Perforated and multiperforated plates in linear acoustic

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Motivation

- A combustion chamber of a Turbo engine of Turbo Meca (SAFRAN Group, - Acknowledgment -).

- Goal: Study the effects of small holes on resonance frequencies using Matching of Asymptotic Expansions (MAE).
The physical problem

Why multiperforated plates in turbo engines?

- Temperature of a combustion chamber: \( \sim 2000K \)
- Temperature of the casing: \( \sim 800K \)
- Injection of "fresh" air from the casing to the combustion chamber to protect the boundary from the combustion (cooling film)

Importance of the acoustic resonance frequencies of the gaz

- For high ratio of fuel-air the combustion is unstable. The combustion can easily be perturbed by the acoustic.
- The diameter (\( \sim .5mm \)) and spacing (\( \sim 5mm \)) of the holes are chosen to ensure the cooling and the solidity of the boundary.
- Holes with small diameters have an impact on the acoustic.
The matching of asymptotic expansions

- A technique of asymptotic analysis require a small parameter
- **Very popular** in fluid mechanics due to Van Dyke (75). A method to study the effect of the boundary layers occurring in the fluids
- **Rigorous** framework for elliptic problems Il’in (92) and for Helmholtz problem Joly and Tordeux (06,08)
- Well-suited for writing approximate coupling condition between domains of different dimension
- Leads to numerical scheme **with no mesh refinement** in the neighborhood of the coupling
The matching of asymptotic expansions

2D-1D coupling: field transmitted through a slot:

The small parameter: the width of the Patch antenna.

- The slot is modelled by a 1D propagative medium
- The exterior is a 2D propagative medium
- A 2D-1D coupling should be written at the edge of the slot

Some references: McLver and Rawlins (93), Joly, Tordeux (06)
3D-2D coupling: field radiated by a Patch antenna

The small parameter: the width of the slot.

- The cavity is modelled by a 2D propagative medium
- The exterior is a 3D propagative medium
- A 3D-2D coupling should be written at the edge of the Patch

Some references: Mclver and Rawlins (93), Bendali, Makhlfouf, ST (10)

\[^1\text{image: http://en.wikipedia.org/wiki/Patch\_antenna}\]
The matching of asymptotic expansions

3D-1D coupling: scattering by a small wire

The small parameter: the width $\varepsilon$ of the wire.
- The wire is seen as a 1D medium
- The exterior is a 3D propagative medium
- A 3D-1D coupling should be written at each point of the wire

Some references: Fedoryuk (85), Il’In (92), Claeys (2009)

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$^3$image: PhD thesis of X. Claeys
The matching of asymptotic expansions

3D-0D coupling: Coupling through small holes

The small parameter: the width $\varepsilon$ of the hole.
- The hole is seen as a pointwise perturbation
- The exterior is a 3D propagative medium
- A 3D-0D coupling should be written

A reference: Gadyl’shin (92)
A non exhaustive bibliography

- **Small holes**: Rayleigh (1877), Rauch and Taylor (75), Tuck (75), Sanchez-Hubert and Sanchez-Palencia (82), Taflov (88), Bonnet-BenDhia, Drissi and Gmati (04), Mendez and Nicoud (08), Gadyl’shin (92).
- **Dumbell problems**: Beale (73), Jimbo and Morita (92) Brown, Hislop and Martinez (95), Arrieta (95), Anné (95).
- **Quasi-mode and min-max**: Bamberger and Bonnet (90), Dauge, Djurdjevic, Faou, and Roessle (99), Bonnaillie-Noel and Dauge (06).
Outline

1. Dirichlet condition in 2D
   - Definition of the asymptotic expansions
   - Error estimates
   - Numerical simulations

2. Neumann condition in 3D
   - Definition of the asymptotic expansions
   - Error estimates
   - Numerical simulations

3. Multiperforated boundary in 3D
   - Problem definition
   - Numerical simulations
Part I

Dirichlet boundary condition in 2D

with Abderrahmane Bendali and Abdelkader Tizaoui (IMT)
A Toy Problem

**Problem:** Find $u^n_\delta \in H^1_0(\Omega^\delta)$ and $\lambda^n_\delta \in \mathbb{R}$ satisfying

$$
\begin{cases}
-\nabla \cdot (a(x, y) \nabla u^n_\delta)(x, y) = \lambda^n_\delta b(x, y) \ u^n_\delta(x, y) \text{ in } \Omega^\delta, \\
 u^n_\delta(x, y) = 0 \text{ on } \partial \Omega^\delta,
\end{cases}
$$

(1)

with $a$ and $b$ two bounded positive regular functions with two sides

$$a_{int}(0) \neq a_{ext}(0) \text{ and } a_{int}(0) \neq a_{ext}(0)$$

(2)

The small parameter $\delta > 0$ is the width of the hole in the domain $\Omega^\delta$.

**Hypothesis:** The eigenvalues of $\Omega$ are simple.
The Matching of Asymptotic Expansions Method

The MAE method is based on a domain decomposition with overlapping 

The solution is described:
- with a **far**-field.
- with a **near**-field.
The Matched Asymptotic Expansions Method

The MAE is based on a domain decomposition with overlapping

\[ \Omega^\delta \]

The solution is described:

- with a \textbf{far}-field.
- with a \textbf{near}-field.
The Matched Asymptotic Expansions Method

The MAE is based on a domain decomposition with overlapping:

\[ \Omega^\delta \]

Matching zone

The solution is described:

- with a far-field.
- with a near-field.
The Asymptotic Expansions: The Eigenvalue Expansion

- The second order asymptotic expansion reads
  \[
  \lambda^\delta = \lambda^0 + \delta \lambda^1 + \delta^2 \lambda^2 + \delta^2 \ln\delta + o(\delta^2) . \tag{3}
  \]
- Polynomial gauge functions? not trivial: at fourth order
  \[
  \lambda^\delta = \lambda^0 + \delta \lambda^1 + \delta^2 \lambda^2 + \delta^3 \lambda^3 + \delta^4 \lambda^{4,0} + \delta^4 \ln\delta \lambda^{4,1} + o(\delta^4) . \tag{4}
  \]
- The coefficients \( \lambda^i \in \mathbb{R} \) and do not depend on \( \delta \)
- A proof is required
The Asymptotic Expansions: The Far-field Expansion

Far-field (Asymptotic Expansion):

$$u^\delta = u^0 + \delta u^1 + \delta^2 u^2 + o(\delta^2)$$

The coefficients of the far-field asymptotic expansion $u^i$ will be
- defined in the far-field domain $\Omega$: The limit of $\Omega^\delta$ when $\delta \to 0$,
- independent of $\delta$. 
Asymptotic Expansion: The Far-Field Expansion

