Universality of charge transport in weakly interacting fermionic systems

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Outline

- Introduction: Integer Quantum Hall Effect.

- Universality of the Hall conductivity for $2d$ interacting fermionic systems.

- Hall transitions in the interacting Haldane model.

- Sketch of the proofs.

- Conclusions.
Introduction
Integer quantum Hall effect

- 2d condensed matter systems display remarkable transport properties.
- Paradigmatic example: Integer quantum Hall effect.
- Setting. Thin samples of suitable insulators, at low temperatures, exposed to strong magnetic field $B$ and weak electric field $E$.  

\[ B \quad z \quad x \quad y \]

$V_H$
Integer quantum Hall effect

- $2d$ condensed matter systems display remarkable transport properties.

- Paradigmatic example: Integer quantum Hall effect.

- $J =$ current generated by weak field $E$. Linear response: $J_i = \sigma_{ij} E_j$.

\[
\sigma_{11} = \sigma_{22} = 0, \quad \sigma_{12} = -\sigma_{21} \in \frac{e^2}{h} \cdot \mathbb{Z}.
\]
Theory: noninteracting particles

Thouless-Kohmoto-Nightingale-Den Nijs '82, Avron-Seiler-Simon '83, '94, Bellissard-van Elst-Schulz-Baldes '94, Aizenman-Graf '98...
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- \( H = H_0 + \lambda W \) = one-particle Schrödinger operator on \( \ell^2(\mathbb{Z}^2) \).
  
  \( H_0 = \) magnetic lattice Laplacian, \( W = \) random local potential.

\[
H_0(x; y) = e^{i\phi_{xy}} \delta_{|x-y|,1}, \quad W(x; y) = w_x \delta_{|x-y|,0},
\]

with \( w_x = \) i.i.d. random variables.
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  - \( H_0(x; y) = e^{i\phi_{xy}} \delta_{|x-y|, 1} \), \( W(x; y) = w_x \delta_{|x-y|, 0} \),
  - with \( w_x \) = i.i.d. random variables.

- Let \( P_\mu = \chi(H \leq \mu) \) = Fermi projector. If \( E|P_\mu(x; y)| \leq Ce^{-c|x-y|}:
  - \( \sigma_{12} = i \text{Tr} P_\mu[[X_1, P_\mu], [X_2, P_\mu]] \in \frac{1}{2\pi} \cdot \mathbb{Z} \)
  - with \( \text{Tr} \cdot = \lim_{|\Lambda| \to \infty} |\Lambda|^{-1} \text{tr} \cdot \chi(x \in \Lambda) \) = trace per unit volume.

- \( P_\mu \) decays exp. if \( \mu \in \text{spectral gap} \), or \( \mu \in \text{mobility gap} \) (strong disorder).
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  - \( H_0 \) = magnetic lattice Laplacian, \( W \) = random local potential.
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    with \( w_x \) = i.i.d. random variables.
- Let \( P_\mu = \chi(H \leq \mu) \) = Fermi projector. If \( \mathbb{E}|P_\mu(x;y)| \leq C e^{-c|x-y|} \):
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    with \( \text{Tr } \cdot = \lim_{|\Lambda| \to \infty} |\Lambda|^{-1} \text{tr } \chi(x \in \Lambda) = \text{trace per unit volume.} \)
- \( P_\mu \) decays exp. if \( \mu \in \text{spectral gap} \), or \( \mu \in \text{mobility gap} \) (strong disorder).
- If no disorder: \( \sigma_{12} = \text{Chern number of Bloch bundle.} \)
Theory: interacting particles

- The previous results do not apply to interacting systems.
- Consider clean, interacting many-body systems.
  - Incompressibility: \( \exists \) gap above the interacting ground state.
  (proven for frustration-free systems, perturbations of classical systems)
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Fröhlich et al. '90. Gauge theory of topological phases of matter.
FQHE as a consequence of the chiral anomaly in cond-mat physics.

