INVERSE VISCOSITY BOUNDARY VALUE PROBLEM FOR THE STOKES EVOLUTIONARY EQUATION

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Contenido

Isakov Result

Consider the parabolic problem

$$\begin{cases} \partial_t u - \text{div}\,(a\nabla u) + cu &= 0 \quad , \quad \text{in } \Omega \times (0,T), \\ u &= g_0 \quad , \quad \text{on } \partial\Omega \times (0,T), \\ u(x,0) &= 0 \quad , \quad \text{in } \Omega, \end{cases}$$

We define the Dirichlet-to-Neumann map as

$$\Gamma_l(g_0) = \frac{\partial u}{\partial n}, \qquad \text{on } \partial\Omega$$

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- The proof is based in make use of stabilization of solutions of parabolic problem when $t \to \infty$, reducing the inverse parabolic problem to inverse elliptic problem with parameter.

Stokes Equations

$$(1) \qquad \begin{cases} u_t - \mathrm{div}\ (\sigma_\mu(u,p)) &= & 0 &, & \text{in } \Omega \times (0,T), \\ \mathrm{div}\ u &= & 0 &, & \text{in } \Omega \times (0,T), \\ u(x,0) &= & u_0(x) &, & \text{in } \Omega, \end{cases}$$

where $u = (u_1, u_2, u_3)$ is the velocity vector field and p is the pressure and

$$\sigma_\mu(u,p) = 2\mu e(u) - pI_3$$

is the stress tensor, where $e(u) = ((\nabla u) + (\nabla u)^T)/2$, and $\mu > 0$ is the viscosity function.

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Identifiability of μ by overdetermined data.

We note that when μ is constant, the problem (1) can be reduced to the following familiar form

$$(2) \qquad \begin{cases} u_t - \mu\Delta u + \nabla p = 0 & , \quad \text{in } \Omega \times (0,T), \\ \text{div } u = 0 & , \quad \text{in } \Omega \times (0,T), \\ u(x,0) = u_0(x) & , \quad \text{in } \Omega, \end{cases}$$

If u is independent of time, then problem (1) can be formulated as

$$(3) \qquad \begin{cases} -\text{div } (\sigma_\mu(u,p)) = 0 & , \quad \text{in } \Omega, \\ \text{div } u = 0 & , \quad \text{in } \Omega \end{cases}$$

- Heck, Li, and Wang (2007): Identifiability for system (3) (Stationary Stokes Equation.)

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- Isakov (1999) : Some inverse problem for the diffusion equation.

Contenido

Mathematical Setup

Let $\Omega \subset \mathbb{R}^3$ be an open bounded connected domain with boundary $\partial\Omega \in C^2$.

Consider the following boundary value problem

$$
(4) \qquad \begin{cases} u_t - \operatorname{div}\,(\sigma_\mu(u,p)) &= 0 \quad , \quad \text{in } \Omega \times (0,T), \\ \operatorname{div}\, u &= 0 \quad , \quad \text{in } \Omega \times (0,T), \\ u &= g \quad , \quad \text{on } \partial\Omega \times (0,T), \\ u(x,0) &= u_0(x) \quad , \quad \text{in } \Omega, \end{cases}
$$

where g satisfies the compatibility condition

$$
\int_{\partial\Omega} g \cdot n ds = 0,
$$

where n is the unit outer normal of $\partial\Omega$,

- Assume that the solution of (4) exists and the trace

$$\sigma_\mu(u,p)\cdot n|_{\partial\Omega}$$

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- Define the set of Cauchy data for (4)

$$S_\mu = \{(u|_{\partial\Omega}, \sigma_\mu(u,p)\cdot n|_{\partial\Omega}) \ : \ (u,p) \text{ solution to } (4)\}.$$

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Theorem 1

Assume that μ_1 and μ_2 are two viscosity functions satisfying $\mu_1, \mu_2 \in C^k(\overline{\Omega})$ for $k \geq 8$ and

$$\mu_i \geq 1, \ \forall i = 1, 2. \tag{5}$$

$$\mu_1(x) = \mu_2(x), \ \forall x \in \partial\Omega. \tag{6}$$

Let S_{μ_1} and S_{μ_2} be the Cauchy data associated with μ_1 and μ_2, respectively. If $S_{\mu_1} = S_{\mu_2}$, then $\mu_1 = \mu_2$.

Contenido

Sketch Proof

- Stabilization of solutions of following Stokes problem

$$(7)\quad \begin{cases} u_t - \text{div}\,(\sigma_\mu(u,p)) + u &= 0 \quad , \quad \text{en } \Omega \times (0,T), \\ \text{div}\, u &= 0 \quad , \quad \text{en } \Omega \times (0,T), \\ u &= g \quad , \quad \text{sobre } \partial\Omega \times (0,T), \\ u(x,0) &= u_0(x) \quad , \quad \text{en } \Omega. \end{cases}$$

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- Identifiability result for the stationary Stokes problem

$$\begin{cases} -\text{div}\,(\sigma_\mu(u_\infty,p_\infty)) + u_\infty &= 0 \quad , \quad \text{en } \Omega, \\ \text{div}\, u_\infty &= 0 \quad , \quad \text{en } \Omega, \\ u &= g(T^*) \quad , \quad \text{sobre } \partial\Omega. \end{cases}$$

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- Conclusion of result.

