ell] \)
- Shannon-McMillan-Breiman: \( S(\mathbb{P}) = -\lim_{T \to \infty} T^{-1} \log \mathbb{P}_T, \mathbb{P}\text{-a.s. and in } L^1(\mathbb{P}) \)
- Gibbs property (Bowen)

\[
C^{-1} e^{-\sum_{t=1}^{T} \varphi \circ \tau^t} \leq \mathbb{P}_T \leq C e^{-\sum_{t=1}^{T} \varphi \circ \tau^t}
\]

for some (Hölder) continuous potential \( \varphi \) (Gallavotti-Cohen chaotic hypothesis) generally fails for repeated measurement processes \( \Rightarrow \) need thermodynamic formalism for non-Gibbsian systems

- Weaker than Gibbs: Upper & Lower Quasi-Bernoulli properties

\[
C^{-1} \mathbb{P}_T \mathbb{P}_{T'} \circ \tau^T \leq \mathbb{P}_{T+T'} \leq C \mathbb{P}_T \mathbb{P}_{T'} \circ \tau^T
\]

implies existence and differentiability of the entropic pressure

\[
\mathbb{R} \ni \alpha \mapsto e(\alpha) = \lim_{T \to \infty} \frac{1}{T} S_\alpha(\mathbb{P}_T | \widehat{\mathbb{P}}_T)
\]

- Assumptions A & B only ensure upper quasi-Bernoulli \( \Rightarrow e(\alpha) \) may develop singularities: dynamical phase transition
Entropy production: Level I

**Theorem I (Entropy production)**

Under Assumptions A & B

- **Mean entropy production rate**
  \[
  \lim_{T \to \infty} \frac{1}{T} \mathbb{P}_T(\sigma_T) = \mathbb{E}_p \geq 0
  \]

- **Strong law of large numbers:** \(\mathbb{P}\text{-a.s.}\)
  \[
  \lim_{T \to \infty} \frac{1}{T} \sigma_T(\omega) = \mathbb{E}_p
  \]  \(\text{(1)}\)

- If \(\mathbb{E}_p < \infty\) then (1) holds in \(L^1(\Omega, \mathbb{P})\)

- **Stein’s exponent:** Let \(s_T(\epsilon) = \min\{\hat{\mathbb{P}}_T(A) \mid A \in \mathcal{A}^T, \mathbb{P}_T(A) \geq \epsilon\}\) for \(\epsilon \in ]0, 1[\)
  \[
  \lim_{T \to \infty} \frac{1}{T} \log s_T(\epsilon) = -\mathbb{E}_p
  \]
**Theorem I (Entropy production)**

Under Assumptions A & B

- Mean entropy production rate

\[ \lim_{T \to \infty} \frac{1}{T} \mathbb{P}_T(\sigma_T) = \text{Ep} \geq 0 \]

- Strong law of large numbers: \( \mathbb{P} \)-a.s.

\[ \lim_{T \to \infty} \frac{1}{T} \sigma_T(\omega) = \text{Ep} \quad (1) \]

- If \( \text{Ep} < \infty \) then (1) holds in \( L^1(\Omega, \mathbb{P}) \)

- Stein’s exponent: Let \( s_T(\epsilon) = \min\{\hat{\mathbb{P}}_T(A) \mid A \in \mathcal{A}_T, \mathbb{P}_T(A) \geq \epsilon\} \) for \( \epsilon \in ]0, 1[ \)

\[ \lim_{T \to \infty} \frac{1}{T} \log s_T(\epsilon) = -\text{Ep} \]

**Message**

- \( \text{Ep} = 0 \iff \hat{\mathbb{P}} = \mathbb{P} \) & \( \text{Ep} > 0 \iff \hat{\mathbb{P}} \perp \mathbb{P} \)

- \( \mathbb{P}_T(A_T) \geq \epsilon > 0 \) for large \( T \) \( \Rightarrow \hat{\mathbb{P}}_T(A_T) \lesssim e^{-TEp} \) exponential separation of the supports of \( \mathbb{P}_T \) and \( \hat{\mathbb{P}}_T \)
Rényi’s relative entropy

\[ e_T(\alpha) = S_\alpha(\mathbb{P}_T \mid \hat{\mathbb{P}}_T) = \log \mathbb{P}_T(e^{-\alpha \sigma_T}) \]