They are solutions of the following problems

\[
\begin{align*}
\begin{cases}
\text{Find } u^0 : \Omega \to \mathbb{R} \text{ and } \lambda^0 \in \mathbb{R} \text{ such that } \\
\quad \nabla \cdot (a \nabla u^0) + \lambda^0 \ b \ u^0 = 0, \quad &\text{in } \Omega, \\
\quad u^0 = 0, \quad &\text{on } \partial \Omega \setminus \{0\}.
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\begin{cases}
\text{Find } u^1 : \Omega \to \mathbb{R} \text{ and } \lambda^1 \in \mathbb{R} \text{ such that } \\
\quad \nabla \cdot (a \nabla u^1) + \lambda^0 \ b \ u^1 = -\lambda^1 \ b \ u^0, \quad &\text{in } \Omega, \\
\quad u^1 = 0, \quad &\text{on } \partial \Omega \setminus \{0\}.
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\begin{cases}
\text{Find } u^2 : \Omega \to \mathbb{R} \text{ and } \lambda^2 \in \mathbb{R} \text{ such that } \\
\quad \nabla \cdot (a \nabla u^2) + \lambda^0 \ b \ u^2 = -\lambda^2 \ b \ u^0 - \lambda^1 \ b \ u^1, \quad &\text{in } \Omega, \\
\quad u^2 = 0, \quad &\text{on } \partial \Omega \setminus \{0\}.
\end{cases}
\end{align*}
\]

Missing information:

- The coefficients $u^0$, $u^1$, and $u^2$ are possibly singular at $x = 0$.
- The problems (5) do not uniquely define $u^1$ and $u^2$. 
Asymptotic Expansion: The Near-Field Expansion

Let $X = \frac{x}{\delta}$, $Y = \frac{y}{\delta}$, and we put $\Pi^\delta(X, Y) = u^\delta(\delta X, \delta Y)$. 

Near-field (Asymptotic Expansion):

$$\Pi^\delta(X, Y) = \Pi^0(X, Y) + \delta \, \Pi^1(X, Y) + \delta^2 \, \Pi^2(X, Y) + \mathcal{O}_0(\delta^2).$$

(6)

These functions will be
- defined on the near-field domain $\hat{\Omega}$.
- independent of $\delta$. 

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Asymptotic Expansion: The Near-Field expansion

The functions \( \Pi^0, \Pi^1 \) and \( \Pi^2 : \hat{\Omega} \rightarrow \mathbb{R} \) are solutions of elliptic problems

\[
\begin{align*}
\nabla \cdot (a^0 \nabla \Pi^0) &= 0, \quad \text{in } \hat{\Omega}, \\
\Pi^0 &= 0, \quad \text{on } \partial \hat{\Omega}. \\
\nabla \cdot (a^0 \nabla \Pi^1 + a^1 \nabla \Pi^0) &= 0, \quad \text{in } \hat{\Omega}, \\
\Pi^1 &= 0, \quad \text{on } \partial \hat{\Omega}. \\
\nabla \cdot (a^0 \nabla \Pi^2 + a^1 \nabla \Pi^1 + a^2 \nabla \Pi^0) + \lambda^0 b^0 \Pi^0 &= 0, \quad \text{in } \hat{\Omega}, \\
\Pi^2 &= 0, \quad \text{on } \partial \hat{\Omega}.
\end{align*}
\]

with the coefficients \( a^0, a^1, a^2 \) defined by part

\[
\begin{align*}
a^0_{\text{ext}}(X) &= a^0_{\text{ext}}(0) \quad \text{and} \quad a^0_{\text{int}}(X) = a^0_{\text{int}}(0), \\
a^1_{\text{ext}}(X) &= \nabla a^0_{\text{ext}}(0) \cdot X \quad \text{and} \quad a^1_{\text{int}}(X) = \nabla a^0_{\text{int}}(0) \cdot X, \\
a^2_{\text{ext}}(X) &= \frac{X \cdot Ha^0_{\text{ext}}(0)X}{2} \quad \text{and} \quad a^2_{\text{int}}(X) = \frac{X \cdot Ha^0_{\text{int}}(0)X}{2}, \\
b^0_{\text{ext}}(X) &= b^0_{\text{ext}}(0) \quad \text{and} \quad b^0_{\text{int}}(X) = b^0_{\text{int}}(0).
\end{align*}
\]

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The missing information for the near-field

The standard way to prove existence and uniqueness of solution in unbounded domains

- Lax-Milgram Theorem in weighted Sobolev spaces

**Missing information**

- The behaviors at infinity of the near-field are missing
- The problems do not uniquely define $\Pi^0$, $\Pi^1$, and $\Pi^2$. 

Matching the asympt. exp. to get the missing information

In order to ensure the uniqueness of $u^0$, $u^1$, $u^2$, $\Pi^0$, $\Pi^1$, and $\Pi^2$ we derive additional matching conditions. We use the following procedure to obtain these extra conditions.

1. We consider the far-field approximation of order $m$ written with $x = \delta X$

$$\sum_{i=0}^{m} \delta^i u^i(\delta X).$$

(7)

2. Then this sum is expanded up to $o(\delta^m)$. This defines the $U^i_m$ in the $X$ coordinates

$$\sum_{i=0}^{m} \delta^i u^i(\delta X) = \sum_{i=0}^{m} \delta^i U^i_m(X) + o(\delta^m).$$

(8)

3. The matching conditions are the following

$$\Pi^i - U^i_m = o\left(\frac{1}{R^{m-i}}\right) \quad \forall i \in \mathbb{Z}^+.$$ 

(9)
Using the **matching condition**, we get the problem defining $u^0$, $\lambda^0$, and $\Pi^0$.

**Far-field.** The function $u^0$ is an eigenfunction of the Dirichlet Laplace:

\[
\begin{cases}
    \text{Find } u^0 \in H^1(\Omega) \text{ such that } \\
    \nabla \cdot (a \nabla u^0) + \lambda^0 b u^0 = 0, & \text{in } \Omega, \\
    u^0 = 0, & \text{on } \partial \Omega.
\end{cases}
\] (10)

Due to Dirichlet conditions $\Pi^0 \equiv 0$. 

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The First Order Asymptotic Expansion

Using the **matching condition**, we get the problem defining $u^1$, $\lambda^1$, and $\Pi^1$.

**Far-field.**

\[
\begin{cases}
\text{Find } u^1 \in H^1(\Omega) \text{ and } \lambda^1 \in \mathbb{R} \text{ such that } \\
\nabla \cdot \left( a \nabla u^1 \right) + \lambda^0 b u^1 = -\lambda^1 b u^0, & \text{in } \Omega, \\
u^1 = 0, & \text{on } \partial \Omega.
\end{cases}
\] (11)

Note that $u^1$ is regular. Due to the Fredholm alternative, the second member has to be orthogonal to $u^0$

\[
\lambda^1 \int_{\Omega} b(x, y) \left( u^0(x, y) \right)^2 \, dx \, dy = 0.
\] (12)

We obtain $\lambda^1 = 0$.

