Linear stability of relative equilibria in n-body-type problems

Riccardo D. Jadanza

Department of Mathematical Sciences
Politecnico di Torino

BCAM (Bilbao)
10 December 2013

Joint work with V. Barutello and A. Portaluri

The n-body problem and its generalizations

Consider $n \geqslant 3$ point particles in $\mathbb{R}^2$ with positive masses $m_1, \ldots, m_n$ and coordinates $q_1(t), \ldots, q_n(t)$ subject to the gravitational potential

$$U(q) := \sum_{\substack{i,j=1 \\ i<j}}^{n} \frac{m_i m_j}{|q_i - q_j|},$$

Newton's equation for the i-th particle:

$$m_i \ddot{q}_i = \frac{\partial U}{\partial q_i} = -\sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{m_i m_j (q_i - q_j)}{|q_i - q_j|^3}.$$

The n-body problem and its generalizations

Consider $n \geqslant 3$ point particles in $\mathbb{R}^2$ with positive masses $m_1, \ldots, m_n$ and coordinates $q_1(t), \ldots, q_n(t)$ subject to the α-homogeneous potential

$$U_\alpha(q) := \sum_{\substack{i,j=1 \\ i<j}}^{n} \frac{m_i m_j}{|q_i - q_j|^\alpha}, \qquad \alpha \in (0,2).$$

Newton's equation for the i-th particle:

$$m_i \ddot{q}_i = \frac{\partial U_\alpha}{\partial q_i} = -\alpha \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{m_i m_j (q_i - q_j)}{|q_i - q_j|^{\alpha+2}}.$$

Gravitational case: $\alpha = 1$.

The n-body problem and its generalizations

Consider $n \geqslant 3$ point particles in $\mathbb{R}^2$ with positive masses $m_1, \ldots, m_n$ and coordinates $q_1(t), \ldots, q_n(t)$ subject to the logarithmic potential

$$U_{\log}(q) := \sum_{\substack{i,j=1 \\ i<j}}^{n} m_i m_j \log \frac{1}{|q_i - q_j|}.$$

Newton's equation for the i-th particle:

$$m_i \ddot{q}_i = \frac{\partial U_{\log}}{\partial q_i} = -\sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{m_i m_j (q_i - q_j)}{|q_i - q_j|^2}.$$

$$\frac{U_\alpha(q) - 1}{\alpha} \sim U_{\log}(q) \text{ as } \alpha \to 0^+.$$

Configuration space

We are interested in *periodic solutions* $\Rightarrow$ no collisions!

- Collision space:

$$\Delta := \bigcup_{i,j=1}^{n} \left\{ q \in \mathbb{R}^{2n} \;\middle|\; q_i = q_j \text{ for some } i \neq j \right\}$$

- Configuration space: $X := \mathbb{R}^{2n} \setminus \Delta$
- Reduced configuration space: $\widehat{X} := \left\{ q \in X \;\middle|\; \sum_{i=1}^{n} m_i q_i = 0 \right\}$
- Newtonian system on X (or $\widehat{X}$):

$$M\ddot{q} = \nabla U(q),$$

with $M := \operatorname{diag}(m_1 I_2, \ldots, m_n I_2)$ and $\nabla := \left(\frac{\partial}{\partial q_1}, \ldots, \frac{\partial}{\partial q_n} \right)^{\mathsf{T}}$.

Central configurations

Definition

A *central configuration (CC)* is a point $\bar{q} \in \widehat{X}$ such that, for some positive smooth function $r : (a, b) \subseteq \mathbb{R} \to \mathbb{R}$,

$$q(t) := r(t)\,\bar{q}$$

is a solution of Newton's equations (namely a *homographic solution*).

Central configurations equation:

$$M^{-1}\nabla U(\bar{q}) + \lambda\bar{q} = 0,$$

where

$$\lambda_\alpha := \frac{\alpha U_\alpha(\bar{q})}{\mathcal{I}(\bar{q})}, \qquad \lambda_{\log} := \frac{1}{\mathcal{I}(\bar{q})} \sum_{\substack{i,j=1 \\ i<j}}^{n} m_i m_j =: \frac{\mathcal{M}}{\mathcal{I}(\bar{q})}$$

Central configurations

Definition

A *central configuration (CC)* is a point $\bar{q} \in \widehat{X}$ such that, for some positive smooth function $r : (a, b) \subseteq \mathbb{R} \to \mathbb{R}$,

$$q(t) := r(t)\,\bar{q}$$

is a solution of Newton's equations (namely a *homographic solution*).

