PRESCRIBED GAUSS CURVATURE ON CLOSED SURFACES OF HIGHER GENUS

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Abstract. For a closed Riemann surface (M, g_0) of genus larger than one we study the existence of conformal metrics of prescribed curvature. We show that whenever for a sign-changing function f there exists a conformal metric $g = e^{2u}g_0$ of Gauss curvature $K_g = f$ which is a relative minimizer of the associated variational integral there also exists a second "large" solution to this problem which is not of minimum type.

1. Introduction

Let (M, g_0) be a closed Riemann surface endowed with a smooth background metric g_0. By the uniformization theorem we may assume that g_0 has constant Gauss curvature $K_{g_0} \equiv k_0$. Moreover, we normalize the volume of (M, g_0) to unity.

A classical problem in differential geometry is the question which smooth functions $f\colon M \to \mathbb{R}$ arise as the Gauss curvature of a conformal metric $g = e^{2u}g_0$ on M. Even when (M, g_0) is closed, this question so far has not been completely settled, aside from the case when the genus of M is one [9], or when (M, g_0) is the projective plane [11]. In particular, both in the case of the standard sphere $(M, g_0) = (S^2, g_{S^2})$, when the above question is referred to as Nirenberg's problem, and in the case when M has genus greater than one so far only partial results are known. In this paper we will focus on the latter case. Clearly, by passing to the oriented double cover, if necessary, we may assume throughout that M is orientable.

In order to state our results we first recall that the Gauss curvature of a conformal metric $g = e^{2u}g_0$ on M is given by

$$K_g = e^{-2u}(-\Delta_{g_0}u + k_0)\ .$$

Given a function $f \in C^\infty(M)$, the problem of finding a conformal metric of prescribed Gauss curvature f then amounts to solving the equation

$$-\Delta_{g_0} u + k_0 = e^{2u} f\ . \tag{1.1}$$

For a solution of this equation, upon integrating (1.1) and using the Gauss-Bonnet Theorem we immediately obtain the identity

$$\int_M f d\mu_g = \int_M k_0 d\mu_{g_0} = k_0 = 2\pi\chi(M)\ , \tag{1.2}$$

where $d\mu_g = e^{2u}d\mu_{g_0}$ is the element of area in the metric g and where

$$\chi(M) = 2(1 - g(M))$$

is the Euler characteristic of the surface M with genus $g(M)$.

Thus we arrive at the following necessary condition.

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Theorem 1.1. *For equation* (1.1) *to admit a solution, we need:*

i) *if* $\chi(M) < 0$*, the function* f *has to be negative somewhere;*
ii) *if* $\chi(M) > 0$*, the function* f *has to be positive somewhere;*
iii) *if* $\chi(M) = 0$*, either* $f \equiv 0$*, or* f *must change sign.*

Surprisingly, as shown by Moser [11], in the case when (M, g_0) is the projective plane $P^2\mathbb{R}$ the condition given in Theorem 1.1 also is sufficient. After passing to the oriented double cover S^2 of $P^2\mathbb{R}$, Moser's result may be phrased as follows.

Theorem 1.2. *Let* $f \in C^\infty(S^2)$ *be even, that is, satisfying* $f(p) = f(-p)$ *for all* $p \in S^2$*. Then* f *is the curvature of an even conformal metric* $g = e^{2u}g_{S^2}$ *on* S^2 *if and only if* $\sup_{S^2} f > 0$*.*

In fact, the case of the projective plane is the only instance where Theorem 1.1 gives the optimal result. In the case when $(M, g_0) = (S^2, g_{S^2})$ further necessary conditions for a general function f to be realized as the curvature of a conformal metric have been obtained by Kazdan-Warner [10], but the gap between these conditions and the sufficient conditions established by Chang-Yang [7], with later improvements by Chang-Liu [8] and others, remains considerable. The general problem in this case remains one of the most intriguing problems in geometric analysis.

Also in the case when $\chi(M) \le 0$ (and therefore also $k_0 \le 0$) the necessary conditions given in Theorem 1.1 can easily be refined. Upon multiplying (1.1) with the function e^{-2u} and integrating by parts, we find

$$\int_M f d\mu_{g_0} = \int_M (-\Delta_{g_0} u + k_0)e^{-2u} d\mu_{g_0} = \int_M (-2|\nabla u|^2_{g_0} + k_0)e^{-2u} d\mu_{g_0} \le 0 \tag{1.3}$$

with equality if and only if $\nabla u = 0$ and $k_0 = 0$, that is, $\chi(M) = 0$.

As was shown by Kazdan-Warner [9], in the case when $\chi(M) = 0$ the combined conditions (1.2) and (1.3) now are both necessary and sufficient for the existence of solutions to (1.1).

