Vanishing viscosity method for solutions to an optimal control problem of hyperbolic conservation laws in the presence of shocks

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I. Optimal control problem and vanishing viscosity limit:

Scalar hyperbolic conservation law:

\[ u_t + F(u)_x = 0, \quad (x,t) \in \mathbb{R} \times (0,T); \quad u(x,0) = u^I(x), \] (1)

where \( F \) is smooth and \( F_{uu} > 0 \). Assume the solution \( u \) is the unique entropy solution with a single shock discontinuity \( \Sigma = \{ x = \varphi(t) \} \), satisfying the Rankine-Hugoniot jump condition on the discontinuity \( x = \varphi(t) \):

\[ \varphi'(t)[u]_{\varphi(t)} = [F(u)]_{\varphi(t)}, \]

and the Oleinik’s one-sided Lipschitz condition (OSLC):

\[ (f(u(x,t)) - f(u(y,t)))(x-y) \leq \alpha(t)(x-y)^2. \]

Given a target function \( u^D \in L^2(\mathbb{R}) \), we consider the cost functional \( J : L^1(\mathbb{R}) \to \mathbb{R} \), which is defined by

\[ J(u^I) = \int_{\mathbb{R}} |u(x,T) - u^D(x)|^2 \, dx, \] (2)
where \( u(x, t) \) is the unique entropy solution to (1) with a single shock.

We introduce the set of admissible initial data \( \mathcal{U}_{ad} \subset L^1(\mathbb{R}) \), which is

\[
\mathcal{U}_{ad} = \{ f \in L^\infty(\mathbb{R}) \mid \text{supp}(f) \subset K, \| f \|_{L^\infty(\mathbb{R})} \leq C \},
\]

(3)

where \( K \subset \mathbb{R} \) is a bounded interval and \( C > 0 \) a constant. Then we shall solve the existence of the following optimization problem:

Find \( u^{I, \text{min}} \in \mathcal{U}_{ad} \) such that \( J(u^{I, \text{min}}) = \min_{u^I \in \mathcal{U}_{ad}} J(u^I) \).

For the conservation law (1), the corresponding viscous problem is

\[
u_t + (F(u))_x = \nu u_{xx}, \text{ in } \mathbb{R} \times (0, T), \quad u(x, 0) = g^\epsilon,
\]

(4)

where \( g^\epsilon \to u^I \) as \( \epsilon \to 0 \) and \( \epsilon = \epsilon(\nu) \) is to be determined.

Similar minimization problem for (4): Find \( g^{\epsilon, \text{min}} \in \mathcal{U}_{ad} \) such that

\[
J_\nu(g^{\epsilon, \text{min}}) = \min_{g^\epsilon \in \mathcal{U}_{ad}} J_\nu(g^\epsilon).
\]

Conclusion 1. \( J_\nu(g^{\epsilon, \text{min}}) \to J(u^{I, \text{min}}) \) as \( \nu \to 0 \) (\( \Gamma \)-convergence).
II. Sensitivity analysis by vanishing viscosity method.

**Question.** Could we carry out the sensitivity analysis for the inviscid minimization problem in the presence of shocks by the vanishing viscosity method?

(1) **Sensitivity analysis of \( J \): Inviscid case**

Let \( \delta u \) be the variations of the solution \( u \) with respect to the initial datum \( u_0 \), and \( \delta \varphi \) be the variation of the shock position \( \varphi(t) \) with respect to \( \varphi(t = 0) \).

**Lemma. (Bressan & Marson 1995)** \((\delta u, \delta \varphi)\) is the solution to the linearized problem

\[
(\delta u)_t + (f(u)\delta u)_x = 0, \text{ in } Q^+ \cup Q^-,
\]

\[
(\delta \varphi)'(t)[u]\varphi(t) + \delta \varphi(t) \left( \varphi'(t)[u_x]\varphi(t) - [f(u)u_x]\varphi(t) \right) = [f(u)\delta u]\varphi(t) - \varphi'(t)[\delta u]\varphi(t), \quad t \in (0, T),
\]

\[
\delta \varphi(0) = \delta \varphi^I,
\]

\[
\delta u(x, 0) = \delta u^I(x), \quad x \in \{x < \varphi^I\} \cup \{x > \varphi^I\}.
\]
**Definition.** (Bressan & Marson 1995) \( J \) is Gateaux differentiable at \( u^I \) in a generalized sense if \( \delta J = \lim_{\varepsilon \to 0} \frac{J(u^{I,\varepsilon}) - J(u^I)}{\varepsilon} \) exists.