Central configurations equation:

$$M^{-1}\nabla U(\bar{q}) + \lambda\bar{q} = 0,$$

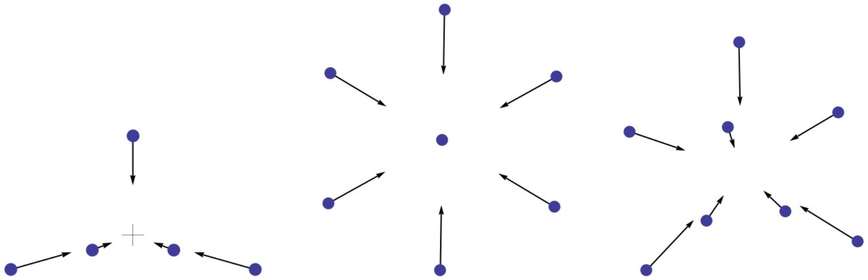
where

$$\lambda_\alpha := \frac{\alpha U_\alpha(\bar{q})}{\mathcal{I}(\bar{q})}, \qquad \lambda_{\log} := \frac{1}{\mathcal{I}(\bar{q})} \sum_{\substack{i,j=1 \\ i<j}}^{n} m_i m_j =: \frac{\mathcal{M}}{\mathcal{I}(\bar{q})}$$

Central configurations

CCs in short

Intuitively, in a CC the position vector lines up with the acceleration vector (with opposite sign) and the proportionality constant is λ for each body.



Central configurations

Two examples of homographic solutions:

Homothetic solution

Lagrange solution

Central configurations

Remark

$$\bar{q} \text{ is a CC } \Rightarrow c\bar{q} \text{ and } R\bar{q} \text{ are CCs,}$$

for any $c \in \mathbb{R} \setminus \{0\}$ and any $2n \times 2n$ block-diagonal matrix R with blocks given by a 2×2 fixed matrix in $\mathrm{SO}(2)$.

We count CCs by:

1. fixing the "scale" $c \Rightarrow$ *ellipsoid of inertia*

$$\mathcal{S} := \left\{ q \in \widehat{X} \;\middle|\; \mathcal{I}(\bar{q}) = 1 \right\}$$

2. identifying all those which are rotationally equivalent

$\Rightarrow$ We take the quotient of $\widehat{X}$ with respect to *homotheties* and *rotations*, i. e. we consider the *shape sphere*

$$\mathbb{S} := \mathcal{S}/\mathrm{SO}(2).$$

Hamiltonian formulation and phase space

Newton's equations can be transformed in the following Hamiltonian system:

$$\begin{cases} \dot{q} = M^{-1} p^{\mathsf{T}} \\ \dot{p}^{\mathsf{T}} = \nabla U(q) \end{cases}$$

defined on the *phase space* $T^*X \cong X \times \mathbb{R}^{2n} \subset \mathbb{R}^{4n}$.

Move to a *uniformly rotating coordinate system*:

$$\begin{cases} x := R(t)\, q \\ y^{\mathsf{T}} := R(t)\, p^{\mathsf{T}} \end{cases}$$

where

$$R(t) := \operatorname{diag}_n(e^{\omega t J_2}, \ldots, e^{\omega t J_2}) \qquad J_{2n} := \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$$

Hamiltonian formulation and phase space

Newton's equations can be transformed in the following Hamiltonian system:

$$\begin{cases} \dot{q} = M^{-1} p^{\mathsf{T}} \\ \dot{p}^{\mathsf{T}} = \nabla U(q) \end{cases}$$

defined on the *phase space* $T^*X \cong X \times \mathbb{R}^{2n} \subset \mathbb{R}^{4n}$.
Move to a *uniformly rotating coordinate system*:

$$\begin{cases} x := R(t)\, q \\ y^{\mathsf{T}} := R(t)\, p^{\mathsf{T}} \end{cases}$$

where

$$R(t) := \operatorname{diag}_n(e^{\omega t J_2}, \ldots, e^{\omega t J_2}) \qquad J_{2n} := \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$$

Relative equilibria

We obtain

$$\begin{cases} \dot{x} = \omega K x + M^{-1} y^{\mathsf{T}} \\ \dot{y}^{\mathsf{T}} = \nabla U(x) + \omega K y^{\mathsf{T}} \end{cases} \tag{1}$$

where $K := \mathrm{diag}_n(J_2, \ldots, J_2)$.