The function $u^1$ is still defined up to its $u^0$-component, which is classical for eigenvalue problems. Then

\[
u^1 = \gamma u^0 \text{ in } \Omega_{int} \text{ and } u^1 = 0 \text{ in } \Omega_{ext}, \quad \text{with } \gamma \in \mathbb{R}.
\] (13)

We add the condition

\[
\int_{\Omega} b(x, y) u^1(x, y) u^0(x, y) \, dx \, dy = 0 \Rightarrow u^1 \equiv 0.
\]
The Second Order Asymptotic Expansion

The matching procedure leads to the following problem

\[
\begin{cases}
\text{Find } u^2 : \Omega \rightarrow \mathbb{R} \text{ and } \lambda^2 \in \mathbb{R} \text{ such that } \\
\quad \nabla \cdot (a \nabla u^2) + \lambda^0 \, b \, u^2 = -\lambda^2 \, b \, u^0, \text{ in } \Omega, \\
\quad u^2 = 0, \text{ on } \partial \Omega \setminus \{0\}. \\
\quad u^2_{\text{int}}(x) - \partial_x u^0_{\text{int}}(0) \frac{1}{8} \frac{a_{\text{int}}(0)}{a_{\text{int}}(0) + a_{\text{ext}}(0)} \frac{\sin(\theta)}{r} = O(r^{-1}), \\
\quad u^2_{\text{ext}}(x) + \partial_x u^0_{\text{int}}(0) \frac{1}{8} \frac{a_{\text{ext}}(0)}{a_{\text{int}}(0) + a_{\text{ext}}(0)} \frac{\sin(\theta)}{r} = O(r^{-1}),
\end{cases}
\] (14)

- Note that \( u^2 \) is singular \( u^2 \notin H^1(\Omega) \).
- Due to the singularity of \( u^2 \), the Fredholm alternative theory cannot be directly applied to obtain \( \lambda^2 \).

Proposition: The problem (14) has solutions. Moreover if \((u^2, \lambda^2)\) and \((u^*_2, \lambda^*_2)\) are solutions, one has

\[
\lambda^2 = \lambda^2_* = -\frac{\pi}{8} \frac{(a_{\text{int}}(0))^2}{a_{\text{int}}(0) + a_{\text{ext}}(0)} \frac{|\partial_x u^0_{\text{int}}(0)|^2}{\int_{\Omega} b(u^0)^2} \quad \text{and} \quad \exists \gamma \in \mathbb{R} : u^*_2 - u^2 = \gamma u^0.
\]
**The Second Order Asymptotic Expansion**

**Sketch of proof:** We introduce the auxiliary function $\omega^2$

$$\omega_{int, ext}^2 = u_{int, ext}^2 - (1 - \chi(r)) \partial_x u_{int, ext}^0(0) \alpha_{int, ext} \frac{\sin(\theta)}{r} \in H^1(\Omega_{int, ext}), \quad (15)$$

with $\chi$ the regular cut-off function:

![Cut-off function graph](image)

Applying Fredholm alternative to $\omega^2$ which is regular, we obtain $\lambda^2$. 
Asymptotic Expansion of the Eigenvalues

Theorem:

Let $\lambda^0$ be a simple eigenvalue of $\Omega$. For all $\delta > 0$, there exists an eigenvalue $\lambda^\delta$ of the Dirichlet laplacian in $\Omega^\delta$, see (8), satisfying

$$\left| \lambda^\delta - (\lambda^0 + \delta^2 \lambda^2) \right| \leq C \delta^3 |\ln(\delta)| \quad (16)$$

with $\lambda^2$ given by

$$\begin{cases} 
\lambda^2 = -\frac{\pi}{8} \frac{(a_{int}(0))^2}{a_{int}(0) + a_{ext}(0)} \frac{|\partial_x u^0_{int}(0)|^2}{\int_\Omega b(u^0)^2}, & \text{if } u^0_{ext} = 0, \\
\lambda^2 = -\frac{\pi}{8} \frac{(a_{ext}(0))^2}{a_{int}(0) + a_{ext}(0)} \frac{|\partial_x u^0_{ext}(0)|^2}{\int_\Omega b(u^0)^2}, & \text{if } u^0_{int} = 0.
\end{cases} \quad (17)$$

Generalization of the result of Gadyl’shin (92) that has looked to the case of constant coefficients.
Do we have missed some eigenvalues?

The last result reveals:

- There exists a $\lambda_n^{\delta}$ in a small neighborhood of each $\lambda_n$
- $0 < \lambda_{\sigma(n)}^{\delta} \leq \lambda_n$ for $\delta$ small enough.

Some questions

- only one $\lambda_n^{\delta}$ in the neighborhood of $\lambda_n$?
- other $\lambda_n^{\delta}$?
Do we have missed some eigenvalues?

The last result reveals:

- There exists a $\lambda_\delta^\delta$ in a small neighborhood of each $\lambda_n$
- $0 < \lambda_\sigma^\delta(n) \leq \lambda_n$ for $\delta$ small enough

Some answers (with min-max)

- only one $\lambda_\delta^\delta$ in the neighborhood of $\lambda_n$? no.
- other $\lambda_\delta^\delta$? no.
Let $\Omega^\delta$ be the domain defined by the following figure with

$$\Omega_{int} = ]-2, 0[ \times ]-2.5, 1.5[ \quad \text{and} \quad \Omega_{ext} = ]0, 2.5[ \times ]-1.5, 1[. \quad (18)$$

A computational mesh.