Hastings-Michalakis '15. Proof of quantization of $\sigma_{12}$ for interacting electrons on a lattice.
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- Today.
  1. Universality of \( \sigma_{ij} \) for weakly interacting fermionic systems.
  2. Hall transitions in the interacting Haldane model (gapless limit).
Universality of conductivity for weakly interacting fermionic systems
Fermions on the lattice

- $\Lambda = 2d$ Bravais lattice, periodic b.c. (e.g. square lattice, honeycomb lattice).
- Fermionic operators $\psi_{x,\alpha}^\pm, \alpha = 1, \ldots, N$ “color” index (e.g. spin, sublattice).
- Hamiltonian: $\mathcal{H} = \mathcal{H}^{(0)} + UV - \mu N$, where
  \[ \mathcal{H}^{(0)} = \sum_{x,y \in \Lambda} \sum_{\alpha,\alpha'} \psi_{x,\alpha}^+ H_{\alpha\alpha'}^{(0)}(x - y) \psi_{y,\alpha'}^-, \quad (H_{\alpha\alpha'}^{(0)}(x) = H_{\alpha'\alpha}^{(0)}(-x)), \]
  \[ V = \sum_{x,y \in \Lambda} \sum_{\alpha,\alpha'} n_{x,\alpha} v_{\alpha\alpha'}(x - y) n_{y,\alpha'}, \quad (n_{x,\alpha} = \psi_{x,\alpha}^+ \psi_{x,\alpha}^-) \]
  $H^{(0)}(x) =$ short-range hopping, $v(x) =$ short-range interaction.

- $\hat{H}^{(0)}(k) =$ Bloch Hamiltonian. In the following, we will assume that its spectrum either has a gap or conical intersections.
Conductivity

- The conductivity is defined starting from Kubo formula \((e^2 = \hbar = 1)\):

\[
\sigma_{ij} := \lim_{\eta \to 0^+} \frac{i}{\eta} \left( \int_{-\infty}^{0} dt \, e^{\eta t} \langle [e^{iH}, J_i e^{-iH}, J_j] \rangle_{\infty} - \langle [J_i, X_j] \rangle_{\infty} \right)
\]

where \(X = \) second quantization of the position operator and

\[
J := i[H, X] = \text{current operator}, \quad \langle \cdot \rangle_{\infty} = \lim_{\beta, \Lambda \to \infty} |\Lambda|^{-1} \langle \cdot \rangle_{\beta, \Lambda}.
\]

- Kubo formula describes the linear response at \(t = 0\), after introducing a weak external field \(e^{\eta t} E \cdot X\) at \(t = -\infty\).

- We take Kubo formula as a definition. Rigorous results on the validity of linear response theory at finite time, for interacting fermions: Bru-Pedra '14.
**Stability of IQHE**

**Theorem 1 (Giuliani, Mastropietro, P. - CMP ’16).**

Suppose that $\hat{H}^{(0)}(k)$ is gapped, $\mu \notin \sigma(\hat{H}^{(0)}(k))$. Then, there exists $U_0 > 0$ s.t.:

$$\sigma_{ij} = \sigma_{ij}|_{U=0} \quad \text{for all } U \in (-U_0, U_0)$$

In particular, $\sigma_{ii} = 0$ and $\sigma_{12} = -\sigma_{21} \in (e^2/h) \cdot \mathbb{Z}$. 

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- **Strategy.**
  1. Construction of Euclidean correlations.
  2. Wick rotation to imaginary times (Euclidean conductivity matrix).
  3. Universality of Euclidean conductivity matrix. Argument inspired by:

    **Coleman-Hill ’85:** “no corrections beyond 1-loop to the topological mass in $\text{QED}_{2+1}$.”
Stability of IQHE

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- $U_0 \equiv U_0(\text{gap})$. Hall transitions?
Hall transitions in the interacting Haldane model
Graphene

- First realization of a 2d crystal (Geim-Novoselov, Nobel prize 2010)
- Hamiltonian:

\[ \mathcal{H}_G^{(0)} = t_1 \sum_{x \in \Lambda, \sigma = \uparrow, \downarrow} \left[ \psi_{x,A,\sigma}^+ \psi_{x,B,\sigma}^- + \psi_{x,A,\sigma}^+ \psi_{x-\ell_1,B,\sigma}^- + \psi_{x,A,\sigma}^+ \psi_{x-\ell_2,B,\sigma}^- + h.c. \right] \]

**Figure:** Dimer \( \rightsquigarrow (\psi_{x,A,\sigma}^\pm, \psi_{x,B,\sigma}^\pm). \)
Graphene

- The spectrum is gapless:

- Fermi level: $\mu = 0$ corresponds to undoped graphene (half-filling).
- Low-energy excitations: 2D massless Dirac fermions ($v \approx c/300$).
- “Relativistic” charge carriers, remarkable transport properties.
The Haldane model

- **Haldane ’88.** Graphene + nnn hopping + staggered potential.

