INVERSE VISCOSITY BOUNDARY VALUE PROBLEM FOR THE STOKES EVOLUTIONARY EQUATION

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Contenido

1 Introduction

2 Main Result

3 Proof Main Result

4 References
Consider the parabolic problem

\[
\begin{aligned}
\frac{\partial u}{\partial t} - \text{div} (a \nabla u) + cu &= 0, \quad \text{in } \Omega \times (0, T), \\
u &= g_0, \quad \text{on } \partial \Omega \times (0, T), \\
u(x, 0) &= 0, \quad \text{in } \Omega,
\end{aligned}
\]

We define the Dirichlet-to-Neumann map as

\[
\Gamma_I(g_0) = \frac{\partial u}{\partial n}, \quad \text{on } \partial \Omega
\]
Isakov Result

Theorem 9.4.1, Isakov [3]
Let $a$ be a scalar matrix. Then the lateral Dirichlet-to-Neumann map $\Gamma_1$ determines $a$ and $b$. 
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Let $a$ be a scalar matrix. Then the lateral Dirichlet-to-Neumann map $\Gamma_l$ determines $a$ and $b$.

- The proof is based in make use of stabilization of solutions of parabolic problem when $t \to \infty$, reducing the inverse parabolic problem to inverse elliptic problem with parameter.
Stokes Equations

\[
\begin{cases}
    u_t - \text{div} \left( \sigma_\mu(u, p) \right) = 0, & \text{in } \Omega \times (0, T), \\
    \text{div } u = 0, & \text{in } \Omega \times (0, T), \\
    u(x, 0) = u_0(x), & \text{in } \Omega,
\end{cases}
\]

where \( u = (u_1, u_2, u_3) \) is the velocity vector field and \( p \) is the pressure and

\[
\sigma_\mu(u, p) = 2\mu e(u) - pl_3
\]

is the stress tensor, where \( e(u) = \frac{1}{2} (\nabla u + (\nabla u)^T) \) and \( \mu > 0 \) is the viscosity function.
We are interested in the following inverse problem:
We are interested in the following inverse problem:

Identifiability of $\mu$ by overdetermined data.
We note that when $\mu$ is constant, the problem (1) can be reduced to the following familiar form

\[
\begin{aligned}
\left\{ 
    & u_t - \mu \Delta u + \nabla p = 0, & \text{in } \Omega \times (0, T), \\
    & \text{div } u = 0, & \text{in } \Omega \times (0, T), \\
    & u(x, 0) = u_0(x), & \text{in } \Omega,
\end{aligned}
\]

(2)

If $u$ is independent of time, then problem (1) can be formulated as

\[
\begin{aligned}
\left\{ 
    & -\text{div } (\sigma_\mu(u, p)) = 0, & \text{in } \Omega, \\
    & \text{div } u = 0, & \text{in } \Omega
\end{aligned}
\]

(3)
Heck, Li, and Wang (2007): **Identifiability** for system (3) (Stationary Stokes Equation.)
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Contenido

1. Introduction
2. Main Result
3. Proof Main Result
4. References
Mathematical Setup

Let $\Omega \subset \mathbb{R}^3$ be an open bounded connected domain with boundary $\partial \Omega \in C^2$.

Consider the following boundary value problem

$$
\begin{aligned}
\begin{cases}
  u_t - \text{div} \left( \sigma_\mu(u, p) \right) &= 0, & \text{in } \Omega \times (0, T), \\
  \text{div } u &= 0, & \text{in } \Omega \times (0, T), \\
  u &= g, & \text{on } \partial \Omega \times (0, T), \\
  u(x, 0) &= u_0(x), & \text{in } \Omega,
\end{cases}
\end{aligned}
$$

(4)

where $g$ satisfies the compatibility condition

$$
\int_{\partial \Omega} g \cdot n ds = 0,
$$

where $n$ is the unit outer normal of $\partial \Omega$, 

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Assume that the solution of (4) exists and the trace

\[ \sigma_\mu(u, p) \cdot n|_{\partial \Omega} \]

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\[ \sigma_\mu(u, p) \cdot n_{|\partial\Omega} \]
is well defined.

Physically, \( \sigma(u, p) \cdot n_{|\partial\Omega} \) is the Cauchy forces acting on the boundary \( \partial\Omega \).
Assume that the solution of (4) exists and the trace
\[ \sigma_\mu(u, p) \cdot n|_{\partial \Omega} \]
is well defined.

Physically, \( \sigma(u, p) \cdot n|_{\partial \Omega} \) is the Cauchy forces acting on the boundary \( \partial \Omega \).

Define the set of Cauchy data for (4)
\[ S_\mu = \{(u|_{\partial \Omega}, \sigma_\mu(u, p) \cdot n|_{\partial \Omega}) : (u, p) \text{ solution to (4)}\}. \]
Inverse Problem: Determine $\mu$ from $S_{\mu}$. 