Definition

A *relative equilibrium (RE)* is an equilibrium solution of System (1).

$\Rightarrow$ A relative equilibrium must satisfy

$$\begin{cases} y^{\mathsf{T}} = -\omega M K x \\ M^{-1} \nabla U(x) + \omega^2 x = 0 \end{cases}$$

CCs equation with $\lambda = \omega^2$.

Relative equilibria

Planar three-body problem

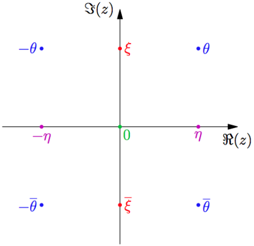
Non-equal masses

Equal masses

Linear and spectral stability

Hamiltonian matrices:

$$\mathfrak{sp}(2n) := \left\{ H \in \mathbb{R}^{2n\times 2n} \;\middle|\; H^{\mathsf{T}} J + JH = 0 \right\}$$

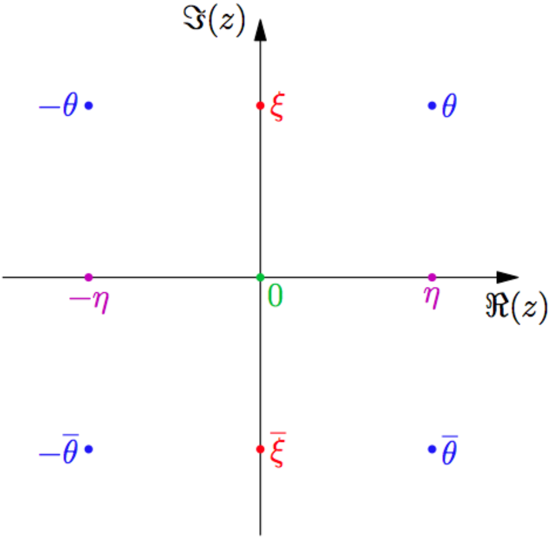


Definition

A Hamiltonian matrix $H \in \mathfrak{sp}(2n)$ is:

- *Spectrally stable* if $\sigma(H) \subset i\mathbb{R}$;
- *Linearly stable* if it is spectrally stable and diagonalisable.

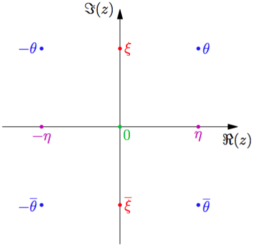
Linear and spectral stability



Linear and spectral stability

Hamiltonian matrices:

$$\mathfrak{sp}(2n) := \left\{ H \in \mathbb{R}^{2n\times 2n} \;\middle|\; H^{\mathsf{T}} J + JH = 0 \right\}$$



Definition

A Hamiltonian matrix $H \in \mathfrak{sp}(2n)$ is:

- *Spectrally stable* if $\sigma(H) \subset i\mathbb{R}$;
- *Linearly stable* if it is spectrally stable and diagonalisable.

Linear and spectral stability

Definition

A *linear autonomous* Hamiltonian system

$$\dot{z} = -JBz,$$

with B a real symmetric matrix, is *spectrally* (resp. *linearly*) *stable* if JB is spectrally (resp. linearly) stable.

Linearising System (1) around a RE and writing $z := (x^{\mathsf{T}}, y)^{\mathsf{T}}$ we have

$$\dot{z} = -JBz, \qquad \text{with } B := \begin{pmatrix} -D^2U(\bar{x}) & -\omega K \\ \omega K & M^{-1} \end{pmatrix}$$

$\Rightarrow$ We study the spectral stability (hence the spectrum) of JB.

Linear and spectral stability

Definition

A *linear autonomous* Hamiltonian system

$$\dot{z} = -JBz,$$

with B a real symmetric matrix, is *spectrally* (resp. *linearly*) *stable* if JB is spectrally (resp. linearly) stable.

Linearising System (1) around a RE and writing $z := (x^{\mathsf{T}}, y)^{\mathsf{T}}$ we have

$$\dot{z} = -JBz, \qquad \text{with } B := \begin{pmatrix} -D^2U(\bar{x}) & -\omega K \\ \omega K & M^{-1} \end{pmatrix}$$

$\Rightarrow$ We study the spectral stability (hence the spectrum) of JB.