\[ H^{(0)}_H = t_1 \sum_{x \in \Lambda, \sigma = \uparrow \downarrow} [\psi_{x,A, \sigma}^+ \psi_{x,B, \sigma}^- + \psi_{x,A, \sigma}^+ \psi_{x-\ell_1,B, \sigma}^- + \psi_{x,A, \sigma}^+ \psi_{x-\ell_2,B, \sigma}^- + h.c.] + t_2 \sum_{x \in \Lambda, \sigma = \uparrow \downarrow} \sum_{a = \pm} \sum_{j = 1, 2, 3} [e^{ia \phi} \psi_{x,A, \sigma}^+ \psi_{x+a \gamma_j, A, \sigma}^- + e^{-ia \phi} \psi_{x,B, \sigma}^+ \psi_{x+a \gamma_j, B, \sigma}^-] + M \sum_{x \in \Lambda, \sigma = \uparrow \downarrow} [\psi_{x,A, \sigma}^+ \psi_{x,A, \sigma}^- - \psi_{x,B, \sigma}^+ \psi_{x,B, \sigma}^-]

- Black: \( t_2 e^{i \phi} \). Red: \( t_2 e^{-i \phi} \)
- Zero net flux.
The Haldane model

- Haldane '88. Graphene + nnn hopping + staggered potential.

\[ \mathcal{H}_H^{(0)} = t_1 \sum_{x \in \Lambda} \sum_{\sigma = \uparrow \downarrow} \left[ \psi_{x,A,\sigma}^+ \psi_{x,B,\sigma}^- + \psi_{x,A,\sigma}^+ \psi_{x-\ell_1,B,\sigma}^- + \psi_{x,A,\sigma}^+ \psi_{x-\ell_2,B,\sigma}^- + h.c. \right] \]

\[ + t_2 \sum_{x \in \Lambda} \sum_{a = \pm} \sum_{\sigma = \uparrow \downarrow} \left[ e^{ia\phi} \psi_{x,A,\sigma}^+ \psi_{x+a\gamma_j,A,\sigma}^- + e^{-ia\phi} \psi_{x,B,\sigma}^+ \psi_{x+a\gamma_j,B,\sigma}^- \right] \]

\[ + M \sum_{x \in \Lambda} \sum_{\sigma = \uparrow \downarrow} \left[ \psi_{x,A,\sigma}^+ \psi_{x,A,\sigma}^- - \psi_{x,B,\sigma}^+ \psi_{x,B,\sigma}^- \right] \]

- Gapped system. Gaps:

\[ \Delta_\pm = |m_\pm|, \quad m_\pm = M \pm 3\sqrt{3}t_2 \sin \phi. \]
Topological phase diagram

- IQHE without net external magnetic flux:

\[ \sigma_{12} = \frac{2e^2}{h} \nu, \quad \nu = \frac{1}{2} \left[ \text{sgn}(m_-) - \text{sgn}(m_+) \right] \]

- Simplest example of topological insulator. Building brick for more complex systems (e.g. Kane-Mele model).
Topological phase diagram

- IQHE without net external magnetic flux:
  \[ \sigma_{12} = \frac{2e^2}{h} \nu, \quad \nu = \frac{1}{2} [\text{sgn}(m_-) - \text{sgn}(m_+)] \]

Figure: Experimental realization (Esslinger group, Nature ’14)

- What is the effect of many-body interactions on the phase diagram?
Phase transitions in the interacting Haldane model

**Theorem 2 (Giuliani, Jauslin, Mastropietro, P. - arXiv 2016).**

*There exists $U_0 > 0$ and a function*

$$m_{R,\omega} = m_\omega + F_\omega(m_\pm; U) \quad \text{where} \quad F_\omega = O(U), \quad F_\omega \big|_{m_\pm = 0} = 0, \quad \omega = \pm$$

*such that, for $U \in (-U_0, U_0)$, choosing $\mu = \mu(m_\pm; U)$:*

$$\lim_{m_{R,\omega} \to 0^+} - \lim_{m_{R,\omega} \to 0^-} \sigma_{12} = \frac{2e^2}{h}\omega$$

$$\sigma_{ii}^c := \lim_{\eta \to 0^+} \lim_{m_{R,\omega} \to 0} \sigma_{ii}(\eta) = \frac{e^2}{h} \frac{\pi}{4}.$$  