Theorem 1.3. *Let* (M, g_0) *be closed with* $\chi(M) = 0$*, and let* $f \in C^\infty(M)$ *be given. Then* f *is the curvature of a conformal metric* $g = e^{2u}g_0$ *on* M *if and only if either i)* $f \equiv 0$*, or ii)* f *changes sign and* $\int_M f\, d\mu_{g_0} < 0$*.*

By contrast, in the case when $\chi(M) < 0$ the sufficient conditions stated, for instance, in the following classical result, are much stronger than what is required by either Theorem 1.1 or (1.3), similar to the case when $(M, g_0) = (S^2, g_{S^2})$.

Theorem 1.4. *Let* (M, g_0) *be closed with* $\chi(M) < 0$*, and let* $0 \ge f \in C^\infty(M)$*,* $f \not\equiv 0$*. Then* f *is the curvature of a conformal metric* $g = e^{2u}g_0$ *on* M*.*

In particular, Theorem 1.4 gives no information what can be said for sign-changing functions f satisfying (1.3). In order to shed some light on this question we now focus our attention on the case when $\chi(M) < 0$.

Recall that solutions u of (1.1) can be characterized as critical points of the functional

$$E_f(u) = \frac{1}{2}\int_M \left(|\nabla u|^2_{g_0} + 2k_0 u - fe^{2u}\right) d\mu_{g_0},\ u \in H^1(M, g_0)\,.$$

Note that E_f is strictly convex and coercive on $H^1(M, g_0)$ when $f \le 0$ does not vanish identically. Hence for such f the functional E_f admits a strict absolute minimizer $u \in H^1(M, g_0)$ solving (1.5).

As our first result we now have the following stability result for relative minimizers of E_f for arbitrary f.

Theorem 1.5. *Let (M, g_0) be closed with $\chi(M) < 0$, and suppose that for some $f \in C^\infty(M)$ the functional E_f admits a relative minimizer $u_f \in H^1(M, g_0)$. Then u_f is a non-degenerate critical point of E_f in the sense that with a constant $c_0 > 0$ there holds*

$$d^2E_f(u_f)(h,h) = \int_M \left(|\nabla h|^2_{g_0} - 2fe^{2u_f}h^2\right)\, d\mu_{g_0} \geq c_0 ||h||^2_{H^1} \tag{1.4}$$

for all $h \in H^1(M, g_0)$, and there exists an open neighborhood U of f in $C^0(M)$ and a smooth map $U \ni \varphi \mapsto u_\varphi \in H^1(M, g_0)$ such that for every $\varphi \in U$ the function u_φ is a strict relative minimizer of E_φ.

As a special case this results includes a stability result of Aubin [1] for functions $f \leq 0$. In particular, it shows that for non-constant functions f with a not too large positive maximum the corresponding functional E_f still admits critical points which can be characterized as *relative* minimizers of E_f. Observe that for functions f with $\max_M f > 0$ the functional E_f is no longer bounded from below, as can be seen by choosing a comparison function $v \geq 0$ supported in the set where $f > 0$ and looking at $E_f(sv)$ for large $s > 0$.

It is not clear how large the maximum of f may be in order to still have existence of a solution u to (1.1). A first result in this direction was achieved by Aubin and Bismuth [2], [4], who established the following result.

Theorem 1.6. *Let (M, g_0) be closed with $\chi(M) < 0$. For a compact subset $K \neq M$ of M let F_K be the set of functions $f \in C^\infty(M)$ such that $K = \{x \in M : f(x) \geq 0\}$. There exists a constant $C = C(K, M) > 0$ such that any function $f \in F_K$ satisfying*

$$\sup_M f \leq C \sup_M(-f)$$

is the scalar curvature of a metric conformal to g_0.

We can see the problem from a different perspective if we embed equation (1.1) into the 1-parameter family of problems

$$-\Delta_{g_0} u + k_0 = e^{2u} f_\lambda \ , \tag{1.5}$$

where for a given non-constant function $f \in C^\infty(M)$ and any $\lambda \in \mathbb{R}$ we let

$$f_\lambda = f + \lambda \ .$$

Solutions u of (1.5) then can be characterized as critical points of the functional $E_\lambda(u) = E_{f_\lambda}(u)$ on $H^1(M, g_0)$, and E_λ is strictly convex and coercive on $H^1(M, g_0)$ as long as $\lambda \leq \lambda_1 := -\max_M f$. Hence for such λ there is a strict absolute minimizer $u_\lambda \in H^1(M, g_0)$ of E_λ, uniquely solving (1.5) and depending smoothly on λ. By Theorem 1.5, moreover, this C^1-branch of *absolute* minimizers $(u_\lambda)_{\lambda \leq \lambda_1}$ extends as a C^1-curve of *relative* minimizers beyond the threshold $\lambda = \lambda_1$.

