Use of the Mellin transform to determine the Calcium channel distribution in the olfactory system

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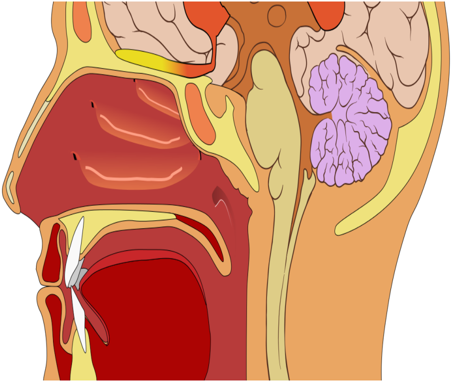
What are the olfactory cilia?

The olfactory epithelium is the part of the nose, which traps smells and communicates them to the brain. The microscopic olfactory cilia play a very important role in the perception of smell.



1. Olfactory bulb
2. Bone
3. Nasal Epithelium
4. Cilia in mucosa

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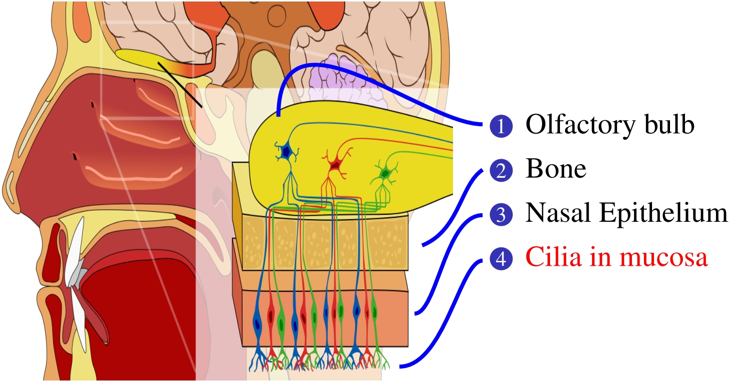
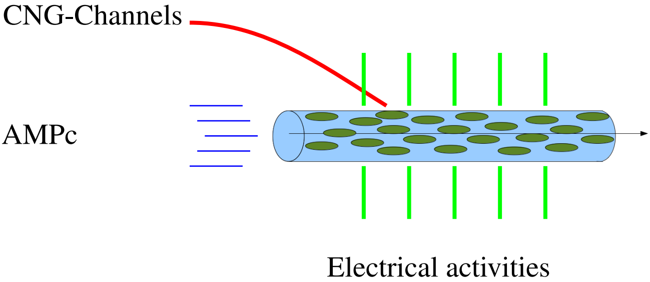


Experimental procedure

Developed by Steven Kleene and Rick Flannery in the College of Medicine University of Cincinnati



Direct problem: Obtain $\mathrm{I}[\rho](t)$, when we know $\rho(x)$

$$\mathrm{I}[\rho](t) = \int_0^L \rho(x) F(w(t,x))dx$$

$$w(t,x) = g\left(\frac{x}{2\sqrt{Dt}}\right)$$

$$g(z) = 1 - \frac{2}{\sqrt{\pi}} \int_0^z e^{-\tau^2} d\tau \text{ (complementary error function).}$$

$$F(x) = \frac{x^n}{x^n + (k_{\frac{1}{2}})^n}, \quad n > 1$$

Linear inverse problem

D. A. French et al. propose a mathematical model

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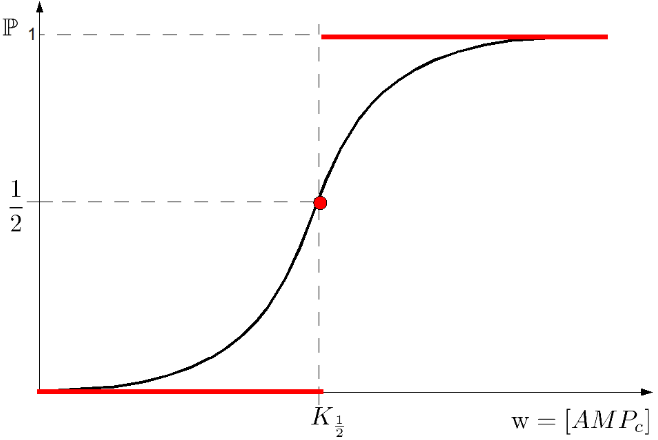
The opening probability F is approximated by

$$F(x) \simeq H(x - K_{\frac{1}{2}}),$$

where

$$H(x) = \begin{cases} 0 & s < 0 \\ \frac{1}{2} & s = 0 \\ 1 & s > 0 \end{cases}$$