Asymptotic Stability Evolutionary Stokes Solution

Consider the following evolutionary Stokes system

$$
(8) \qquad \begin{cases} u_t - \text{div}\,(\sigma_\mu(u,p)) + u = 0 & , \quad \text{in } \Omega \times (0,T), \\ \text{div}\, u = 0 & , \quad \text{in } \Omega \times (0,T), \\ u = g & , \quad \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & , \quad \text{in } \Omega. \end{cases}
$$

Then, we can prove the following result for solutions of this systems

Theorem 2

Let (u, p) be a solution of (8). Assume that g do not depend on t, for all $t \in (T^*/2, T)$ for some $T^* > 0$. Then, there exists constants $C_1, C_2 > 0$ such that

$$\|u(t) - u_\infty\|_{H^1(\Omega)} \leq \|u(T^*/2) - u_\infty\|^2_{L^2(\Omega)} e^{-2C_1 t} + \|\nabla u(T^*/2) - \nabla u_\infty\|^2_{L^2(\Omega)} e^{\frac{-2C_2 t}{\mu}}, \ \forall t > \frac{T^*}{2},$$

where $(u_\infty, p_\infty) \in H^2(\Omega) \times H^1(\Omega)$ is a solution to the (stationary) Stokes problem

$$(9) \qquad \begin{cases} -\mathrm{div}\ (\sigma_\mu(u_\infty, p_\infty)) + u_\infty = 0 & , \quad \text{in } \Omega, \\ \mathrm{div}\ u_\infty = 0 & , \quad \text{in } \Omega, \\ u_\infty = g(T^*) & , \quad \text{on } \partial\Omega. \end{cases}$$

Identifiability of Viscosity: Stationary Case

Let (u, p) be the solution to the stationary Stokes problem

$$
(10) \qquad \left\{ \begin{array}{rcll} -\mathrm{div}\,(\sigma_\mu(v_\infty, q_\infty)) + v_\infty & = & 0 & , \quad \text{in } \Omega, \\ \mathrm{div}\, v_\infty & = & 0 & , \quad \text{in } \Omega, \\ v_\infty & = & g^0 & , \quad \text{on } \partial\Omega. \end{array} \right.
$$

Theorem 3

Let (u, p) be the solution to the stationary Stokes problem (10). Assume that $\mu_1(x)$ and $\mu_2(x)$ are two viscosity function satisfying

$$\mu_1, \mu_2 \in C^k(\overline{\Omega}),\ \forall k \geq 8.$$
$$\mu_i \geq 1,\ \forall i = 1, 2.$$
$$\mu_1(x) = \mu_2(x),\ \forall x \in \partial\Omega.$$

Let $S^E_{\mu_1}$ and $S^E_{\mu_2}$ be the Cauchy data associated with μ_1 and μ_2, respectively. If $S^E_{\mu_1} = S^E_{\mu_2}$, then $\mu_1 = \mu_2$.

Proof Theorem 1

- Consider the substitution $u = ve^{\lambda t}$, $p = qe^{\lambda t}$, with $\lambda > 0$. Then

$$(11) \qquad \left\{ \begin{array}{rcccl} v_t - \text{div}\,(\sigma_\mu(v,q)) + \lambda v & = & 0 & , & \text{en } \Omega \times (0,T), \\ \text{div}\ v & = & 0 & , & \text{en } \Omega \times (0,T). \end{array} \right.$$

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- Let $g = g^0\phi$, where g^0 any function in $C^2(\overline{\Omega})$ and $\phi(t) \in C^\infty(\mathbb{R})$ satisfies the conditions $\phi(t) = 0$ on $(\infty, T^*/4)$ and $\phi(t) = e^{\lambda t}$ on $(T^*/2, \infty)$.

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- The boundary data of the equation (11) is independent of $t > T^*/2$.

- We can define the Cauchy data, S_μ^E, associated with the stationary problem (10), using Theorem 2.

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- Then $S_{\mu_1} = S_{\mu_2}$ then $S_{\mu_1}^{E} = S_{\mu_2}^{E}$.
- Theorem 3 implies that $\mu_1 = \mu_2$.

Contenido

References

- G. Eskin and J. Ralston. *On the inverse boundary value problem for linear isotropic elasticity*, Inverse Problems, **18** (2002), 907-921.
- H. Heck, R. Li, and J.N. Wang, *Identification of Viscosity in an Incompressible Fluid*, Indiana Univ. Math. J. 56 (2007), no. 5, 2489?2510.
- V. Isakov, *Some inverse problems for the diffusion equation*, Inverse Problems **15** (1999) 3-10.
- R. Lai, G.Uhlmann, and J-N. Wang, *Inverse Boundary Value Problem for the Stokes and the Navier-Stokes equations in the plane*, arXiv febrero 2014.

Thank you for your attention