A symplectic decomposition

First integrals of the n-body-type problems $\Rightarrow$ invariant symplectic subspaces

$$T^*X = E_1 \oplus E_2 \oplus E_3$$

- E_1: position and linear momentum of the centre of mass, $\dim E_1 = 4$;
- E_2: dilations and rotations, $\dim E_2 = 4$;
- $E_3 := (E_1 \oplus E_2)^{\perp_\Omega}$, $\dim E_3 = 4n - 8$.

$$JB = JB_1 \oplus JB_2 \oplus JB_3$$

A symplectic decomposition

JB_1 and JB_2 always yield the same 8 eigenvalues:

Potential		Eigenvalue	Multiplicity
U_α	JB_1	$i\omega$	2
		$-i\omega$	2
	JB_2	0	2
		$i\omega\sqrt{2-\alpha}$	1
		$-i\omega\sqrt{2-\alpha}$	1
$U_{\log}$	JB_1	$i\omega$	2
		$-i\omega$	2
	JB_2	0	2
		$i\omega\sqrt{2}$	1
		$-i\omega\sqrt{2}$	1

$\Rightarrow$ We focus on JB_3, the essential part of the dynamics.

A symplectic decomposition

Example: the equilateral triangle with equal masses

- *α-homogeneous case:* the eigenvalues on E_3 are

$$\lambda_{9,10} := \pm\frac{1}{2}\sqrt{6\alpha^2 + 12\alpha(i\sqrt{2\alpha} - 1)},$$
$$\lambda_{11,12} := \pm\frac{1}{2}\sqrt{6\alpha^2 - 12\alpha(i\sqrt{2\alpha} + 1)}.$$

  $\Rightarrow$ Spectrally unstable but diagonalisable.

- *Logarithmic case:* the eigenvalues on E_3 are

$$\lambda_{9,10} := \pm i\sqrt{3}, \qquad \lambda_{11,12} := \pm i\sqrt{3}.$$

  $\Rightarrow$ Spectrally stable but not diagonalisable.

In both cases there is linear instability but for opposite reasons.

Main theorem

Let $\bar{x} \in \mathbb{S}$ be a CC.

$$B_3 := \begin{pmatrix} -\mathcal{D} & -\omega K \\ \omega K & I_{2n-4} \end{pmatrix},$$

$$\begin{pmatrix} I & \omega K \\ 0 & I \end{pmatrix} B_3 \begin{pmatrix} I & 0 \\ -\omega K & I \end{pmatrix} = \begin{pmatrix} -\left(\mathcal{D} + \omega^2 I\right) & 0 \\ 0 & I \end{pmatrix} =: N_3.$$

Remark

- $\mathcal{D} = D^2 U(\bar{x})|_{E_3}$
- $\mathcal{D} + \omega^2 I = D^2 U|_{\mathbb{S}}(\bar{x})|_{E_3}$

Definition

- *Nullity:* $\nu(\bar{x}) := \nu\left(\mathcal{D} + \omega^2 I\right) = \nu(B_3)$;
- *Morse index:* $i_{\text{Morse}}(\bar{x}) := i_{\text{Morse}}\left(\mathcal{D} + \omega^2 I\right) = i_{\text{Morse}}(B_3)$.

Main theorem

Theorem

Assume that $\nu(\bar{x})$ is even. If $i_{\text{Morse}}(\bar{x})$ is odd, then the RE associated with $\bar{x}$ is spectrally unstable.

Corollary

If $i_{\text{Morse}}(\bar{x})$ or $\nu(\bar{x})$ are odd, then the RE associated with $\bar{x}$ is linearly unstable.

Remark

These results are independent of the potential! The only requirement is SO(2)-invariance (for RE to exist).

Main theorem

Theorem

Assume that $\nu(\bar{x})$ is even. If $i_{\mathrm{Morse}}(\bar{x})$ is odd, then the RE associated with $\bar{x}$ is spectrally unstable.

Corollary

If $i_{\mathrm{Morse}}(\bar{x})$ or $\nu(\bar{x})$ are odd, then the RE associated with $\bar{x}$ is linearly unstable.

Remark

These results are independent of the potential! The only requirement is $\mathrm{SO}(2)$-invariance (for RE to exist).