We recall that the eigenmodes of the limit problem in a domain $[0, a] \times [0, b]$ are

$$\lambda_{nm} = \pi \sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2}}, \quad U_{nm}(x, y) = \sin \left(\frac{n\pi}{a} x\right) \sin \left(\frac{m\pi}{b} y\right). \quad (19)$$
The $\lambda_n^2, \delta$ are numerically computed

The $\lambda_n$ and $\lambda_n^2$ are analytically computed

<table>
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<th>$n$</th>
<th>$\lambda_n$</th>
<th>$\lambda_n^2$</th>
</tr>
</thead>
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<tr>
<td>0</td>
<td>2.38</td>
<td>−0.087</td>
</tr>
<tr>
<td>1</td>
<td>3.08</td>
<td>−0.207</td>
</tr>
<tr>
<td>2</td>
<td>4.80</td>
<td>−0.135</td>
</tr>
<tr>
<td>3</td>
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<tr>
<td>4</td>
<td>7.12</td>
<td>−0.347</td>
</tr>
<tr>
<td>5</td>
<td>8.02</td>
<td>−0.036</td>
</tr>
</tbody>
</table>
The Relative Errors

\[ |\lambda^n_0 - \lambda^n_{0,\delta}| / \lambda^n_0 \]

\[ n = 0 \]

\[ |\lambda^n_2 - \lambda^n_{2,\delta}| / \lambda^n_2 \]

\[ n = 2 \]

\[ |\lambda^n_4 - \lambda^n_{4,\delta}| / \lambda^n_4 \]

\[ n = 4 \]

\[ |\lambda^n_1 - \lambda^n_{1,\delta}| / \lambda^n_1 \]

\[ n = 1 \]

\[ |\lambda^n_3 - \lambda^n_{3,\delta}| / \lambda^n_3 \]

\[ n = 3 \]

\[ |\lambda^n_5 - \lambda^n_{5,\delta}| / \lambda^n_5 \]

\[ n = 5 \]

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Second experiment

With continuous varying coefficients

\[
\begin{align*}
  a(x, y) &= 2 - \cos(x) \sin(y) \\
  b(x, y) &= 3 + \cos(x + y)
\end{align*}
\]
Third experiment

With **discontinuous constant coefficients**

\[
\begin{aligned}
a|_{\Omega_{\text{ext}}} &= 2, & a|_{\Omega_{\text{int}}} &= 1, \\
b|_{\Omega_{\text{ext}}} &= 0.5, & b|_{\Omega_{\text{int}}} &= 1,
\end{aligned}
\]
Part II

Neumann boundary condition in 3D

with Bendali (IMT), Fares (Cerfacs), Tizaoui (IMT)
The model problem

**The problem:**

Find \( u_\delta^n \in H^1(\Omega^\delta) \) and \( \lambda_\delta^n \in \mathbb{R} \) such that

\[
\begin{align*}
- \nabla \cdot (a \nabla u_\delta^n) &= \lambda_\delta^n b u_\delta^n \text{ in } \Omega^\delta, \\
\partial_n u_\delta^n &= 0 \text{ on } \partial \Omega^\delta,
\end{align*}
\]

with \( a \) and \( b \) two regular positive functions with two sides

\( a|_{\Omega_{\text{int}}(0)} \neq a|_{\Omega_{\text{ext}}(0)} \) and \( b|_{\Omega_{\text{int}}(0)} \neq b|_{\Omega_{\text{ext}}(0)} \)

- The small parameter \( \delta > 0 \) is the diameter of the hole.
- The hole is autosimilar \( \Sigma_\delta = \delta \Sigma \).
The limit problem:

The problem:

Find \( u_n \in H^1(\Omega) \) et \( \lambda_n \in \mathbb{R} \) such that

\[
\begin{align*}
-\nabla \cdot (a \nabla u_n) &= \lambda_n b u_n \text{ in } \Omega, \\
\partial_n u_n &= 0 \text{ on } \partial \Omega,
\end{align*}
\]

- The limit corresponds to \( \delta = 0 \)
- The first eigenvalue is 0 and has multiplicity 2.
- \textbf{no hypothesis on the multiplicity}
Classical theory of eigenvalue problems

The $\lambda_n$ and $\lambda_\delta$ are countable:

\[
\begin{cases}
0 = \lambda_\delta^1 < \lambda_\delta^2 \leq \lambda_\delta^3 \leq \cdots \leq \lambda_\delta^{n-1} \leq \lambda_\delta^n 
\rightarrow +\infty \\
0 = \lambda_1 = \lambda_2 < \lambda_3 \leq \cdots \leq \lambda_{n-1} \leq \lambda_n 
\rightarrow +\infty
\end{cases}
\]

The min-max principle:

\[\lambda_\delta^\delta \geq \lambda_n \quad \forall \delta > 0 \quad \forall n \in \mathbb{N}\]

Some natural questions:

\[\lambda_\delta^n \rightarrow \lambda_n ?\]

Could we derive an efficient method to compute the $\lambda_\delta^n$ without local mesh refinement?
The expansion of the eigenvalues

- The first order expansion reads
  \[ \lambda^\delta = \lambda^0 + \delta \lambda^1 + o(\delta). \]

- The coefficients \( \lambda^i \in \mathbb{R} \) are independent of \( \delta \).

- Logarithmic gauge function at order 2
  \[ \lambda^\delta = \lambda^0 + \delta \lambda^1 + \delta^2 \lambda^{2,0} + \delta^2 \ln \delta \lambda^{2,1} + o(\delta^2). \]

- Very surprising in 3D (variable coefficients)
The far-field expansion

\[ u^\delta = u^0 + \delta u^1 + o(\delta) \]

The coefficients \( u^i \) are
- defined on the limit domain \( \Omega \)
- independent of \( \delta \)
The far-field coefficients

The coefficients are solution of

\[
\begin{align*}
\text{Find } u^0 : \Omega &\rightarrow \mathbb{R} \text{ and } \lambda^0 \in \mathbb{R} \text{ such that } \\
\nabla \cdot (a \nabla u^0) + \lambda^0 b u^0 &= 0 \text{ in } \Omega, \\
\partial_n u^0 &= 0 \text{ on } \partial \Omega \setminus \{0\}.
\end{align*}
\]

\[
\begin{align*}
\text{Find } u^1 : \Omega &\rightarrow \mathbb{R} \text{ and } \lambda^1 \in \mathbb{R} \text{ such that } \\
\nabla \cdot (a \nabla u^1) + \lambda^0 b u^1 &= -\lambda^1 b u^0 \text{ in } \Omega, \\
\partial_n u^1 &= 0 \text{ on } \partial \Omega \setminus \{0\}.
\end{align*}
\]

- The \( u^i \) can be singular at 0.
- The missing information: the behavior of \( u^i \) at 0.
The near-field expansion

The first order expansion:

\[ \Pi^\delta(X) = u^\delta(X\delta) \quad \text{and} \quad \Pi^\delta = \Pi^0 + \delta \Pi^{1,0} + \delta \ln \delta \Pi^{1,1} + o(\delta) \]

The coefficients \( \Pi^i \) are

- defined on the limit domain \( \hat{\Omega} \)
- independent of \( \delta \)
The near-field coefficients

Find $\Pi^0 \in H^1_{\text{loc}}(\hat{\Omega})$ such that
\[
\nabla \cdot (a_0 \nabla \Pi^0) = 0 \text{ in } \hat{\Omega},
\]
\[
\partial_n \Pi^0 = 0 \text{ on } \partial \hat{\Omega}.
\]