- is a convex function of \( \alpha \)
- has left/right derivatives \( \partial^\pm e(\alpha) \) where finite
- is non-positive for \( \alpha \in [0, 1] \)
- is non-negative for \( \alpha \in \mathbb{R} \setminus [0, 1] \)
- vanishes at \( \alpha = 0 \) and \( \alpha = 1 \)
- satisfies \( e_T(1 - \alpha) = e_T(\alpha) \) (the Gallavotti-Cohen symmetry)

All these properties will be preserved in the limit (\( = \) entropic pressure)

\[ e(\alpha) = \lim_{T \to \infty} \frac{1}{T} e_T(\alpha) \]

whenever it exists.
Suppose that Assumptions A & B hold and denote by $\mathcal{P}_\tau$ the set of $\tau$-invariant probability measures on $\Omega$.

- The entropic pressure $e(\alpha)$ exists for all $\alpha \in [0, 1]$.
- Either $e(\alpha) = -\infty$ for all $\alpha \in ]0, 1[$, or $e(\alpha) > -\infty$ for all $\alpha \in ]0, 1[$.
- The limit
  \[ f(Q) = \lim_{T \to \infty} \frac{1}{T} Q(\log P_T - \log Q_T) \]
  exists for all $Q \in \mathcal{P}_\phi$ and satisfies
  \[ \alpha f(Q) + (1 - \alpha) f(\hat{Q}) \geq \limsup_{T \to \infty} \frac{1}{T} e_T(\alpha) \]
  for all $\alpha \in \mathbb{R}$.
- For $\alpha \in [0, 1]$
  \[ e(\alpha) = \sup_{Q \in \mathcal{P}_\tau} \alpha f(Q) + (1 - \alpha) f(\hat{Q}) \]
Theorem II (thermodynamic formalism, cont’d)

- If \( e(\alpha) \) is finite for \( \alpha \in [0, 1] \) then
  \[
P_{eq}(\alpha) = \{ Q \in P_\tau | \alpha f(Q) + (1 - \alpha)f(\hat{Q}) = e(\alpha) \}
  \]
  is a non-empty compact convex subset of \( P_\tau \), a Choquet simplex and a face of \( P_\tau \) whose extreme points are ergodic.

- For \( \alpha \in ]0, 1[ \)
  \[
  \partial^- e(\alpha) = \inf_{Q \in P_{eq}(\alpha)} f(\hat{Q}) - f(Q) \leq \sup_{Q \in P_{eq}(\alpha)} f(\hat{Q}) - f(Q) = \partial^+ e(\alpha)
  \]

- \( P_{eq}(0) = \{ P \} \), \( P_{eq}(1) = \{ \hat{P} \} \) and
  \[-\partial^+ e(0) = Ep = \partial^- e(1) \]

Remark

- If \( f(Q) \) and \( f(\hat{Q}) \) are finite, then \( f(\hat{Q}) - f(Q) = -\lim_{T \to \infty} \frac{1}{T} Q(\sigma_T) \)
Assumption C (weaker than lower quasi-Bernoulli)

There exists $T^* > 0$ and $C_{T^*} > 0$ such that

$$\max_{|\xi| \leq T^*} \frac{P(\omega \xi \nu) \hat{P}(\omega \xi \nu)}{P(\omega) \hat{P}(\nu) \hat{P}(\omega) \hat{P}(\nu)} \geq C_{T^*}$$

for all finite words $\omega, \nu$ (i.e., cylinder sets)

Remarks

- Minimal assumption for Theorem III
- Often easy to check in concrete models
- Irreducibility of $\sum_a \Phi_a \otimes \hat{\Phi}_a \Rightarrow C$
- Simple algebraic criterion in terms of Kraus representations
Theorem III (Differentiability on \([0, 1]\))

Under Assumptions A,B & C

- \(\alpha \in [0, 1] \Rightarrow \mathcal{P}_{eq}(\alpha)\) is a singleton: \(e(\alpha)\) is differentiable on \([0, 1]\).
- For any open set \(O \subset \] - \text{Ep}, \text{Ep}[\) the local LDP

\[
\lim_{T \to \infty} \frac{1}{T} \mathbb{P} \left( \left\{ \omega \in \Omega \mid \frac{1}{T} \sigma_T(\omega) \in O \right\} \right) = - \inf_{s \in O} l(s),
\]

holds with rate function \(l(s) = - \inf_{\alpha \in [0,1]} (\alpha s + e(\alpha))\) satisfying the fluctuation relation

\[l(-s) = l(s) + s\]