A useful inequality

Theorem

Let $\bar{x} \in \mathbb{S}$ be a CC for U_α. If the following inequality holds:

$$\sum_{\substack{i,j=1 \\ i<j}}^{n} \frac{m_i + m_j}{|\bar{x}_i - \bar{x}_j|^{\alpha+2}} > \frac{2n+\alpha-4}{\alpha} \, U_\alpha(\bar{x}) \tag{1}$$

then the associated RE is spectrally unstable.

Remark

- The RE can be degenerate (but not completely).
- There cannot be such a result for $U_{\log}$: as $\alpha \to 0^+$ the solution set of (1) shrinks to $\varnothing$.

A useful inequality

Theorem

Let $\bar{x} \in \mathbb{S}$ be a CC for U_α. If the following inequality holds:

$$\sum_{\substack{i,j=1 \\ i<j}}^{n} \frac{m_i + m_j}{|\bar{x}_i - \bar{x}_j|^{\alpha+2}} > \frac{2n + \alpha - 4}{\alpha} \, U_\alpha(\bar{x}) \tag{1}$$

then the associated RE is spectrally unstable.

Remark

- The RE can be degenerate (but not completely).
- There cannot be such a result for $U_{\log}$: as $\alpha \to 0^+$ the solution set of (1) shrinks to $\varnothing$.

A useful inequality

Application: regular *n*-gon

A rough approximation of the inequality gives the following:
For every $n \geqslant 8$ there exists a real number

$$\bar{\alpha}(n) := \frac{2\pi^2(n^2 - 3n + 2)}{n^3 - \pi^2 n + \pi^2} \in (0, 2)$$

such that for any $\alpha \in \big(\bar{\alpha}(n), 2\big)$ the regular *n*-gon is *spectrally unstable*.
Moreover, $\bar{\alpha}(n)$ monotonically tends to 0 as $n \to +\infty$.

The End

Images and animations from:

- http://www.scholarpedia.org/article/Three_body_problem
- http://www.scholarpedia.org/article/N-body_choreographies
- http://www.d.umn.edu/~mhampton/threebodies123.gif

Spectral flow

Definition

Let $T : [a, b] \subseteq \mathbb{R} \to \mathbb{C}^{4n\times 4n}$ be a path of Hermitian matrices with invertible endpoints. Its *spectral flow* on the interval $[a, b]$ is

$$\mathrm{sf}\,\big(T, [a, b]\big) := n^-\big(T(a)\big) - n^-\big(T(b)\big).$$

Concatenation property: If T and S are paths such that $T(b) = S(a)$, then $\mathrm{sf}\,\big(T * S, [a, b]\big) = \mathrm{sf}\,\big(T, [a, b]\big) + \mathrm{sf}\,\big(S, [a, b]\big)$.

Definition

A *crossing instant* for the path T is a number $t_* \in [a, b]$ for which $\ker T(t_*) \neq \{0\}$. The *crossing form* relative to the crossing t_* is

$$\Gamma(T, t_*) := Q\dot{T}(t_*)Q\big|_{\ker T(t_*)},$$

where Q is the orthogonal projection onto $\ker T(t_*)$. A crossing t_* is called *regular* if $\Gamma(T, t_*)$ is non-degenerate.

Spectral flow

Proposition

Assume that $t_* > 0$ is an isolated (possibly non-regular) crossing instant for the path of Hermitian matrices

$$D(t) := A + itJ, \qquad t \in [0, +\infty),$$
$$= -J(JA - itI)$$

with A real and symmetric. Then, for $\delta > 0$ small enough,

$$\mathrm{sf}\big(D, [t_* - \delta, t_* + \delta]\big) = \mathrm{sgn}\langle iJ\,\cdot, \cdot\rangle\big|_{\mathcal{H}_{t_*}},$$

where $\mathcal{H}_{t_*}$ is the generalised eigenspace associated with the eigenvalue $it_* \in \sigma(JA)$.

Main theorem: sketch of the proof

Assume spectral stability. Consider the affine path of Hermitian matrices in $\mathbb{C}^{(4n-8)\times(4n-8)}$

$$\begin{aligned} D(t) &:= B_3 + itJ, \qquad\qquad t \in [0, +\infty) \\ &= -J(JB_3 - itI) \end{aligned}$$

Then $\ker D(t) = \ker(JB_3 - itI)$, whence

$$t_* \text{ crossing} \iff it_* \in \sigma(JB_3) \cap i[0, +\infty)$$

Remark: Every crossing is *isolated*.