Find $\Pi^1 \in H^1_{\text{loc}}(\hat{\Omega})$ such that
\[
\nabla \cdot (a_0 \nabla \Pi^1) = -\nabla \cdot (a_1 \nabla \Pi^0) \text{ in } \hat{\Omega},
\]
\[
\partial_n \Pi^1 = 0 \text{ on } \partial \hat{\Omega}.
\]

- $a_0$ and $a_1$ are functions with two sides
  \[
  a_0|_{\hat{\Omega}_{\text{int}}} (X) = a|_{\Omega_{\text{int}}}(0)
  \]
  \[
  a_0|_{\hat{\Omega}_{\text{ext}}} (X) = a|_{\Omega_{\text{ext}}}(0),
  \]
  \[
  a_1|_{\hat{\Omega}_{\text{int}}} (X) = \nabla a|_{\Omega_{\text{int}}}(0) \cdot X,
  \]
  \[
  a_1|_{\hat{\Omega}_{\text{ext}}} (X) = \nabla a|_{\Omega_{\text{ext}}}(0) \cdot X.
  \]

- The behavior at $+\infty$ are missing.
The limit coefficients

The far-field limit

\[ \begin{cases} \text{Find } u^0 \in H^1(\Omega) \text{ and } \lambda^0 \in \mathbb{R} \text{ such that} \\ \nabla \cdot (a \nabla u^0) + \lambda^0 b u^0 = 0 \text{ in } \Omega, \\ \partial_n u^0 = 0 \text{ on } \partial \Omega \setminus \{0\}. \end{cases} \]

To simplify simple eigenvalue and \( u^0|_{\Omega_{\text{ext}}} = 0 \).

The near field limit

\[ \begin{cases} \Pi^0 \in H^1_{\text{loc}}(\hat{\Omega}) \text{ such that} \\ \nabla \cdot (a_0 \nabla \Pi^0) = 0 \text{ in } \hat{\Omega}, \\ \partial_n \Pi^0 = 0 \text{ on } \partial \hat{\Omega}, \\ \Pi^0|_{\Omega_{\text{int}}(X)} = u^0|_{\Omega_{\text{int}}(0)} + o(1), \\ \Pi^0|_{\Omega_{\text{ext}}(X)} = o(1). \end{cases} \]

\[ \text{A laplacian with two sides} \]
The limit coefficients

Solution of the near-field problem

- The coefficient $a_0$ is piecewise constant. We can use a boundary element formulation.
- By linearity, the problem can be put in a canonical form

\[
\begin{cases}
\text{Find } \Pi^* \in H^1_{\text{loc}}(\hat{\Omega}) \text{ such that } \\
\quad \nabla \cdot (a_0 \nabla \Pi^*) = 0 \text{ in } \hat{\Omega}, \\
\quad \partial_n \Pi^* = 0 \text{ on } \partial \hat{\Omega}, \\
\quad \Pi^*|_{\hat{\Omega}_{\text{int}}(X)} = 1 + o(1), \\
\quad \Pi^*|_{\hat{\Omega}_{\text{ext}}(X)} = o(1), \\
\end{cases}
\]

- **Outgoing fields** are identified

\[
\Pi^*|_{\hat{\Omega}_{\text{int}}} = 1 + \Pi^*|_{\hat{\Omega}_{\text{int}}} \text{ and } \Pi^*|_{\hat{\Omega}_{\text{ext}}} = \Pi^*|_{\hat{\Omega}_{\text{ext}}}
\]
The limit coefficients

Solution of the near field problem

\[
\begin{aligned}
\Delta \Pi^*_{\text{out}} &= 0 \quad \text{in } \hat{\Omega}_{\text{int}}, \\
\Delta \Pi^*_{\text{out}} &= 0 \quad \text{in } \hat{\Omega}_{\text{int}}, \\
\partial_n \Pi^*_{\text{out}} &= 0 \quad \text{on } \partial \hat{\Omega}.
\end{aligned}
\]

Integral representation in \( \hat{\Omega}_{\text{int}} \) and in \( \hat{\Omega}_{\text{ext}} \) (image principle)

\[
\begin{aligned}
\Pi^*_{\text{out}} &= -2S \frac{\partial \Pi^*}{\partial z} \bigg|_{\Sigma_{\text{int}}} \quad \iff \quad \Pi^* = 1 - 2S \frac{\partial \Pi^*}{\partial z} \bigg|_{\Sigma_{\text{int}}}, \\
\Pi^*_{\text{out}} &= 2S \frac{\partial \Pi^*}{\partial z} \bigg|_{\Sigma_{\text{ext}}} \quad \iff \quad \Pi^* = 2S \frac{\partial \Pi^*}{\partial z} \bigg|_{\Sigma_{\text{ext}}}.
\end{aligned}
\]

with \( S \) the simple layer operator

\[
S : (H^1_2(\Sigma))^* \mapsto H^1_2(\Sigma), \quad \lambda \mapsto S\lambda(X) = \frac{1}{4\pi} \int_{\Sigma} \frac{\lambda(X')}{\|X - X'\|} dX'.
\]
The limit coefficients

Solution of the near-field problem

transmission condition through $\Sigma$

$$\Pi^*|_{\Sigma_{\text{int}}} = \Pi^*|_{\Sigma_{\text{ext}}} \text{ and } a|_{\Omega_{\text{int}}(0)} \partial_z \Pi^*|_{\Sigma_{\text{int}}} = a|_{\Omega_{\text{ext}}(0)} \partial_z \Pi^*|_{\Sigma_{\text{ext}}}$$

$\implies$ obtention of the unknown potential $\lambda^*$

$$\partial_z \Pi^*|_{\Sigma_{\text{int}}} = -\frac{a|_{\Omega_{\text{ext}}(0)}}{a|_{\Omega_{\text{ext}}(0)} + a|_{\Omega_{\text{int}}(0)}} \frac{\lambda^*}{2}$$

$$\partial_z \Pi^*|_{\Sigma_{\text{ext}}} = -\frac{a|_{\Omega_{\text{int}}(0)}}{a|_{\Omega_{\text{int}}(0)} + a|_{\Omega_{\text{ext}}(0)}} \frac{\lambda^*}{2}.$$
The limit coefficients

The near-field $\Pi^*$ is given by an integral representation

$$
\Pi^* = \begin{cases} 
1 - \frac{a|\Omega_{\text{ext}}(0)}{a|\Omega_{\text{ext}}(0) + a|\Omega_{\text{int}}(0)} \left( \frac{1}{4\pi} \int_\Sigma \frac{\lambda^*(X')}{||X - X'||} dX' \right) & \text{in } \hat{\Omega}_{\text{int}}, \\
+ \frac{a|\Omega_{\text{int}}(0)}{a|\Omega_{\text{ext}}(0) + a|\Omega_{\text{int}}(0)} \left( \frac{1}{4\pi} \int_\Sigma \frac{\lambda^*(X')}{||X - X'||} dX' \right) & \text{in } \hat{\Omega}_{\text{ext}}.
\end{cases}
$$

Behavior at infinity

$$
\Pi^*|_{\hat{\Omega}_{\text{int}}}(X) = 1 - \frac{a|\Omega_{\text{ext}}(0)}{a|\Omega_{\text{ext}}(0) + a|\Omega_{\text{int}}(0)} \frac{\alpha}{||X||}
$$

$$
\Pi^*|_{\hat{\Omega}_{\text{ext}}}(X) = \frac{a|\Omega_{\text{int}}(0)}{a|\Omega_{\text{ext}}(0) + a|\Omega_{\text{int}}(0)} \frac{\alpha}{||X||}
$$

with $\alpha = \frac{1}{4\pi} \int_\Sigma \lambda^*(X') dX'$. 