- Chernoff exponent: \(\lim_{T \to \infty} \frac{1}{T} \log(2 - \|\mathbb{P}_T - \hat{\mathbb{P}}_T\|_{\text{var}}) = e(1/2)\)
- Hoeffding exponent: for \(s \geq 0\)

\[
\inf_{\{A_T \subset A_T^T\}} \left\{ \limsup_{T} \frac{1}{T} \log \mathbb{P}_T(A_T) \mid \limsup_{T} \frac{1}{T} \log \mathbb{P}_T(A_T^T \setminus A_T) < -s \right\}
\]

\[= - \sup_{\alpha \in [0,1]} \frac{-s\alpha - e(\alpha)}{1 - \alpha}\]
Remarks

- Assuming the lower Quasi-Bernoulli property one can show that $e(\alpha)$ is differentiable on $\mathbb{R}$. As a consequence, the LDP for the mean entropy production rate holds for all open sets $O \subset \mathbb{R}$.
- $\Phi_a$ positivity improving for all $a \in \mathcal{A}$ $\Rightarrow$ lower quasi-Bernoulli ($\sim$ Gallavotti-Cohen chaotic hypothesis)
- We have simple examples of repeated measurement processes for which Assumptions A, B & C hold but the lower quasi-Bernoulli property fails in a strong way

\[
\frac{\mathbb{P}_T(\omega_T \nu_T)}{\mathbb{P}_T(\omega_T)\mathbb{P}_T(\nu_T)} \sim e^{-\gamma T}, \quad \gamma > 0
\]

Nevertheless, in these example the entropic pressure $e(\alpha)$ exist and is finite for all $\alpha \in \mathbb{R}$. It exhibits a second order phase transition at $\alpha = 0/1$. 
A Markov instrument: $\Phi_{(i,j)}(X) = p_{ij}|i\rangle\langle j|X|j\rangle\langle i|$ 

- $p = (p_{ij})$ stochastic matrix $p_{ij} > 0 \Rightarrow$ unique invariant state $\pi p = \pi$
- Time-reversal $\theta(i,j) = (j,i)$
- $\{|i\}\}$ ON-basis: A holds with $\rho = \sum_i \pi_i |i\rangle\langle i|$ 
- B holds and $Ep = \sum_{i,j} \pi_i p_{ij} \log \frac{p_{ij}}{p_{ji}}$
- $Ep = 0$ iff detailed balance $\pi_i p_{ij} = \pi_j p_{ji}$ holds
- C holds. Entropic pressure $e(\alpha)$ is given by the spectral radius of the matrix $m(\alpha) = (p_{ij}^{1-\alpha} p_{ji}^\alpha)$
- Lower quasi-Bernoulli fails, nevertheless $\mathbb{R} \ni \alpha \mapsto e(\alpha)$ is real analytic
A Bernoulli instrument (ancilla measurement of $S_{1/2}$ in $\mathcal{S}_\ell \otimes S_{1/2}$)

Let $\vec{S}^{(s)}$ and $\vec{S}^{(p)}$ denote spin $\ell$ and spin $1/2$ operators, $\epsilon, \omega, \lambda, t \in \mathbb{R}, \eta \in ]0, 1[$

\[
H = \epsilon S_3^{(p)} + \omega S_3^{(s)} + \lambda \vec{S}^{(p)} \cdot \vec{S}^{(s)}
\]

\[
\rho^{(p)} = \frac{1}{2} + (2\eta - 1) S_3^{(p)}, \quad P_{\pm} = \frac{1}{2} \pm S_3^{(p)}
\]

\[
\Phi^*_\pm(\rho) = (\text{Id}^{(s)} \otimes \text{tr}^{(p)})(((1 \otimes P_{\pm}) e^{-itH}(\rho \otimes \rho^{(p)})) e^{itH}), \quad \theta(\pm) = \mp
\]

Assumptions A,B & C hold. $\mathbb{P}$ is Bernoulli,

\[
\mathbb{E}_p = (2\eta - 1) \log \frac{\eta}{1 - \eta}, \quad e(\alpha) = \log(\eta^\alpha (1 - \eta)^{1-\alpha} + \eta^{1-\alpha} (1 - \eta)^\alpha)
\]
A quasi-Bernoulli perfect Kraus instrument: \( \Phi_\pm(X) = V_\pm XV_\pm^* \)

\[
V_- = \begin{pmatrix}
0 & -\sin \theta/2 \\
\cos \theta/2 & 0 \\
\end{pmatrix}, \quad V_+ = \begin{pmatrix}
\cos \theta/2 & 0 \\
0 & \sin \theta/2 \\
\end{pmatrix}, \quad \theta(\pm) = \mp e^{i\sigma_3 \otimes \sigma_2}/2
\]