Heaviside function



$\mathbb{P}$
1
$\frac{1}{2}$
$K_{\frac{1}{2}}$
$\mathrm{w} = [AMP_c]$

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Heaviside function

Result

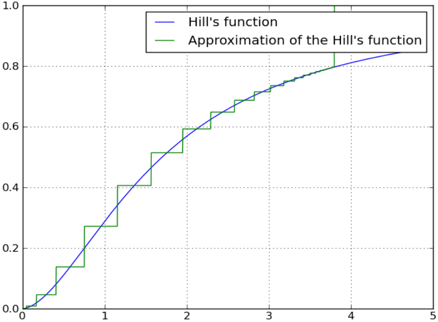
$$\mathrm{I}[\rho](t) = \int_0^{\beta\sqrt{t}} \rho(x)dx$$

$$\rho(y) = \frac{2\mathrm{I}'((y/\beta)^2)y}{\beta^2}$$

F is approximated by

$$F(x) \simeq F_m(x) = F(w_0) \sum_{j=1}^{m} a_j H(x - \alpha_j), \quad \forall x \in [0, w_0]$$

$a_j, \alpha_j > 0$ and $\sum_{j=1}^{m} a_j = 1$, (we can think that α_j are increasing),



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$$\mathrm{I}_m[\rho](t) = \int_0^L \rho(x) F_m(w(t,x))dx$$

A estimate of the linear inverse problem

C. Conca, R. L., J. H. Ortega, and L. Rosier.

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Theorem, stability for operator I_m

Let $\rho : [0, L] \to \mathbb{R}$ be a function in $L^2(0, L)$. Then, for all $\gamma > \gamma_0$, there exists a positive constant $C > 0$ such that

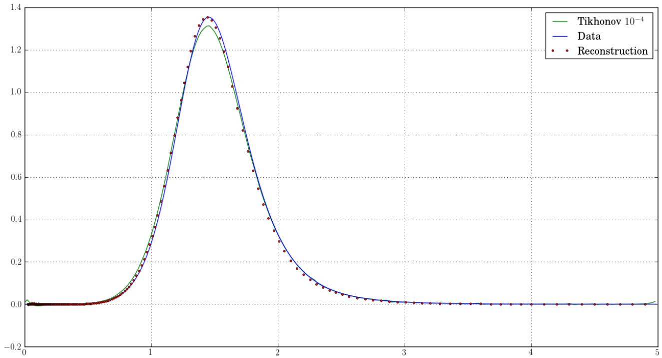
$$\|\rho\|_{-1,\gamma+2} \leq C \|\mathrm{I}_m[\rho]\|_{1,\frac{\gamma}{2}+\frac{1}{4}}.$$

Where $\|f\|_{l,k} = \|x^k f(x)\|_{H^l}$

Numerical reconstruction

our recovery method

$$\rho(x) = 2k^2 \frac{x}{x^2+k^2}$$



F is approximated by Taylor polynomial expansion of degree m

$$F(x) \simeq PF_m(x) = \sum_{j=1}^{m} a_j (x - w_0)^j, \quad \forall x \in [0, w_0]$$

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Theorem, Identificability for operator $P\mathrm{I}_m$

Let $m \leq 8$, and $\rho : [0, L] \to \mathbb{R}$ be a function in $L^2(0, L)$, such that

$$P\mathrm{I}_m[\rho] = 0, \quad \forall t > 0.$$

Then $\rho \equiv 0$.

Polynomial approximation of Hill's function
C. Conca, R. L., J. H. Ortega, and L. Rosier.

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Use Mellin transform
T. Bourgeron, C. Conca, R. L.

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$a_j, \alpha_j > 0$ and $\sum_{j=1}^{m} a_j = 1$, (we can think that α_j are increasing),

Theorem

There exists a constant q_0, such that, for $q < -2q_0 + 1$ there exist constants $C, C' > 0$ such that

$$C' \|(\mathrm{I}_m[\rho])'\|_{L^2_{\frac{q+1}{2}}} \leq \|\rho\|_{L^2_q} \leq C\|(\mathrm{I}_m[\rho])'\|_{L^2_{\frac{q+1}{2}}}.$$

The inequality in the right-hand side is true for $q < 1$.

Proposition

Let $q < 1$.