When $\Sigma$ is a circle with radius $\rho$, $\alpha = \frac{2}{\pi} \rho$.

In classical theory of acoustic, $\alpha \delta$ is the effective size of the hole.
Approximation of the effective size of the hole $\alpha \delta$

- The computation of a numerical approximation of $\alpha$ requires a boundary element code.
- Approximation of the effective size by the of the effective size of the circle with same area

$$\alpha_{\text{app}} \delta = \frac{2}{\pi} \sqrt{\frac{A}{\pi}}$$  with $A$ the area of the hole.

- The error is small for not too elongated structure.

![Graphs showing the error in effective size approximation](image)
The first order coefficients

First order far-field

Find $u_n^1 : \Omega \rightarrow \mathbb{R}$,

\[ \nabla \cdot (a \nabla u_n^1) + \lambda_n b u_n^1 = -\lambda_n^1 b u_n \text{ in } \Omega, \]

\[ \partial_n u_n^1 = 0 \text{ on } \partial \Omega \setminus \{0\}, \]

\[ u_n^1 + \frac{a_{\mid \Omega_{\text{ext}}(0)}}{a_{\mid \Omega_{\text{int}}(0)} + a_{\mid \Omega_{\text{ext}}(0)}} u_n_{\mid \Omega_{\text{int}}(0)} \frac{1}{\|x\|} \in H^1(\Omega_{\text{int}}), \]

\[ u_n^1 - \frac{a_{\mid \Omega_{\text{int}}(0)}}{a_{\mid \Omega_{\text{int}}(0)} + a_{\mid \Omega_{\text{ext}}(0)}} u_n_{\mid \Omega_{\text{int}}(0)} \frac{1}{\|x\|} \in H^1(\Omega_{\text{ext}}), \]

We can apply the Fredholm alternative

\[ v = u_n^1 + \frac{a_{\mid \Omega_{\text{ext}}(0)}}{a_{\mid \Omega_{\text{int}}(0)} + a_{\mid \Omega_{\text{ext}}(0)}} u_n_{\mid \Omega_{\text{int}}(0)} \frac{1}{\|x\|} \in H^1(\Omega_{\text{ext}}). \]
The first order coefficients

The coefficient $u_n^1$ exists and is uniquely defined modulo $u_n$ and

$$\lambda_n^1 = 2 \pi \alpha \frac{a|_{\Omega_{\text{int}}}(0)a|_{\Omega_{\text{ext}}}(0)}{a|_{\Omega_{\text{int}}}(0) + a|_{\Omega_{\text{ext}}}(0)} \frac{(\bar{u}_n(0))^2}{\int_{\Omega} b(u_n)^2}.$$ 

Asymptotic expansion in the neighborhood of 0 of $u_n^1$ (**Kondratiev theory**)

$$u_n^1|_{\Omega_{\text{int}}}(x) = -\frac{a|_{\Omega_{\text{ext}}}(0) \alpha}{a|_{\Omega_{\text{int}}}(0) + a|_{\Omega_{\text{ext}}}(0)} u_n|_{\Omega_{\text{int}}}(0) \left(\frac{1}{r} - \frac{s_n^1(x)}{2} + r_n^1(x)\right), \quad (20)$$

with

$$s_n^1|_{\Omega_{\text{int}}}(x) = \frac{\nabla a|_{\Omega_{\text{int}}}(0)}{a|_{\Omega_{\text{int}}}(0)} \cdot \frac{x}{||x||} + \frac{\partial z a|_{\Omega_{\text{int}}}(0)}{a|_{\Omega_{\text{int}}}(0)} \ln \left(\frac{||x|| - z}{2}\right)$$

and $r_n^1 : \Omega_{\text{int}} \rightarrow \mathbb{R}$ a continuous function.

For the first order near field coefficient, it becomes much more complicated!
The main results

Theorem: simple eigenvalue

If $\lambda_n$ is an eigenvalue with multiplicity 1 of the limit problem, then $\lambda_\delta^n$ can be expanded with the form

$$\lambda_\delta^n = \lambda_n + 2 \pi \frac{a|_{\Omega_{\text{int}}(0)}a|_{\Omega_{\text{ext}}(0)}}{a|_{\Omega_{\text{int}}(0)} + a|_{\Omega_{\text{ext}}(0)}} \frac{(\bar{u}_n(0))^2}{\int_{\Omega} b(u_n)^2} \alpha \delta + O(\delta^2 \ln \delta),$$

with $\alpha$ a coefficient only dependent of the form of the hole and with the convention

$$\bar{u}(0) = u|_{\Omega_{\text{ext}}(0)} \quad \text{if } u|_{\Omega_{\text{int}}} = 0,$$

$$\bar{u}(0) = u|_{\Omega_{\text{int}}(0)} \quad \text{if } u|_{\Omega_{\text{ext}}} = 0.$$

All these quantities do not require any mesh refinement.
The main result

**Theorem: multiple eigenvalue**

If $\lambda_n$ is an eigenvalue of the limit problem with multiplicity $N + 1$ ($\lambda_n = \cdots = \lambda_{n+N}$), then

\[
\begin{align*}
\lambda_\delta &= \lambda_n + O(\delta^2), \\
\lambda_{\delta,n+1} &= \lambda_n + O(\delta^2), \\
\lambda_{\delta,n+N-1} &= \lambda_n + O(\delta^2), \\
\lambda_{\delta,n+N} &= \lambda_n + 2\pi \frac{a_{\Omega_{\text{int}}}(0) a_{\Omega_{\text{ext}}}(0)}{a_{\Omega_{\text{int}}}(0) + a_{\Omega_{\text{ext}}}(0)} \sum_{p=0}^{N} \left( \bar{u}_{n+p}(0) \right)^2 \alpha \delta + O(\delta^2 \ln \delta),
\end{align*}
\]

with $\alpha$ a coefficient only dependent of the form of the hole.
Numerical simulations

