Asymptotic FOEL for the Heisenberg model on boxes

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Shannon Starr

University of Alabama at Birmingham

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- Bruno Nachtergaele, U. C. Davis,
- Wolfgang Spitzer, FernUniversität Hagen.
(1) Review Heisenberg model on graph $G = (V, E)$

(2) Define the “ferromagnetic ordering of energy levels” condition (FOEL)

(3) Linear spin wave approximation for $\{1, \ldots, L\}^d \subseteq \mathbb{Z}^d$ at energies on the order of $1/L^2$
1. Define Heisenberg model

Graph: \( G = (V, E), \quad |V| < \infty, \)

\[ E \subseteq \{ \{x, y\} : x \in V, \ y \in V, \ x \neq y \}. \]

Let: \( V = \{x_1, \ldots, x_N\} \) with \( |V| = N. \)

Hilbert space: \( \mathcal{H}_V = (\mathbb{C}^2)^\otimes N = \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2. \)

\[ S^{(1)} = \begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix}, \quad S^{(2)} = \begin{bmatrix} 0 & -i/2 \\ i/2 & 0 \end{bmatrix}, \quad S^{(3)} = \begin{bmatrix} 1/2 & 0 \\ 0 & -1/2 \end{bmatrix} \]

Operators \( S_{x_k}^{(a)} : \mathcal{H}_V \to \mathcal{H}_V, \)

\[ S_{x_k}^{(a)} = 1_{\mathbb{C}^2} \otimes \cdots \otimes 1_{\mathbb{C}^2} \otimes S^{(a)} \otimes 1_{\mathbb{C}^2} \otimes \cdots \otimes 1_{\mathbb{C}^2} = 1_{\mathbb{C}^2} \otimes \cdots \otimes 1_{\mathbb{C}^2} \otimes S^{(a)} \otimes 1_{\mathbb{C}^2} \otimes \cdots \otimes 1_{\mathbb{C}^2} \]

for \( a \in \{1, 2, 3\}, \ k \in \{1, \ldots, N\}. \)
\[
\left[ S_x^{(1)}, S_y^{(2)} \right] = \frac{i}{2} \delta_{x,y} S_x^{(3)}, \quad \text{and cyclic permutations.}
\]

Define \( S_V^{(a)} = \sum_{x \in V} S_x^{(a)} \).

For each \( \{x, y\} \in E \), define \( h_{x,y}: \mathcal{H}_V \rightarrow \mathcal{H}_V \),

\[
h_{x,y} = \frac{1}{4} 1 - \vec{S}_x \cdot \vec{S}_y = \frac{1}{4} 1 - \sum_{a=1}^{3} S_x^{(a)} S_y^{(a)}.
\]

Then the total Hamiltonian is

\[
H_G = \sum_{\{x,y\} \in E} h_{x,y}.
\]

This is a model such that \( \left[ S_V^{(a)}, H_G \right] = 0 \) for \( a = 1, 2, 3 \).
Brief review: history for Heisenberg model

  - classical Heisenberg ferromagnet
  - proved LRO for \( d \geq 3 \) and sufficiently small \( T > 0 \)
  - invented reflection positivity technique to prove Gaussian domination for 2-point correlation functions.


Usual ordering of energy levels: Lieb-Mattis theorem

Suppose $G$ is bipartite: $G = (V, E)$ with $V = A \sqcup B$ and

$$E \subseteq \Big\{ \{x, y\} : x \in A, \ y \in B \Big\},$$

and assume $|A| = |B|$, balanced.

Then Lieb and Mattis (J. Math. Phys., 1962) proved that, for the antiferromagnet $-H_G$, the ground state is a spin singlet.

More generally, they allow antiferromagnetic couplings between the two parts and ferromagnetic couplings with single parts

$$H_V(J) = \sum_{\{x,y\} \in E} J_{x,y} h_{x,y}.$$

$J_{x,y} \leq 0$ if $(x, y) \in (A \times B) \cup (B \times A)$,

$J_{x,y} \geq 0$ if $(x, y) \in (A \times A) \cup (B \times B)$. 
For each \( s \in \{0, 1, \ldots, N/2\} \), define the total-spin \( s \) subspace

\[
\mathcal{H}_{V}^{\text{tot sp}}(s) = \ker \left( \sum_{a=1}^{3} \left( S_{V}^{(a)} \right)^{2} - s(s + 1) \mathbb{I} \right).
\]

Then \( H_{G} \mathcal{H}_{V}^{\text{tot sp}}(n) \subseteq \mathcal{H}_{V}^{\text{tot sp}} \) by \( SU(2) \) symmetry.