Case 1: $t_* = 0$ is a crossing.

Then $\exists\, T > \varepsilon > 0$ such that 0 is the only crossing in $[0, \varepsilon]$ and $\mathrm{sf}\big(D, [\varepsilon, T_1]\big) = \mathrm{sf}\big(D, [\varepsilon, T_2]\big)\ \forall\, T_1, T_2 \geqslant T$. Therefore

$$\begin{aligned} \mathrm{sf}\big(D, [\varepsilon, T]\big) &\overset{\text{def}}{=} n^-\big(D(\varepsilon)\big) - n^-\big(D(T)\big) \\ &= n^-(B_3) + \frac{\dim \mathcal{H}_0}{2} - 2n \end{aligned}$$

Main theorem: sketch of the proof

Since for all crossings $t_* > 0$

$$\mathrm{sf}\big(D, [t_* - \delta, t_* + \delta]\big) = \mathrm{sgn}\langle iJ\cdot, \cdot\rangle\big|_{\mathcal{H}_{t_*}} \equiv \dim \mathcal{H}_{t*} \qquad \mod 2,$$

by concatenation we have

$$\mathrm{sf}\big(D, [\varepsilon, T]\big) \equiv \sum_{it_* \in \sigma(JA) \cap i(0,+\infty)} \dim \mathcal{H}_{t_*}$$

$$= 2n - \frac{\dim \mathcal{H}_0}{2} \equiv \frac{\dim \mathcal{H}_0}{2} \qquad \mod 2.$$

Hence

$$n^-(B_3) \equiv 0 \qquad \mod 2.$$

Main theorem: sketch of the proof

By Sylvester's Law of Inertia (*):

$$n^-(B_3) \overset{*}{=} n^-(N_3) = n^- \begin{pmatrix} -(\mathcal{D} + \omega^2 I) & 0 \\ 0 & I \end{pmatrix}$$
$$= 2n - i_{\mathrm{Morse}}(\bar{x}) - \underbrace{\nu(\bar{x})}_{\text{even}}.$$

It follows that

$$i_{\mathrm{Morse}}(\bar{x}) \equiv 0 \qquad \mod 2.$$

Case 2: $t_* = 0$ is *not* a crossing.
This means that $D(0) = B_3$ is invertible ($\nu(\bar{x}) = 0$) $\Rightarrow$ do as before but compute the spectral flow on $[0, T]$. □

Proof of the inequality

Theorem

Let $\bar{x} \in \mathbb{S}$ be a CC for U_α (resp. $U_{\log}$) and let it rotate with angular velocity $\omega = \sqrt{\alpha U_\alpha(\bar{x})}$ (resp. $\omega = \sqrt{\mathcal{M}}$). Consider the matrix JB with eigenvalues λ_i $(i = 1, \ldots, 4n)$. Then we have

i) *α-homogeneous case:*

$$\sum_{i=1}^{4n} (\lambda_i)^2 = 2\alpha^2 \sum_{\substack{i,j=1 \\ i<j}}^{n} \frac{m_i + m_j}{|\bar{x}_i - \bar{x}_j|^{\alpha+2}} - 4n\alpha U_\alpha(\bar{x});$$

ii) *Logarithmic case:*

$$\sum_{i=1}^{4n} (\lambda_i)^2 = -4n\mathcal{M}.$$

Proof of the inequality

Assume spectral stability. Therefore

$$\sum_{i=9}^{4n}(\lambda_i)^2 \leqslant 0.$$

Adding to both sides the first eight eigenvalues we obtain

$$\sum_{i=1}^{4n}(\lambda_i)^2 \leqslant \sum_{i=1}^{8}(\lambda_i)^2.$$

Since $\sum_{i=1}^{8}(\lambda_i)^2 = (2\alpha - 8)\omega^2 = 2\alpha(\alpha - 4)U_\alpha(\bar{x})$, by the previous theorem we get

$$2\alpha^2 \sum_{\substack{i,j=1 \\ i<j}}^{n} \frac{m_i + m_j}{|\bar{x}_i - \bar{x}_j|^{\alpha+2}} - 4n\alpha U_\alpha(\bar{x}) \leqslant 2\alpha(\alpha - 4)U_\alpha(\bar{x}).$$

Solving for the summation yields the result.