- Two series of simulations with two geometries and two sets of functions $a$ and $b$
- 10 values for $\delta$ from $10^{-2}$ to 1
  
  $$\delta = 10^{-\frac{n}{5}} \text{ with } n = 0, \cdots, 10$$

- **computation code**: a mesher and a parallel code from CERFACS
  - **Altair Hypermesh**:
  - **CESC**: (Fares and Bendali)
    - 3D $P_1$-finite elements.
    - 3D $P_1$-boundary elements.
  - **ARPACK**: eigenvalue solver (Sorensen and al.)
- Computation of $\lambda_n^\delta$ with mesh refinement ($\#dof \simeq 4 \times 10^6$)
- Computation of the $\lambda_n$ without mesh refinement
First experiment

The classical laplacian

\[ a = 1 \quad \text{and} \quad b = 1. \]  \hspace{1cm} (21)

Definition of the computational domain: two straight cavities

\[ \Omega_{\text{int}} = \left[ -\frac{\ell^\text{int}_x}{2}, \frac{\ell^\text{int}_x}{2} \right] \times \left[ -\frac{\ell_y}{3}, \frac{2\ell_y}{3} \right] \times [0, \ell_z], \]
\[ \Omega_{\text{ext}} = \left[ -\frac{\ell^\text{ext}_x}{4}, \frac{3\ell^\text{ext}_x}{4} \right] \times \left[ -\frac{\ell_y}{3}, \frac{2\ell_y}{3} \right] \times [0, \ell_z]. \]

with \( \ell^\text{int}_x = .6, \ell^\text{ext}_x = 1., \ell_y = .8, \ell_z = .3. \)

The hole \( \Sigma \): the polygon linking the points

points \( A = (0, 0), B = (0, .1), \)
\( C = (-.08, .1), D = (-.08, -.08), \)
\( E = (.1, -.08) \) and \( F = (.1, 0) \)

\[ \alpha = 0.0578 \]
The geometry

- Dirichlet condition in 2D
- Neumann condition in 3D
- Multiperforated boundary in 3D
- Definition of the asymptotic expansions
- Error estimates
- Numerical simulations

S. Tordeux
Perforated and multiperforated plates in linear acoustic
The first results

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\lambda_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>9.87</td>
</tr>
<tr>
<td>4</td>
<td>15.4</td>
</tr>
<tr>
<td>5</td>
<td>15.4</td>
</tr>
</tbody>
</table>

$\lambda_n + \delta \lambda^1_n$

$\lambda_n^\delta$

$\lambda_n$

$\lambda_n$ and $\lambda_n^\delta$ for different values of $n$. The table shows the values of $\lambda_n$ for $n = 1, 2, 3, 4, 5$. The graph illustrates the asymptotic expansions for $n = 1, 2, 3, 4, 5$. The equation $\lambda_n + \delta \lambda^1_n$ is plotted for each $n$. The graph also shows the difference between $\lambda_n$ and $\lambda_n^\delta$. The table and graph are used to analyze the behavior of the asymptotic expansions for different values of $n$. The results are obtained through numerical simulations.
The relative errors

First eigenvalues

The eigenvector 1 has multiplicity 2 for the limit problem
\[ \lambda_1 = \lambda_2 = 0. \]
and multiplicity 1 for the model problem
\[ \lambda_1^\delta = \lambda_1 = 0 \]
The relative errors

The relative errors for the first eigenvalues

- $\lambda_3^{\delta} - \lambda_3^{1,\delta} / \lambda_3$
  - $1 - 10^{-1}$
  - $10^{-2}$
  - $10^{-3}$
  - $10^{-4}$
  - $10^{-5}$
  - $10^{-6}$
  - $\delta$

- $\lambda_4^{\delta} - \lambda_4^{1,\delta} / \lambda_4$
  - $1 - 10^{-1}$
  - $10^{-2}$
  - $10^{-3}$
  - $10^{-4}$
  - $10^{-5}$
  - $10^{-6}$
  - $\delta$

- $\lambda_5^{\delta} - \lambda_5^{1,\delta} / \lambda_5$
  - $1 - 10^{-1}$
  - $10^{-2}$
  - $10^{-3}$
  - $10^{-4}$
  - $10^{-5}$
  - $\delta$

Simple eigenvalue $\lambda_3 = 9.87$

Double eigenvalue $\lambda_4 = \lambda_5 = 15.4$

Accuracy problem of the solver.
Second experiment

A laplacian with two sides

\[ a|_{\Omega_{\text{int}}} = 2, \quad a|_{\Omega_{\text{ext}}} = 1, \quad b|_{\Omega_{\text{int}}} = 1, \quad b|_{\Omega_{\text{ext}}} = 2. \]

The computational domain: a straight cavity and a pyramid

\[ \Omega_{\text{int}} = \left[ -\frac{\ell_x}{4}, \frac{3\ell_x}{4} \right] \times \left[ -\frac{\ell_y}{3}, \frac{2\ell_y}{3} \right] \times \ell_z, 0[, \]

\[ \Omega_{\text{ext}} = \text{simplex}(A, B, C, D, E). \]

with

- \( \ell_x = 1, \ell_y = 0.8, \ell_z = 0.3. \)
- \( A = (0.3, -0.4, 0), \quad B = (-0.7, -0.4, 0), \)
  \( \quad C = (-0.5, 0.2, 0) \) and \( D = (0.3, 0.2, 0) \) et
  \( \quad E = (0.1, 0.1, 0.7). \)

The hole \( \Sigma \): a circle with radius \( 0.1. \)

\[ \alpha = 0.0636 \]
The geometry
First results

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<td>4</td>
<td>16.7</td>
</tr>
<tr>
<td>5</td>
<td>19.7</td>
</tr>
</tbody>
</table>

\[ \lambda_n + \delta \lambda_n^1 \]

\[ \lambda_n^\delta \]

\[ \lambda_n \]
The relative errors

The relative errors for the double eigenvalue

The eigenvector 1 has multiplicity 2 for the limit problem

$$\lambda_1 = \lambda_2 = 0.$$ 

1 is also an eigenvector for the model problem

$$\lambda_1^\delta = \lambda_1 = 0$$

$$\lambda_2 = 0$$
The relative errors

The relative errors for the simple eigenvalues

\[ |\lambda_3^\delta - \lambda_3^{1,\delta}|/\lambda_3 \]
\[ |\lambda_4^\delta - \lambda_4^{1,\delta}|/\lambda_4 \]
\[ |\lambda_5^\delta - \lambda_5^{1,\delta}|/\lambda_5 \]

\[ \lambda_3 = 9.24 \]
\[ \lambda_4 = 16.7 \]
\[ \lambda_5 = 19.7 \]