Lieb and Mattis also proved

\[
\min \text{spec} \left( H_{G} \upharpoonright \mathcal{H}_{V}^{\text{tot sp}}(0) \right) \leq \ldots \leq \min \text{spec} \left( H_{G} \upharpoonright \mathcal{H}_{V}^{\text{tot sp}}\left( \frac{1}{2}N \right) \right).
\]

More generally if \( |A| \) and \( |B| \) are not balanced, then they proved the ground state has total spin \( \frac{1}{2}||A| - |B|| \) and

\[
\min \text{spec} \left( H_{G} \upharpoonright \mathcal{H}_{V}^{\text{tot sp}}\left( \frac{||A| - |B||}{2} \right) \right) \leq \ldots \leq \min \text{spec} \left( H_{G} \upharpoonright \mathcal{H}_{V}^{\text{tot sp}}\left( \frac{N}{2} \right) \right).
\]
Ferromagnetic ordering of energy levels

Denote $E_0(G, s) = \min \text{spec} \left( H_G \upharpoonright \mathcal{H}_{V}^{\text{totsp}}(s) \right)$.

For $G = (V, E)$, $N = |V|$ the ground state energy is $E_0(G, \frac{1}{2} N)$.

For open chains

$$V = \{1, \ldots, L\} \subset \mathbb{Z}, \quad E = \left\{ \{k, k+1\} : 1 \leq k \leq L - 1 \right\},$$

Koma and Nachtergaele (Lett. Math. Phys., 1997) proved that the spectral gap is $E_0([1, L], \frac{1}{2} L - 1)$.

So

$$E_0\left([1, L], \frac{1}{2} L\right) \leq E_0\left([1, L], \frac{1}{2} L - 1\right) \leq \min_{s < \frac{1}{2} L - 1} E_0\left([1, L], s\right).$$
Consider increasing chain from $[1, L]$ to $[1, L + 1]$. 

The definition of each pair interaction 

$$h_{x,y} = \frac{1}{4} - \vec{S}_x \cdot \vec{S}_y = \begin{pmatrix} 0 & 1/2 & -1/2 \\ -1/2 & 1/2 & 0 \end{pmatrix} \otimes \bigotimes_{z \notin \{x,y\}} (1_{\mathbb{C}^2})_z ,$$

is positive semi-definite and has ground state energy 0.

So $\mathcal{E}_0([1, L], \frac{1}{2} L) = 0$.

Koma and Nachtergaele calculated 

$$\mathcal{E}_0([1, L], \frac{1}{2} L - 1) = 1 - \cos(\pi/L) = 2 \sin^2 \left( \frac{\pi}{2L} \right),$$

strictly decreasing.
Suppose $\psi \in \mathcal{H}^{\text{totsp}}_{[1,L+1]}(\frac{1}{2}(L + 1) - 2)$ is an eigenvector of $H_{[1,L+1]}$.

Then

$$\exists \psi_1, \psi_2 \in \mathcal{H}^{\text{totsp}}_{[1,L]} \left( \frac{1}{2} L - 2 \right) + \mathcal{H}^{\text{totsp}}_{[1,L]} \left( \frac{1}{2} L - 1 \right),$$

such that $\psi = \psi_1 \otimes |\uparrow\rangle + \psi_2 \otimes |\downarrow\rangle$,

where $|\uparrow\rangle$ and $|\downarrow\rangle$ denote the standard basis of $\mathbb{C}^2$.

So, since $h_{L,L+1} \geq_{\text{psd}} 0$, this means $H_{[1,L]} \leq_{\text{psd}} H_{[1,L+1]}$.

$$\langle \psi, H_{[1,L+1]} \psi \rangle \geq \langle \psi, H_{[1,L]} \psi \rangle = \langle \psi_1, H_{[1,L]} \psi_1 \rangle + \langle \psi_2, H_{[1,L]} \psi_2 \rangle.$$