- Satisfies Assumptions A,B & C for \( \theta \in ]0, \pi/2[ \).
- \( \mathbb{P} \) is quasi-Bernoulli but not Bernoulli.
- Entropic pressure is real analytic.
A Non quasi-Bernoulli perfect Kraus instrument: $\Phi_{\pm}(X) = V_{\pm}XV_{\mp}^*$

$$V_- = \begin{pmatrix} \sqrt{\cos \theta} & -\sin \theta/2 \\ -\sin \theta/2 & 0 \end{pmatrix}, \quad V_+ = \begin{pmatrix} -\sin \theta/2 & 0 \\ -\sqrt{\cos \theta} & -\sin \theta/2 \end{pmatrix}, \quad \theta(\pm) = \mp$$

Satisfies Assumptions A, B & C for $\theta \in [0, \pi/2[$ but is not Lower Quasi-Bernoulli

$$\lim_{T \to \infty} \frac{1}{2T + 1} \log \frac{\mathbb{P}_{2T+1}(- \cdots - + - \cdots -)}{\mathbb{P}_{T+1}(- \cdots - ) \mathbb{P}_T(- \cdots - )} = -\xi = - \log \frac{1 + \sqrt{1 - 4 \sin^4 \theta/2}}{2 \sin^2 \theta/2} < 0$$
Examples

A Non quasi-Bernoulli perfect Kraus instrument: \( \Phi_{\pm}(X) = V_{\pm}XV_{\pm}^* \)

\[
V_- = \begin{pmatrix}
\sqrt{\cos \theta} & -\sin \theta/2 \\
-\sin \theta/2 & 0
\end{pmatrix}, \quad V_+ = \begin{pmatrix}
-\sin \theta/2 & 0 \\
-\sqrt{\cos \theta} & -\sin \theta/2
\end{pmatrix}, \quad \theta(\pm) = \mp
\]

Satisfies Assumptions A, B & C for \( \theta \in ]0, \pi/2[ \) but is not Lower Quasi-Bernoulli

\[
\lim_{T \to \infty} \frac{1}{2T + 1} \log \frac{P_{2T+1}(\cdots - + \cdots -)}{P_{T+1}(\cdots - +)P_T(\cdots -)} = -\xi = -\log \frac{1 + \sqrt{1 - 4 \sin^4 \theta/2}}{2 \sin^2 \theta/2} < 0
\]

Entropic pressure and its derivative for \( \theta = \pi/3 \)
A Non quasi-Bernoulli perfect Kraus instrument: $\Phi_{\pm}(X) = V_{\pm} X V_{\pm}^*$

$$
V_- = \begin{pmatrix}
\sqrt{\cos \theta} & - \sin \theta / 2 \\
- \sin \theta / 2 & 0 \\
\end{pmatrix}, \quad V_+ = \begin{pmatrix}
- \sin \theta / 2 & 0 \\
- \sqrt{\cos \theta} & - \sin \theta / 2 \\
\end{pmatrix}, \quad \theta(\pm) = \mp
$$

Satisfies Assumptions A, B & C for $\theta \in ]0, \pi/2[$ but is not Lower Quasi-Bernoulli

$$
\lim_{T \to \infty} \frac{1}{2T + 1} \log \frac{\mathbb{P}_{2T+1}(- \cdots - + - \cdots -)}{\mathbb{P}_{T+1}(- \cdots - +) \mathbb{P}_{T}(- \cdots -)} = -\xi = - \log \frac{1 + \sqrt{1 - 4 \sin^4 \theta / 2}}{2 \sin^2 \theta / 2} < 0
$$

Central limit theorem fails: as $T \to \infty$

$$
\frac{\sigma_T - \mathbb{P}(\sigma_T)}{\sqrt{T}} \Rightarrow \frac{\xi}{\cosh \xi} (u - |v|)
$$

with $u, v \sim \mathcal{N}(0, 1)$. 
Further develop the thermodynamic formalism for non-Gibbsian systems using results from the subadditive ergodic theory.

Criteria for analyticity, occurrence of first order phase transitions?

Investigate the physical meaning of phase transition beyond the failure of CLT. Occurrence of anomalous scaling?

Special measurements, e.g., thermal probes.

Continuous measurements/monitoring.

...
Thank you!