- $|m_{R,\omega}| = \text{renormalized mass. } m_{R,\pm} = 0 : \text{renormalized transition curves.}$
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- $|m_{R,\omega}| = \text{renormalized mass}$. $m_{R,\pm} = 0$: renormalized transition curves.
- If $m_{R,+} = m_{R,-} = 0$, $\sigma_{ii}^c = (e^2/h)(\pi/2)$ (same as graphene, GMP '12).
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such that, for $U \in (-U_0, U_0)$, choosing $\mu = \mu(m_\pm; U)$:

$$\left[ \lim_{m_{R,\omega} \to 0^+} - \lim_{m_{R,\omega} \to 0^-} \right] \sigma_{12} = \frac{2e^2}{h} \omega$$

$$\sigma_{ii}^c := \lim_{\eta \to 0^+} \lim_{m_{R,\omega} \to 0} \sigma_{ii}(\eta) = \frac{e^2 \pi}{h 4}.$$ 

Proof relies on rigorous RG, combined with Ward identities and Wick rotation.

[Brydges-Battle-Federbush, Gawedzki-Kupiainen, Lesniewski, Benfatto-Gallavotti-Mastropietro, Feldman-Knörer-Trubowitz, Magnen-Rivasseau-Sénéor, Pedra-Salmhofer, ... ]
Renormalized transition curves

Figure: Red: $U = 0$. Blue: $U > 0$.

- Away from the transition line the correlations decay exponentially fast.
- On the transition line the decay is algebraic.
Sketch of the proofs

[Wick rotation & Hall transitions]
Let us define the Euclidean conductivity matrix as:

\[
\sigma_{ij} := - \lim_{\eta \to 0^+} \frac{1}{\eta} \left[ \hat{K}_{ij}(\eta) - \hat{K}_{ij}(0) \right]
\]

where, setting \( J(-it) := e^{Ht} J e^{-Ht} = \) imaginary time evolution of \( J \):

\[
\hat{K}_{ij}(\eta) = \lim_{\beta, |\Lambda| \to \infty} \frac{1}{|\Lambda|} \int_{-\beta/2}^{\beta/2} dt \, e^{-int} \langle T J_i(-it) ; J_j(-is) \rangle_{\beta, \Lambda}
\]

\( (T = \) fermionic time ordering).
Euclidean Kubo formula

- Let us define the Euclidean conductivity matrix as:

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\bar{\sigma}_{ij} := - \lim_{\eta \to 0^+} \frac{1}{\eta} \left[ \hat{K}_{ij}(\eta) - \hat{K}_{ij}(0) \right]
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where, setting \( \mathcal{J}(-it) := e^{\mathcal{H}t} \mathcal{J} e^{-\mathcal{H}t} \) = imaginary time evolution of \( \mathcal{J} \):

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\hat{K}_{ij}(\eta) = \lim_{\beta, |\Lambda| \to \infty} \frac{1}{|\Lambda|} \int_{-\beta/2}^{\beta/2} dt e^{-i\eta t} \langle T \mathcal{J}_i(-it); \mathcal{J}_j(-is) \rangle_{\beta, \Lambda}
\]

(\( T = \) fermionic time ordering).

- Euclidean correlations can be studied via cluster expansion and RG methods. For instance, for weak interactions:

\[
\left| \frac{1}{|\Lambda|} \langle T \mathcal{J}_i(-it); \mathcal{J}_j(-is) \rangle_{\beta, \Lambda} \right| \leq \frac{C_M}{1 + |t - s|^M} \quad \text{unif. in } \beta, \Lambda
\]

\( \forall M > 0 \) if \( \mu \notin \sigma(\hat{H}^{(0)}(k)) \) or \( M = 2 \) for conical intersections.
We would like to show that, for $U \in (-U_0, U_0)$:

$$\overline{\sigma}_{ij} := -\lim_{\eta \to 0^+} \frac{1}{\eta} \int_{-\infty}^{\infty} dt \left( e^{-i\eta t} - 1 \right) \langle \mathbf{T} e^{tH} \mathcal{J}_i e^{-tH} ; \mathcal{J}_j \rangle_{\infty}$$

$$= \lim_{\eta \to 0^+} \frac{i}{\eta} \left( \int_{-\infty}^{0} dt e^{\eta t} \left\langle [e^{iHt} \mathcal{J}_i e^{-iHt} , \mathcal{J}_j] \right\rangle_{\infty} - \left\langle [\mathcal{J}_i , \mathcal{X}_j] \right\rangle_{\infty} \right)$$