But for $k \in \{1, 2\}$,

$$\frac{\langle \psi_k, H_{[1,L]} \psi_k \rangle}{\|\psi_k\|^2} \geq \min \left( \left\{ \mathcal{E}_0 \left( [1, L], \frac{1}{2} L - 2 \right), \mathcal{E}_0 \left( [1, L], \frac{1}{2} L - 1 \right) \right\} \right).$$
So this proves

\[
\mathcal{E}_0 \left([1, L + 1], \frac{1}{2}(L + 1) - 2\right) \\
\geq \min \left( \left\{ \mathcal{E}_0 \left([1, L], \frac{1}{2}L - 2\right), \mathcal{E}_0 \left([1, L], \frac{1}{2}L - 1\right) \right\} \right)
\]

If, as an induction hypothesis, we assume

\[
\mathcal{E}_0 \left([1, L], \frac{1}{2}L - 1\right) \leq \mathcal{E}_0 \left([1, L], \frac{1}{2}L - 2\right),
\]

then this means

\[
\mathcal{E}_0 \left([1, L + 1], \frac{1}{2}(L + 1) - 2\right) \geq \mathcal{E}_0 \left([1, L], \frac{1}{2}L - 1\right).
\]
But Koma and Nachtergaele showed

\[ \mathcal{E}_0 \left( [1, L + 1], \frac{1}{2}(L + 1) - 1 \right) \leq \mathcal{E}_0 \left( [1, L], \frac{1}{2}L - 1 \right), \]

i.e., \( \sin^2\left(\frac{\pi}{2(L+1)}\right) \leq \sin^2\left(\frac{\pi}{2L}\right) \).

So we deduce the induction step

\[ \mathcal{E}_0 \left( [1, L + 1], \frac{1}{2}(L + 1) - 2 \right) \geq \mathcal{E}_0 \left( [1, L + 1], \frac{1}{2}(L + 1) - 1 \right). \]
Aldous’s conjec; Caputo, Liggett, Richthammer’s thm

Handjani and Jungreis rediscovered this argument (J. Theor. Prob., 1996).

The Heisenberg ferromagnet is unitarily equivalent to the SEP.

Aldous conjectured, for any graph $G = (V, E)$, with $N = |V|$,

$$\mathcal{E}_0 \left( G, \frac{1}{2} N - 1 \right) \leq \min_{s < \frac{1}{2} N - 1} \mathcal{E}_0(G, s).$$

Caputo, Liggett and Richthammer proved this (J. A. M. S., 2010).
CLR’s network reduction

Caputo, Liggett and Richthammer consider weighted graphs.

Rates $c_{xy} \equiv$ coupling constants $J_{xy}$.

Network reduction $G = (V, c) \longrightarrow G_z = (V \setminus \{z\}, \tilde{c}^{(z)})$

$$\forall x, y \in V \setminus \{z\}, \quad \tilde{c}_{xy}^{(z)} = c_{xy} + \frac{c_{xz} c_{yz}}{\sum_{w \in V \setminus \{z\}} c_{wz}}$$

Example:
The definition of property “FOEL-$n$”

We say a graph $G = (V, E)$, with $N = |V|$, satisfies FOEL-$n$ if

$$\mathcal{E}_0 \left( G, \frac{1}{2} N - n \right) \leq \min_{s < \frac{1}{2} N - n} \mathcal{E}_0(G, s).$$

The FOEL-$n$ property is a property that $G$ may or may not satisfy, for each $n \in \{0, \ldots, \lfloor \frac{1}{2} N \rfloor \}$.

By the same reasoning as Koma and Nachtergaele’s argument,

- if we can grow our graph one vertex at a time $G_{2n}, G_{2n+1}, \ldots, G_N = G$,
- and if $\mathcal{E}_0(G_{2n}, 0) \geq \mathcal{E}_0(G_{2n+1}, \frac{1}{2}) \geq \cdots \geq \mathcal{E}_0(G, \frac{1}{2} N - n)$,
- then $G$ satisfies FOEL-$n$. 
Example [1, L] á la Koma and Nachtergaele

Using this argument, the graphs Koma and Nachtergaele considered, \([1, L] \subseteq \mathbb{Z}\), satisfy FOEL-\(n\) for each \(n \leq \left\lfloor \frac{1}{2}L \right\rfloor\).

*Note*, by the CLR theorem, all graphs satisfy FOEL-1. (By Lieb-Mattis all graphs also satisfy FOEL-0.)

Koma and Nachtergaele actually considered quantum group \(SU_q(2) = \mathcal{U}_q(sl_2)\) symmetric XXZ model for \(q \in (0, \infty)\).