**Lemma.** (Castro, Palacios & Zuazua 2008) Assume that \( u^D \) is continuous at \( x = \varphi(T) \). Then

\[
\delta J = 2 \int_{\{x < \varphi^I\} \cup \{x > \varphi^I\}} p(x, 0) \delta u^I(x) dx + 2q(0) [u^I]_{\varphi^I} \delta \varphi^I,
\]

where the adjoint state pair \((p, q)\) satisfies the adjoint problem:

\[
-\partial_t p - f(u) \partial_x p = 0, \text{ in } Q^- \cup Q^+,
\]

\[
[p]_{\Sigma} = 0,
\]

\[
q(t) = p(\varphi(t), t), \ t \in (0, T'),
\]

\[
q'(t) = 0, \ t \in (0, T),
\]

\[
p(x, T) = u(x, T) - u^D(x), \ x \in \{x < \varphi(T)\} \cup \{x > \varphi(T)\},
\]

\[
q(T') = \frac{[(u(x, T) - u^D)^2]_{\varphi(T)}}{[u]_{\varphi(T)}},
\]
(2) Viscose case and Vanishing viscosity limit

Viscous conservation law:

\[ u_t + F(u)_x = \nu u_{xx}, \quad u(x, 0) = g^\epsilon(x). \]  

Viscous linearized equation (let \( v = \delta u \)):

\[ v_t + (f(u)v)_x = \nu v_{xx}, \quad v(x, 0) = h^\epsilon(x). \]  

and its adjoint problem:

\[ -p_t - f(u)p_x = \nu p_{xx}, \quad p(x, t = T) = p^T_\epsilon(x). \]

Proposition. On \( x = \varphi(t) \), \( \delta u \) and \( \delta \varphi \) satisfy

\[
[u]_{\varphi(t)} \delta \varphi'(t) = \delta \varphi(t) \left( -[u_x]_{\varphi(t)} \varphi'(t) + [f(u)u_x]_{\varphi(t)} \right) \\
+ \left( -[\delta u]_{\varphi(t)} \varphi'(t) + [f(u)\delta u]_{\varphi(t)} \right) + \frac{1}{\sigma} \left( [u_x]_{\varphi(t)} - ([w]_{\varphi(t)} \varphi'(t) - [f(u)w]_{\varphi(t)}) \right).
\]
Proposition. $p^\nu \rightarrow \text{constant as } \nu \rightarrow 0 \text{ in a “triangular region”}$.

Main Theorems: Assume that \( \max_{x \in [-R,R]} p^T(x) \leq C \) for some constant \( R > 0 \) and

\[
\int_0^T \int_{\{x \neq \varphi(t), t \in [0,T]\}} \sum_{i=1}^6 \left( |\partial_x^i u(x,t)|^2 + |\partial_x^i v(x,t)|^2 + |\partial_x^i p(x,t)|^2 \right) \, dx \, dt \leq C. \tag{18}
\]

Then the viscous solutions \((u^\nu, v^\nu, p^\nu) \rightarrow (u, v, p)\) in \([L^\infty(0, T; L^2(\mathbb{R}))]^3\) as \( \nu \rightarrow 0 \), and the following estimates hold

\[
\sup_{0 \leq t \leq T} \left( \|u^\nu(t) - u(t)\| + \|v^\nu(t) - v(t)\| + \|p^\nu(t) - p(t)\| \right) \leq C \eta \nu^\eta, \tag{19}
\]

where \( \eta = \frac{1+\gamma}{2} \) and \( \eta \in \left(\frac{5}{6}, 1\right) \). Moreover, as \( \nu \rightarrow 0 \), we have the interior estimate

\[
\sup_{0 \leq t \leq T} \|p^\nu - p\|_{L^\infty(\Omega_h)} \leq C_h \nu \rightarrow 0, \tag{20}
\]

where

\[
\Omega_h = \{(x, t) \in Q_T \mid |x - \varphi(t)| > h\}.
\]