$$\equiv \sigma_{ij} .$$
Wick rotation

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$$
\sigma_{ij} := - \lim_{\eta \to 0^+} \frac{1}{\eta} \int_{-\infty}^\infty dt \left( e^{-i\eta t} - 1 \right) \langle \mathbf{T} e^{t\mathcal{H}} \mathcal{J}_i e^{-t\mathcal{H}} ; \mathcal{J}_j \rangle_{\infty}
$$

$$
= \lim_{\eta \to 0^+} \frac{i}{\eta} \left( \int_{-\infty}^0 dt e^{\eta t} \left[ \left[ e^{i\mathcal{H}t} \mathcal{J}_i e^{-i\mathcal{H}t} , \mathcal{J}_j \right] \right]_{\infty} - \left[ \mathcal{J}_i , \mathcal{X}_j \right]_{\infty} \right)
$$

$$
\equiv \sigma_{ij}.
$$

- Proof based on complex deformation, for $\eta > 0$. We’ll show:

$$
\frac{1}{\eta} \int_{-\infty}^\infty dt e^{-i\eta t} \langle \mathbf{T} e^{t\mathcal{H}} \mathcal{J}_i e^{-t\mathcal{H}} ; \mathcal{J}_j \rangle_{\infty} = \frac{i}{\eta} \int_{-\infty}^0 dt e^{\eta t} \langle \left[ e^{i\mathcal{H}t} \mathcal{J}_i e^{-i\mathcal{H}t} , \mathcal{J}_j \right] \rangle_{\infty}
$$

(part of the statement is that the r.h.s. exists.)
Wick rotation

Fix $T, \eta \in \mathbb{R}^+$. We have:

$$\int_{-T}^{T} dt \ e^{-i\eta t} \langle T \ J_i(-it); J_j \rangle_\infty$$

$$= \int_{-T}^{0} dt \ e^{-i\eta t} \langle J_j J_i(-it) \rangle_\infty + \int_{0}^{T} dt \ e^{-i\eta t} \langle J_i(-it) J_j \rangle_\infty \equiv I_1 + I_2.$$
Wick rotation

- Fix $T, \eta \in \mathbb{R}^+$. We have:

$$
\int_{-T}^{T} dt \ e^{-i\eta t} \left\langle T \mathcal{J}_i(-it) ; \mathcal{J}_j \right\rangle_\infty \\
= \int_{-T}^{0} dt \ e^{-i\eta t} \left\langle \mathcal{J}_j \mathcal{J}_i(-it) \right\rangle_\infty + \int_{0}^{T} dt \ e^{-i\eta t} \left\langle \mathcal{J}_i(-it) \mathcal{J}_j \right\rangle_\infty \\
\equiv I_1 + I_2 .
$$

- Consider $I_2$. Fix $\varepsilon > 0$. Claim: up to $O(\varepsilon)$,

$$
I_2 = \left[ -i \int_{-T+\varepsilon}^{0} e^{\eta(t-i\varepsilon)} \left\langle \mathcal{J}_i(t-i\varepsilon) \mathcal{J}_j \right\rangle_\infty + \text{boundary term} \right]
$$
Fix $T, \eta \in \mathbb{R}^+$. We have:

$$\int_{-T}^{T} dt \, e^{-i\eta t} \langle T \mathcal{J}_i(-it) \mathcal{J}_j \rangle_{\infty} = \int_{-T}^{0} dt \, e^{-i\eta t} \langle \mathcal{J}_j \mathcal{J}_i(-it) \rangle_{\infty} + \int_{0}^{T} dt \, e^{-i\eta t} \langle \mathcal{J}_i(-it) \mathcal{J}_j \rangle_{\infty} \equiv I_1 + I_2.$$
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- Consider $I_2$. Fix $\varepsilon > 0$. Claim: up to $O(\varepsilon)$,

$$I_2 = \left[ -i \int_{-T+\varepsilon}^{0} e^{\eta(t-i\varepsilon)} \langle \mathcal{J}_i(t-i\varepsilon) \mathcal{J}_j \rangle_\infty + \text{boundary term} \right]$$