Accuracy problem of the solver.
Part III

Multiperforated plates in 3D

with A. Bendali (IMT), M. Fares (Cerfacs), S. Laurens (IMT)
The propagation domain

A wave guide with rectangular section

Reproduce the test bench B2A of ONERA (collaboration with Franck Simon and Estelle Piot) $h_2 = 5cm$ et $h_3 = 5cm$
The geometry of the multiperforated plate

Perforations with radius \( \delta \) and spacing de \( \eta = (\eta_1, \eta_2) \)

\[
\chi_{m,n} = \left(0, \frac{\eta_2}{2}, \frac{\eta_3}{2}\right) + \left(0, (m + \alpha n)\eta_2, n\eta_3\right) \tag{22}
\]

\( \alpha = 0 \): straight grating (left); \( \alpha \in ]0, 1[ \): inclined grating (right)

Perforation with similar size to the real perforation of SNECMA

\( \eta_2 = \eta_3 = 5\, mm \) et \( \delta = .5\, mm \)
The model problem in time domain

Wave equation for the pressure

\[
\frac{\partial^2 p^{\eta,\delta}}{\partial t^2}(x, t) - c^2 \Delta p^{\eta,\delta}(x, t) = 0 \quad \text{in } \Omega^{\eta,\delta}
\]  \hspace{1cm} (23)

The fluid velocity field \( \mathbf{v} \) satisfies

\[
\rho \frac{\partial \mathbf{v}^{\eta,\delta}}{\partial t}(x, t) + \nabla p^{\eta,\delta}(x, t) = 0
\]  \hspace{1cm} (24)

Rigid wall boundary condition

\[
\mathbf{v}^{\eta,\delta} \cdot \mathbf{n} = 0
\]  \hspace{1cm} (25)

with \( \mathbf{n} \) the outgoing normal
A Helmholtz problem

Harmonic regime (wave number $k = \frac{\omega}{c}$)

$p^{\eta,\delta}(\mathbf{x}, t) = p^{\eta,\delta}(\mathbf{x}) \exp(-i\omega t)$ and $\mathbf{v}^{\eta,\delta}(\mathbf{x}, t) = \mathbf{v}^{\eta,\delta}(\mathbf{x}) \exp(-i\omega t)$.

Helmholtz equation for pressure

$$\begin{cases} 
\Delta p^{\eta,\delta}(\mathbf{x}) + k^2 p^{\eta,\delta}(\mathbf{x}) = 0 \quad \text{in } \Omega^{\eta,\delta}, \\
\frac{\partial p^{\eta,\delta}}{\partial n}(\mathbf{x}) = 0 \quad \text{on } \partial \Omega^{\eta,\delta}.
\end{cases}$$

The velocity field can be obtained by

$$-i\omega \rho \mathbf{v}^{\eta,\delta}(\mathbf{x}) + \nabla p^{\eta,\delta}(\mathbf{x}) = 0$$

Outgoing wave equation

$$u^{\eta,\delta}(\mathbf{x}) = \exp(ikx_1) + \text{outgoing}.$$
The approximate model

This model is based on the hypothesis

1. The holes are isolated $\delta \ll \eta$ (no hole-hole interaction);
2. The wave length $\lambda$ is much bigger than the characteristic lengths of the holes
3. The linear acoustic is a valid model in the vicinity of the perforations (no aeroacoustic effect and no viscous effect).

$$\lambda >> \eta >> \delta$$ (26)

This model can be entirely justified by a rigorous asymptotic analysis
The Rayleigh coefficient

local quantity: ratio **flux of fluid** through one perforation / **pressure jump**

\[ i \omega \rho \phi = K_R (p_+ - p_-). \]

\( K_R \) is rigorously defined via the solution of an unbounded problem

\[
\begin{cases}
\Delta p^\delta = 0 & \text{dans } \Omega^\delta, \\
\partial_n p^\delta = 0 & \text{sur } \partial \Omega^\delta, \\
p^\delta = p_+ + o \left( \frac{1}{\|x\|} \right) & \text{pour } x_1 > 0, \\
p^\delta = p_- + o \left( \frac{1}{\|x\|} \right) & \text{pour } x_1 < 0,
\end{cases}
\]

Rayleigh Coefficient

\[
K_R = \frac{\int_{x_2^2 + x_3^2 \leq \delta} \partial_1 p(0, x_2, x_3) \, dx_2 \, dx_3}{p_+ - p_-} = 2\delta.
\]
homogenized wall condition

- The flux is distributed over one cell with area $A = \eta_1 \eta_2$

  $$\mathbf{v}_1^+ = \mathbf{v}_1^- = \frac{\Phi}{A} \text{ en } x = 0.$$ 

- Translation in term of pressure

  $$
  \frac{\partial p^+}{\partial x_1} = \frac{\partial p^-}{\partial x_1} = \frac{K_R}{A} \left( p^+ - p^- \right) \text{ en } x = 0.
  $$

1D Problem $\implies$ analytical resolution

$$
\begin{cases}
  p(x) = \exp(ikx_1) + R \exp(-ikx_1) \quad \text{pour } x_1 < 0, \\
  p(x) = T \exp(ikx_1) \quad \text{pour } x_1 > 0.
\end{cases}
$$
Direct numerical solution

Boundary element method:

- Refined mesh in the neighborhood of one hole

Full matrix $\#dof \simeq 1.6 \times 10^5$
Direct numerical solution

Boundary element method:

- Refined mesh in the neighborhood of one hole

- Full matrix $\text{#dof} \simeq 1.6 \times 10^5$
Direct numerical solution

Boundary element method:

- Refined mesh in the neighborhood of one hole
- Full matrix $\#dof \simeq 1.6 \times 10^5$
Validation of the approximate model: straight grating

Comparison of the modules of the reflexion coefficients (direct computation and approximate model)
Validation of the approximate model: straight grating

Comparison of the phases of the reflected wave

\[ \phi(R) \]

frequence \( s^{-1} \)

S. Tordeux
Perforated and multiperforated plates in linear acoustic
Validation of the approximate model: the inclined grating

Comparison of the modules of the reflection coefficients (direct computation and approximate model)
Validation du modèle approché : cas incliné

\[ \phi(R) \]

Comparison of the phases of the reflected wave

Frequency \( s^{-1} \)
Conclusion

Main results: an approximation of eigenvalues with

- A theoretical background
- No mesh refinement required

Some publications:

- Bendali, Huard, Tizaoui, Tordeux and Vila, CRAS (2010)
- Bendali, Fares, Tizaoui, Tordeux, CICP (2011)
- Bendali, Fares, Laurens, Tordeux, submitted

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Thank you!