- For any $k \in \mathbb{N}$, there exists $C \geq 0$ such that:

$$\|\rho\|_{L^2_q} \leq C \|(\mathrm{I}[\rho])^{(k)}\|_{L^2_{2k+\frac{q-3}{2}}}.$$

- Furthermore assume that $\rho \in \mathcal{E}$. Then there is no $k \in \mathbb{N}$, $C \in \mathbb{R}$ such that:

$$\|\rho\|_{L^2_q} \geq C \|(\mathrm{I}[\rho])^{(k)}\|_{L^2_{2k+\frac{q-3}{2}}}.$$

Mellin transform

Definition of Mellin transform

Let f in L^1_q. The Mellin transform of f is a complex valued function defined on the vertical strip $q^* + i\mathbb{R}$, where $q^* = q + 1$

$$\mathbf{M}f(s) = \int_0^{+\infty} x^{s-1} f(x) dx, \quad \forall s = q^* + it, \quad t \in \mathbb{R}$$

Riemann-Lebesgue

The Mellin transform is a linear continuous map of L^1_q into $C_0(q^* + i\mathbb{R}) \subset L^\infty(q^* + i\mathbb{R})$, where $q^* = q + 1$, its norm is 1.

where $\|f\|_{L^p_q} = \left(\int_{\mathbb{R}} |f(x)|^p |x|^q dx\right)^{1/p}$

Mellin transform

Inversion theorem

If f is in L^1_q and if $\|Mf\|_{L^1(q^*+i\mathbb{R})}$ is finite, where $q^* = q+1$, then almost everywhere $x > 0$ we have:

$$f = M_{q^*}^{-1}(Mf),$$

where

$$M_q^{-1}\phi(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \phi(q+it)x^{-q-it}dt.$$

Definition

For two functions f, g we define the multiplicative convolution $f * g$ as:

$$(f * g)(x) = \int_0^{+\infty} f(y)g\left(\frac{x}{y}\right)\frac{dy}{y}.$$

Mellin transform

Proposition

$$M(f * g)(s) = Mf(s)Mg(s),$$

whenever this expression is well defined.

Proposition

For a function f in $L^1_{q-1} \cap L^2_{2q-1}$ we have:

$$\|f\|_{L^2_{2q-1}} = (2\pi)^{-1/2}\|Mf\|_{L^2(q+i\mathbb{R})}.$$

Theorem

According to the previous formula the Mellin transform can be extended, in a unique manner, to an isometry (up to the multiplicative constant $(2\pi)^{-1/2}$) of L^2_q onto $L^2(q^* + i\mathbb{R})$, where $q^* = \frac{q+1}{2}$.

$$\mathrm{I}[\rho](t) = \int_0^L \rho(x) F(w(t,x))dx = \int_0^L \rho(x) F\left(g\left(\frac{x}{2\sqrt{Dt}} \right) \right) dx$$

$g(z) = 1 - \frac{2}{\sqrt{\pi}} \int_0^z e^{-\tau^2} d\tau$ (complementary error function).

$$F(x) = \frac{x^n}{x^n + (k_{\frac{1}{2}})^n}, \quad n > 1$$

Relation with Mellin Transform

$$\mathrm{I}[\rho](t^2) = \int_0^L \rho(x) F \circ g \left(\frac{x}{2t\sqrt{D}} \right) dx = \int_0^L x\rho(x) G \left(\frac{t}{x} \right) \frac{dx}{x}$$

$$\mathrm{I}[\rho](t^2) = \left(x\rho(x) \mathbf{1}_{x \in [0,L]} \right) * G$$

where

$$G(z) = F \circ g \left(\frac{1}{2z\sqrt{D}} \right)$$

Sketch of proof

Transform of F and g

$$g(x) = \frac{2}{\sqrt{\pi}} \int_x^{+\infty} \exp(-t^2)dt \quad \Rightarrow Mg(s) = \frac{1}{\sqrt{\pi}s}\Gamma\left(\frac{s+1}{2}\right).$$

$$F(x) = \frac{x^n}{x^n + K_{\frac{1}{2}}^n} \quad \Rightarrow MF(s) = -\frac{\pi\left(K_{\frac{1}{2}}\right)^s}{n\sin\left(\frac{\pi s}{n}\right)}.$$

We recall the Faà di Bruno formula

$$(f \circ g)^{(k)} = \sum_{\substack{(m_1,\ldots,m_k)\in\mathbb{N}^k \\ \sum_{j=1}^k jm_j = k}} \frac{k!}{\prod_{j=1}^k m_j!\, j!^{m_j}}\, f^{\left(\sum_{j=1}^k m_j\right)} \circ g \prod_{j=1}^k \left(g^{(j)}\right)^{m_j}$$

Thank you for your attention!