- Bound for correlations at complex times: for $\text{Im} \, z \leq 0$,

$$\left| \frac{1}{|\Lambda|} \langle \mathcal{J}_j(z) \mathcal{J}_i(0) \rangle_{\beta,\Lambda} \right|^2 \leq \left| \frac{1}{|\Lambda|} \langle \mathcal{J}_j(i\text{Im} \, z) \mathcal{J}_i(0) \rangle_{\beta,\Lambda} \right| \left| \frac{1}{|\Lambda|} \langle \mathcal{J}_i(i\text{Im} \, z) \mathcal{J}_i(0) \rangle_{\beta,\Lambda} \right| \leq \frac{C_M}{1 + |\text{Im} \, z|^M} \quad (M \geq 2)$$
Wick rotation

\[ \left| \frac{1}{|\Lambda|} \langle \mathcal{J}_j(-iz)\mathcal{J}_i(0) \rangle_{\beta,\Lambda} \right| \leq \frac{C_M}{1 + |\text{Re} \, z|^M} \quad (M \geq 2, \quad \text{Re} \, z \geq 0) \quad (*) \]

- Using also \( e^{-i\eta z} = e^{\eta \text{Im} \, z} e^{-i\eta \text{Re} \, z} \), the boundary term vanishes for \( T \rightarrow \infty \).
Wick rotation

\[ \left| \frac{1}{\Lambda} \langle \mathcal{J}_j(-iz) \mathcal{J}_i(0) \rangle_{\beta,\Lambda} \right| \leq \frac{C_M}{1 + |\text{Re } z|^M} \quad (M \geq 2, \quad \text{Re } z \geq 0) \quad (\ast) \]

- Using also \( e^{-i\eta z} = e^{\eta \text{Im } z} e^{-i\eta \text{Re } z} \), the boundary term vanishes for \( T \to \infty \).
- Analyticity in the right-half complex plane follows from:
  1. analyticity for finite \( \beta, \Lambda \)
  2. the bound \((\ast)\), which is uniform in \( \beta, \Lambda \)
  3. existence of the \((\beta, \Lambda) \to \infty \) limit on positive real axis.
Then, we use Vitali’s theorem.
Wick rotation

\[ \left| \frac{1}{|\Lambda|} \langle \mathcal{J}_j(-iz)\mathcal{J}_i(0) \rangle_{\beta,\Lambda} \right| \leq \frac{C_M}{1 + |\text{Re} z|^M} \quad (M \geq 2, \quad \text{Re} z \geq 0) \quad (*) \]

- Using also \( e^{-i\eta z} = e^{\eta \text{Im} z} e^{-i\eta \text{Re} z} \), the boundary term vanishes for \( T \to \infty \).

- Analyticity in the right-half complex plane follows from:
  1. analyticity for finite \( \beta, \Lambda \)
  2. the bound \((*)\), which is uniform in \( \beta, \Lambda \)
  3. existence of the \((\beta, \Lambda) \to \infty\) limit on positive real axis.

Then, we use Vitali’s theorem.

- Repeating the same analysis for \( I_1 \):

\[
I_1 + I_2 = -i \lim_{\epsilon \to 0} \int_{-\infty}^{0} dt \ e^{\eta t} \langle [\mathcal{J}_i(t + i\epsilon), \mathcal{J}_j] \rangle_{\infty} = -i \int_{-\infty}^{0} dt \ e^{\eta t} \langle [\mathcal{J}_i(t), \mathcal{J}_j] \rangle_{\infty}
\]

by Lieb-Robinson bounds, \((*)\) and dominated convergence.
Hall transitions in the interacting Haldane model

- **Noninteracting theory.** Let \( m_\pm \neq 0 \). Euclidean conductivity:

\[
\bar{\sigma}_{12} = - \lim_{\eta \to 0^+} \frac{1}{\eta} \int_{-\infty}^{\infty} dt \left( e^{-i\eta t} - 1 \right) \langle T J_1(-it) ; J_2 \rangle_\infty
\]

\[
\equiv -\partial_\eta \hat{K}_{12}(0) , \quad \hat{K} = \text{Fourier transfo of } \langle T J_1(-it) ; J_2 \rangle_\infty .
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\]

- **The state is quasi-free.** By Wick rule:

\[
\overline{\sigma}_{12} = \int_{\mathbb{R} \times \mathbb{T}^2} \text{tr} \Gamma_1(k) \partial_{k_0} g(k) \Gamma_2(k) g(k)
\]

\[
g(k)^{-1} = -\begin{pmatrix} ik_0 - m(k) & t_1 \Omega^*(k) \\ t_1 \Omega(k) & ik_0 + m(k) \end{pmatrix}, \quad k = (k_0, k),
\]

where, for $k'$ small:

\[
\Omega(k' + k_F^\omega) \simeq \frac{3}{2} \left( ik_1' + \omega k_2' \right) \quad m(k' + k_F^\omega) \simeq m_\omega \quad \Gamma_i \simeq \text{Pauli matrices}
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\Omega(k' + k_F^\omega) \approx \frac{3}{2} (ik_1' + \omega k_2') \quad m(k' + k_F^\omega) \approx m_\omega \quad \Gamma_i \simeq \text{Pauli matrices}
\]