Theorem 1
Assume that $\mu_1$ and $\mu_2$ are two viscosity functions satisfying $\mu_1, \mu_2 \in C^k(\Omega)$ for $k \geq 8$ and $\mu_i \geq 1$, $\forall i = 1, 2$. (5)

$\mu_1(x) = \mu_2(x), \forall x \in \partial \Omega$. (6)

Let $S_{\mu_1}$ and $S_{\mu_2}$ be the Cauchy data associated with $\mu_1$ and $\mu_2$, respectively. If $S_{\mu_1} = S_{\mu_2}$, then $\mu_1 = \mu_2$. 

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Inverse Problem: Determine $\mu$ from $S_\mu$.

Identifiability: if $S_{\mu_1} = S_{\mu_2}$ then $\mu_1 = \mu_2$?
Inverse Problem : Determine $\mu$ from $S_\mu$.

Identifiablity : if $S_{\mu_1} = S_{\mu_2}$ then $\mu_1 = \mu_2$?

**Theorem 1**

Assume that $\mu_1$ and $\mu_2$ are two viscosity functions satisfying $\mu_1, \mu_2 \in C^k(\bar{\Omega})$ for $k \geq 8$ and

\begin{align*}
\mu_i & \geq 1, \forall i = 1, 2. \\
\mu_1(x) & = \mu_2(x), \forall x \in \partial\Omega.
\end{align*}

Let $S_{\mu_1}$ and $S_{\mu_2}$ be the Cauchy data associated with $\mu_1$ and $\mu_2$, respectively. If $S_{\mu_1} = S_{\mu_2}$, then $\mu_1 = \mu_2$. 

Contenido

1. Introduction
2. Main Result
3. Proof Main Result
4. References
Sketch Proof

- Stabilization of solutions of following Stokes problem

\[
\begin{align*}
\left\{ \begin{array}{ll}
    u_t - \text{div} \left( \sigma_{\mu}(u, p) \right) + u &= 0 , & \text{en } \Omega \times (0, T), \\
    \text{div } u &= 0 , & \text{en } \Omega \times (0, T), \\
    u &= g , & \text{sobre } \partial \Omega \times (0, T), \\
    u(x, 0) &= u_0(x) , & \text{en } \Omega.
\end{array} \right.
\end{align*}
\]
Sketch Proof

- Stabilization of solutions of following Stokes problem

\[
\begin{aligned}
 u_t - \text{div} \left( \sigma_{\mu}(u, p) \right) + u &= 0 \quad , \quad \text{en } \Omega \times (0, T), \\
 \text{div} u &= 0 \quad , \quad \text{en } \Omega \times (0, T), \\
 u &= g \quad , \quad \text{sobre } \partial \Omega \times (0, T), \\
 u(x, 0) &= u_0(x) \quad , \quad \text{en } \Omega.
\end{aligned}
\]  

- Identifiability result for the stationary Stokes problem

\[
\begin{aligned}
 -\text{div} \left( \sigma_{\mu}(u_\infty, p_\infty) \right) + u_\infty &= 0 \quad , \quad \text{en } \Omega, \\
 \text{div} u_\infty &= 0 \quad , \quad \text{en } \Omega, \\
 u &= g(T^*) \quad , \quad \text{sobre } \partial \Omega.
\end{aligned}
\]
Sketch Proof

▶ Stabilization of solutions of following Stokes problem

\[
\begin{cases}
    u_t - \text{div} \ (\sigma_\mu(u, p)) + u = 0, & \text{en } \Omega \times (0, T), \\
    \text{div} \ u = 0, & \text{en } \Omega \times (0, T), \\
    u = g, & \text{sobre } \partial \Omega \times (0, T), \\
    u(x, 0) = u_0(x), & \text{en } \Omega.
\end{cases}
\] (7)

▶ Identifiability result for the stationary Stokes problem

\[
\begin{cases}
    -\text{div} \ (\sigma_\mu(u_\infty, p_\infty)) + u_\infty = 0, & \text{en } \Omega, \\
    \text{div} \ u_\infty = 0, & \text{en } \Omega, \\
    u = g(T^*), & \text{sobre } \partial \Omega.
\end{cases}
\]

▶ Conclusion of result.
Consider the following evolutionary Stokes system

\[
\begin{aligned}
  u_t - \text{div} \left( \sigma_\mu(u, p) \right) + u &= 0, \quad \text{in } \Omega \times (0, T), \\
  \text{div } u &= 0, \quad \text{in } \Omega \times (0, T), \\
  u &= g, \quad \text{on } \partial \Omega \times (0, T), \\
  u(x, 0) &= u_0(x), \quad \text{in } \Omega.
\end{aligned}
\]

(8)