This representation comes with dual canonical basis, that Temperley and Lieb called “Hulthèn bracket basis.” In this basis one has \(tree-like^1(?)\) behavior of \(\mathcal{E}_0([1, L], \frac{1}{2}L - n)\) for all \(n\).


\(^1\)Wielandt minimax for Perron-Frobenius roots
Counterexamples for $n \geq 2$

Since FOEL-1 (as well as FOEL-0) is true for all graphs, is FOEL-2?

No, not for the hexagon.

Let $C_N$ be $G = (V, E)$ for $V = \{1, \ldots, N\}$ and
\[ E = \big\{ \{1, 2\}, \{2, 3\}, \ldots, \{N-1, N\}, \{1, N\} \big\}. \]

Then $C_{2n}$ violates FOEL-$(n - 1)$ for $n > 2$:
true for $n = 3$,
numerically verified by Lanczos iteration for $n = 4, 5, 6, 7$ (Tran, Spitzer, S; J. Math. Phys., 2012),

Asymptotic FOEL-\(n\)

If you have a sequence of graphs \(G_{2n}, G_{2n+1}, \ldots\), and if

\[
\mathcal{E}_0 \left( G_N, \frac{1}{2} N - n \right) \leq \min_{M \in \{2n, \ldots, N\}} \mathcal{E}_0 \left( G_M, \frac{1}{2} M - n \right),
\]

then \(G_N\) satisfies FOEL-\(n\).

**Lemma** Suppose there are constants \(p > 0\) and \(0 = C_0 < C_1 < \ldots\) such that the sequence of graphs above satisfy

\[
\mathcal{E}_0 \left( G_N, \frac{1}{2} N - n \right) \sim C_n N^{-p}, \quad \text{as } N \to \infty,
\]

for each \(n \in \{0, 1, \ldots\}\). Then, for each \(n \in \{0, 1, \ldots\}\), there is \(N_n\) such that \(G_N\) satisfies FOEL-\(n\) for each \(N \geq N_n\).
LSW approximation for energies of order $1/L^2$

Consider a box $B^d(L) = \{1, \ldots, L\}^d \subset \mathbb{Z}^d$.

Then the energies of order $1/L^2$ in $\mathcal{H}^{\text{tot sp}}(B^d(L), \frac{1}{2}L^d - n)$ are asymptotically

$$\frac{\pi^2}{2L^2} \left( \|\vec{k}_1\|^2 + \cdots + \|\vec{k}_n\|^2 \right)$$

ranging over ordered $n$-tuples $(\vec{k}_1, \ldots, \vec{k}_n) \in (\mathbb{Z}^d \setminus \{0\})^n$, modulo the action of $S_n$.

In particular, $\mathcal{E}_0 \left( B^d(L), \frac{1}{2}L^d - n \right) \sim \frac{n\pi^2}{2L^2}$.

This means $\mathcal{E}_0(G_N, \frac{1}{2}N - n) \sim C_n N^{-p}$ for $C_n = n\pi^2/2$, $p = 2/d$. 
Disclaimer

If you prove the claim, you do not actually need the inductive argument for FOEL.

Correggi, Giuliani and Seiringer (Comm. Math. Phys., 2015), “Validity of the Spin-Wave Approximation for the Free Energy of the Heisenberg Ferromagnet,” proved $\exists c > 0$ such that for all $L$ and all $n$

$$\mathcal{E}_0 \left( B^d(L), \frac{1}{2} L^d - n \right) \geq \frac{cn}{L^2}.$$
Graph Laplacian of the configuration graph

Given a graph $G = (V, E)$, the graph Laplacian is the operator 

$$-\Delta_G : \ell^2(V) \to \ell^2(V)$$

$$-\Delta_G f(x) = \frac{1}{2} \sum_{y \in V} 1_E(\{x, y\})[f(x) - f(y)],$$

so that

$$\langle f, -\Delta_G f \rangle = \frac{1}{2} \sum_{\{x, y\} \in E} \|f(x) - f(y)\|^2.$$

One may define a new graph $\Phi^n(G) = (V^n, \Phi^n(E))$, where $\Phi^n(E)$ is the set of all $\{(x_1, \ldots, x_n), (y_1, \ldots, y_n)\}$,

$$\exists k \in \{1, \ldots, n\}, \quad \text{s.t.} \quad \{x_k, y_k\} \in E, \quad \text{and} \quad x_j = y_j \quad \text{for} \quad j \neq k.$$
The configuration graph