- In presence of interactions, this is just the zero-th order.
Hall transitions in the interacting Haldane model

- **Interacting theory.** Let $m_{R,\pm} \neq 0$. Euclidean conductivity:

  \[
  \bar{\sigma}_{12} = - \lim_{\eta \to 0^+} \frac{1}{\eta} \int_{-\infty}^{\infty} dt \left( e^{-i\eta t} - 1 \right) \langle T J_1(-it); J_2 \rangle_{\infty} \\
  \equiv -\partial_\eta \hat{K}_{12}(0), \quad \hat{K} = \text{Fourier transfo of} \langle T J_1(-it); J_2 \rangle_{\infty}.
  \]

- The state is not quasi-free.
Hall transitions in the interacting Haldane model

- **Interacting theory.** Let \( m_{R, \pm} \neq 0 \). Euclidean conductivity:

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\bar{\sigma}_{12} = - \lim_{\eta \to 0^+} \frac{1}{\eta} \int_{-\infty}^{\infty} dt \left( e^{-i\eta t} - 1 \right) \left\langle \mathbf{T} \mathcal{J}_1(-it) ; \mathcal{J}_2 \right\rangle_{\infty} \\
\equiv -\partial_\eta \hat{K}_{12}(0), \quad \hat{K} = \text{Fourier transfo of} \ \left\langle \mathbf{T} \mathcal{J}_1(-it) ; \mathcal{J}_2 \right\rangle_{\infty}.
\]

- The state is **not** quasi-free. By RG methods:

\[
\bar{\sigma}_{12} = \int_{\mathbb{R} \times T^2} \text{tr} \, \Gamma_{1,R}(k) \partial_{k_0} g_{R}(k) \, \Gamma_{2,R}(k) \, g_{R}(k) + \text{h.o.t.}
\]

\[
g_{R}(k)^{-1} \simeq - \begin{pmatrix} iZ_1 k_0 - m_{R}(k) & v_R \Omega^*(k) \\ v_R \Omega(k) & iZ_2 k_0 + m_{R}(k) \end{pmatrix}
\]

where, for \( k' \) small:

\[
m_{R}(k' + \mathbf{k}_F^\omega) \simeq m_{R,\omega} \quad \Gamma_{i,R} \simeq \text{“renormalized” Pauli matrices}
\]

\[
(Z_i, v_R, m_{R,\omega}) = (1, t_1, m_\omega) + \text{convergent series in } U
\]
Hall transitions in the interacting Haldane model

- Let $0 < |m_{R,\omega}| \ll |m_{R,-\omega}|$. Let $\varepsilon > 0$ small. Convenient rewriting:

$$\bar{\sigma}_{12} = \int_{\mathbb{R} \times B_\varepsilon(k_\omega)} \text{tr} \Gamma_{1,R}(k) \partial_k g_R(k) \Gamma_{2,R}(k) g_R(k) + \tilde{\sigma}_{12} \equiv I_1 + I_2$$
Hall transitions in the interacting Haldane model

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$I_1$: Integrand scales as $[|k - k_F^\omega|^2 + m_{R,\omega}^2]^{-\frac{3}{2}} \Rightarrow$ integral not continuous in $m_{R,\omega}$

$I_2$: $\tilde{\sigma}_{12}$ is continuous in $m_{R,\omega}$ (interaction irrelevant in RG sense).
Hall transitions in the interacting Haldane model

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$\Rightarrow \Delta_\omega = \left[ \lim_{m_{R,\omega} \to 0^+} - \lim_{m_{R,\omega} \to 0^-} \right] \bar{\sigma}_{12}$ only determined by term $I_1$. 
Hall transitions in the interacting Haldane model

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$$\Rightarrow \Delta_\omega = \left[ \lim_{m_{R,\omega} \to 0^+} - \lim_{m_{R,\omega} \to 0^-} \right] \sigma_{12} \text{ only determined by term } I_1.$$

- Still, term $I_1$ depends on $Z_{i,R}, v_R, m_{\omega,R}, \Gamma_{i,R}$. Crucial ingredient:

$$\Gamma_{i,R}(k_F^\omega) = - \partial_i g_R(k_F^\omega)^{-1} \quad \text{(Ward identity)}$$

following from $U(1)$ gauge invariance. All parameters cancel out!

$$\Delta_\omega = \frac{\omega}{\pi} \quad \text{(e}^2 = \hbar = 1)$$
Conclusions

We discussed:

- the universality of $\sigma_{12}$ for general interacting gapped fermionic systems,
- the Hall transitions for the interacting Haldane model.