Then, we can prove the following result for solutions of this systems
Theorem 2

Let \((u, p)\) be a solution of (8). Assume that \(g\) do not depend on \(t\), for all \(t \in (T^*/2, T)\) for some \(T^* > 0\). Then, there exists constants \(C_1, C_2 > 0\) such that

\[
\|u(t) - u_\infty\|_{H^1(\Omega)} \leq \|u(T^*/2) - u_\infty\|^2_{L^2(\Omega)} e^{-2C_1 t} + \\
\|\nabla u(T^*/2) - \nabla u_\infty\|^2_{L^2(\Omega)} e^{-2C_2 t} \mu, \quad \forall t > \frac{T^*}{2},
\]

where \((u_\infty, p_\infty) \in H^2(\Omega) \times H^1(\Omega)\) is a solution to the (stationary) Stokes problem

\[
\begin{aligned}
- \text{div} \left( \sigma_\mu(u_\infty, p_\infty) \right) + u_\infty & = 0, \quad \text{in } \Omega, \\
\text{div } u_\infty & = 0, \quad \text{in } \Omega, \\
u_\infty & = g(T^*), \quad \text{on } \partial \Omega.
\end{aligned}
\]

(9)
Let \((u, p)\) be the solution to the stationary Stokes problem

\[
\begin{align*}
-\text{div} \left( \sigma_{\mu}(v_\infty, q_\infty) \right) + v_\infty &= 0, \quad \text{in } \Omega, \\
\text{div } v_\infty &= 0, \quad \text{in } \Omega, \\
v_\infty &= g^0, \quad \text{on } \partial\Omega.
\end{align*}
\]
**Theorem 3**

Let \((u, p)\) be the solution to the stationary Stokes problem (10). Assume that \(\mu_1(x)\) and \(\mu_2(x)\) are two viscosity function satisfying

\[
\mu_1, \mu_2 \in C^k(\bar{\Omega}), \quad \forall k \geq 8.
\]

\[
\mu_i \geq 1, \quad \forall i = 1, 2.
\]

\[
\mu_1(x) = \mu_2(x), \quad \forall x \in \partial \Omega.
\]

Let \(S^E_{\mu_1}\) and \(S^E_{\mu_2}\) be the Cauchy data associated with \(\mu_1\) and \(\mu_2\), respectively. If \(S^E_{\mu_1} = S^E_{\mu_2}\), then \(\mu_1 = \mu_2\).
Proof Theorem 1

Consider the substitution $u = ve^{\lambda t}$, $p = qe^{\lambda t}$, with $\lambda > 0$. Then

$$\begin{aligned}
\begin{cases}
\nu_t - \text{div} \left( \sigma_{\mu}(\nu, q) \right) + \lambda \nu &= 0, & \text{en } \Omega \times (0, T), \\
\text{div } \nu &= 0, & \text{en } \Omega \times (0, T).
\end{cases}
\end{aligned}$$

(11)
Proof Theorem 1

Consider the substitution $u = ve^{\lambda t}$, $p = qe^{\lambda t}$, with $\lambda > 0$. Then

$$
\begin{aligned}
& v_t - \text{div} \left( \sigma_{\mu}(v, q) \right) + \lambda v = 0, \quad \text{en } \Omega \times (0, T), \\
& \text{div } v = 0, \quad \text{en } \Omega \times (0, T).
\end{aligned}
$$

Let $g = g^0 \phi$, where $g^0$ any function in $C^2(\overline{\Omega})$ and $\phi(t) \in C^\infty(\mathbb{R})$ satisfies

the conditions $\phi(t) = 0$ on $(\infty, T^*/4)$ and $\phi(t) = e^{\lambda t}$ on $(T^*/2, \infty)$. 

Proof Theorem 1

Consider the substitution $u = ve^{\lambda t}$, $p = qe^{\lambda t}$, with $\lambda > 0$. Then

\[
\begin{aligned}
\nu_t - \text{div} (\sigma_{\mu}(\nu, q)) + \lambda \nu &= 0, & \text{en } \Omega \times (0, T), \\
\text{div } \nu &= 0, & \text{en } \Omega \times (0, T).
\end{aligned}
\]

Let $g = g^0 \phi$, where $g^0$ any function in $C^2(\Omega)$ and $\phi(t) \in C^\infty(\mathbb{R})$ satisfies the conditions $\phi(t) = 0$ on $(\infty, T^*/4)$ and $\phi(t) = e^{\lambda t}$ on $(T^*/2, \infty)$.

The boundary data of the equation (11) is independent of $t > T^*/2$. 
We can define the Cauchy data, $S^E_\mu$, associated with the stationary problem (10), using Theorem 2.
We can define the Cauchy data, $S^E_\mu$, associated with the stationary problem (10), using Theorem 2.

Then $S_{\mu_1} = S_{\mu_2}$ then $S^E_{\mu_1} = S^E_{\mu_2}.$
Proof Main Result

We can define the Cauchy data, $S^E_\mu$, associated with the stationary problem (10), using Theorem 2.

Then $S_{\mu_1} = S_{\mu_2}$ then $S^E_{\mu_1} = S^E_{\mu_2}$.

Theorem 3 implies that $\mu_1 = \mu_2$. 
Contenido

1. Introduction
2. Main Result
3. Proof Main Result
4. References


Thank you for your attention