The $n$ particle configuration graph is

$$\Theta^{(n)}(G) = (\Theta^{(n)}(V), \Theta^{(n)}(E)),$$

where

$$\Theta^{(n)}(V) = \left\{ (x_1, \ldots, x_n) \in V^n : j \neq k \Rightarrow x_j \neq x_k \right\},$$

and $\Theta^{(n)}(E)$ is the set of edges

$$\left\{ (x_1, \ldots, x_n), (y_1, \ldots, y_n) \right\} \in \Phi^{(n)}(E),$$

such that $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in \Theta^{(n)}(V)$. 

\[ \text{The configuration graph} \]
Then, for any function \( f \in \ell^2(\Theta^{(n)}(V)) \) which is symmetric

\[
\forall \pi \in S_n, \quad f(x_{\pi(1)}, \ldots, x_{\pi(n)}) = f(x_1, \ldots, x_n),
\]

we can map \( T_V^{(n)} : \ell^2_{\text{sym}}(\Theta^{(n)}(V)) \to \mathcal{H}_V \cap \ker(S_V^{(3)} - \frac{1}{2}N + n) \) by

\[
T_V^{(n)} f = \frac{1}{\sqrt{n!}} \sum_{(x_1, \ldots, x_n) \in \Theta^{(n)}(V)} f(x_1, \ldots, x_n) S_{x_1}^- \cdots S_{x_n}^- \left( \bigotimes_{x \in V} |\uparrow\rangle_x \right),
\]

where \( S_x^\pm = S_x^{(1)} \pm iS_x^{(2)} \).

Then \( T_V^{(n)} \) is a unitary transformation, and

\[
H_G T_V^{(n)} f = T_V^{(n)} \left( -\Delta_{\Theta^{(n)}(G)} f \right).
\]
Another tool from CGS, and “filling in”

If we know the spectrum of $-\Delta_G$ then it is trivial to determine the spectrum of $-\Delta_{\Phi(n)(G)}$.

Given a $f \in \ell^2(\Theta^{(n)}(V))$ with “low energy” we could extend by zero to $\ell^2(\Phi^{(n)}(V))$.

Then we have to worry about

$$\sum_{x \in \Theta^{(n)}(V)} \sum_{y \in \Phi^{(n)}(V) \setminus \Theta^{(n)}(V)} 1_{\Phi^{(n)}(E)}(\{x, y\}) \| f(x) - f(y) \|^2.$$  

For $G = \mathbb{B}^d(L)$, with $d \geq 3$, Correggi, Giuliani and Seiringer have another powerful result: for any eigenfunction $\Psi$ of $H_G$ with eigenvalue $\lambda$,

$$\| \rho \|_\infty \leq C \lambda^d \| \rho \|_1,$$

where $\rho(x, y) = \langle \Psi , S_x^- S_x^+ S_y^- S_y^+ \Psi \rangle$ is the 2-particle density.
If $d \leq 2$, then we can “fill in.”

For any $x \in \Phi^{(n)}(V) \setminus \Theta^{(n)}(V)$, just define $f(x)$ to be an average of $f(y)$ for nearby $y \in \Theta^{(n)}(V)$. 
Review Heisenberg model
Ferromagnetic ordering of energy levels
Linear Spin Wave Approx

Energies of $O(1/L^2)$ on boxes $\mathcal{B}^d(L)$
Toth representation
Filling in
We raise the energy by a constant factor (depending on $n$).

But we just use the energy bound with the Chebyshev inequality to restrict the number of eigenmodes we must keep to have a good approximation to $f$ in the eigenvector decomposition relative to $-\Delta \Phi^{(n)}(B^d(L))$.

We already have uniform bounds for eigenfunctions of $-\Delta \Phi^{(n)}(B^d(L))$ and their norms restricted to $\Phi^{(n)}(B^d(L)) \setminus \Theta^{(n)}(B^d(L))$ are small.

Then eigenfunctions $f$ of $-\Delta \Theta^{(n)}(B^d(L))$ with energy $C/L^2$ must be close in norm to linear combinations of eigenfunctions of $-\Delta \Phi^{(n)}(B^d(L))$.

So the min-max theorems prove that the restrictions of the spectra to $[0, C/L^2]$ are close.
Thanks for your attention!