Tools: LR bounds, determinant bounds, rigorous RG, Ward identities.

Open questions:

- Extension to time-reversal invariant 2d topological insulators (for example, interacting Kane-Mele model)?
- Bulk-edge correspondence for interacting fermionic systems?
- Universality in interacting disordered systems?
- ...

...
Thank you!
Universality of $\overline{\sigma}_{i,j}$: Schwinger-Dyson equation

- For $U$ in the analyticity domain, the Schwinger-Dyson equation holds:

$$\hat{K}^{(U)}_{i,j}(p) = \hat{K}^{(0)}_{i,j}(p)$$

$$+ \int_{0}^{U} dU' \int d\mathbf{q} \hat{v}(p) \hat{K}^{(U')}_{i,j,0,0}(p,-p,q)$$

$$+ 2 \int_{0}^{U} dU' \hat{v}(0) \hat{K}^{(U')}_{i,j,0}(p,-p) \hat{K}^{(U')}_{0}$$

$$+ 2 \int_{0}^{U} dU' \hat{v}(p) \hat{K}^{(U')}_{i,0}(p) \hat{K}^{(U')}_{j,0}(-p)$$

with $p = (\eta, p) \in \mathbb{R}^3$ and

$$\hat{K}^{(U')}_{i,j,0,0}(p,-p,q) = \langle T J_{i,p} ; J_{j,-p} ; n_{q} ; n_{-q} \rangle .$$
For $U$ in the analyticity domain, the **Schwinger-Dyson equation** holds:

\[
\begin{align*}
\sigma_{ij}^{(k)} &= \sigma_{ij}^{(k-1)} + \sigma_{ij}^{(k-1-m)} + \sigma_{ij}^{(m)} \\
\end{align*}
\]
Universality of $\overline{\sigma}_{ij}$: Ward identities

- **Continuity equation** (recall $O(-it) = e^{t\mathcal{H}}Oe^{-t\mathcal{H}}$):

$$\partial_t n_p(-it) := [\mathcal{H}, n_p(-it)] = p \cdot J_p(-it)$$
Universality of $\bar{\sigma}_{ij}$: Ward identities

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  \[
  \partial_t n_p(-it) := [\mathcal{H}, n_p(-it)] = p \cdot J_p(-it)
  \]

- It implies relations among correlations: **Ward identities.** E.g.:
  \[
  \eta \hat{K}_{0,0}(p) + p_i \hat{K}_{i,0}(p) = 0 \Rightarrow \hat{K}_{j,0}(p) = -\eta \frac{\partial}{\partial p_j} \hat{K}_{0,0}(p) - p_i \frac{\partial}{\partial p_j} \hat{K}_{i,0}(p)
  \]
  \[
  = O(p).
  \]

  Similarly, $\hat{K}_{i,j,0,0}(p, -p, q) = O(p^2)$, $\hat{K}_{i,j,0}(p, -p) = O(p^2)$. 

Universality of $\overline{\sigma}_{ij}$: Ward identities

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Similarly, $\hat{K}_{i,j,0,0}(p, -p, q) = O(p^2)$, $\hat{K}_{i,j,0}(p, -p) = O(p^2)$. Therefore,
\[
\hat{K}^{(U)}_{i,j}(p) - \hat{K}^{(0)}_{i,j}(p) = O(p^2).
\]
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  \[
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  \]

- Since $\bar{\sigma}^{(U)}_{ij} = \lim_{\eta \to 0^+} (-1/\eta) [\hat{K}^{(U)}_{i,j}(\eta, 0, 0) - \hat{K}^{(U)}_{i,j}(0)] \equiv -\partial_\eta \hat{K}^{(U)}_{i,j}(0)$,
  \[
  \bar{\sigma}^{(U)}_{ij} = \bar{\sigma}^{(0)}_{ij} .
  \]