By (1.3), however, this branch of relative minimizers must end before λ attains the value $\lambda_2 := -\fint_M f\, d\mu_{g_0}$. In fact, the next result gives indication that the branch of relative minimizers will be met at some point $\lambda_1 < \lambda^* < \lambda_2$ by a branch of solutions u^λ of saddle-type, bifurcating from infinity at $\lambda = \lambda_1$. Note that we have $\lambda_1 < \lambda_2$ for any non-constant function f.

Theorem 1.7. *Let (M, g_0) be closed with $\chi(M) < 0$, and let $f \in C^\infty(M)$ be non-constant. Suppose that for some $\lambda_1 < \lambda < \lambda_2$ the functional E_λ admits a relative minimizer $u_\lambda \in H^1(M, g_0)$. Then there exists a second critical point $u^\lambda \neq u_\lambda$ of E_λ not of minimum-type.*

In particular, if for any sign-changing function f the functional E_f admits a relative minimizer $u_f \in H^1(M, g_0)$, there also exists a second critical point $u^f \neq u_f$ of E_f not of minimum-type. This result is similar in spirit to the famous *"Rellich's conjecture"*, solved independently by Brezis-Coron [5], [6] and Struwe [12], [13] in the early 1980's. Moreover, in order to overcome the possible lack of compactness in our problem we make use of the *"monotonicity trick"* from [14], [15] in a way similar to [16].

Unfortunately, the argument we present cannot insure that the *"large"* solutions u^λ that we obtain lie on a smooth curve, nor does it answer the question of multiplicity. A further interesting question worth pursuing is the study of the degeneration of u^λ as $\lambda \downarrow \lambda_1$. A similar question arises in the case of surfaces (M, g_0) with $\chi(M) = 0$ when we consider the solutions $u = u_\lambda$ of the equation

$$-\Delta_{g_0} u = e^{2u} f_\lambda \, , \tag{1.6}$$

for a non-constant function f and $\lambda \uparrow \lambda_0 = -\fint_M f \, d\mu_{g_0}$. We hope to come back to these questions elsewhere.

2. Nondegeneracy and stability of relative minimizers

Throughout the remainder of this paper we assume that (M, g_0) is closed with $\chi(M) < 0$. In this section we present the proof of Theorem 1.5. First we establish (1.4).

Proposition 2.1. *Suppose that for some $f \in C^\infty(M, g_0)$ the functional E_f admits a relative minimizer $u_f \in H^1(M, g_0)$. Then u_f is a non-degenerate critical point of E_f in the sense of (1.4).*

For a relative minimizer $u_f \in H^1(M, g_0)$ of E_f we have

$$d^2E_f(u_f)(h, h) = \int_M \left(|\nabla h|^2_{g_0} - 2fe^{2u_f} h^2\right) d\mu_{g_0} \geq 0 \tag{2.1}$$

for all $h \in H^1(M, g_0)$. Therefore

$$\alpha := \inf_{\|h\|_{H^1}=1} d^2E_f(u_f)(h, h) \geq 0.$$

The claim in Proposition 2.1 is equivalent to the claim that $\alpha > 0$. Otherwise $\alpha = 0$, and the following two lemmas will lead to a contradiction.

Lemma 2.2. *If $\alpha = 0$ there exists $h \in H^1(M, g_0)$ such that*

$$d^2E_f(u_f)(h, h) = 0 \quad \textit{and} \quad \|h\|_{H^1} = 1.$$

Proof. Let $(h_k)_{k \in \mathbb{N}}$ with $\|h_k\|_{H^1} = 1$ such that $d^2E_f(u_f)(h_k, h_k) \to 0$ as $k \to \infty$. Since (h_k) is bounded in H^1, we may assume that $h_k \rightharpoonup h$ weakly in $H^1(M, g_0)$ and strongly in L^p for any $p < \infty$ for some $h \in H^1(M, g_0)$. Since u_f is smooth, then we also have convergence $f e^{2u_f} h_k^2 \to f e^{2u_f} h^2$ in L^1, and from (2.1) it follows that

$$\begin{aligned}\|\nabla h_k\|_{L^2}^2 &= d^2E_f(u_f)(h_k,h_k) + 2\int_M f e^{2u_f} h_k^2 d\mu_{g_0} \to 2\int_M f e^{2u_f} h^2 d\mu_{g_0} \\ &\le d^2E_f(u_f)(h,h) + 2\int_M f e^{2u_f} h^2 d\mu_{g_0} = \|\nabla h\|_{L^2}^2 \quad \text{as } k\to\infty\,.\end{aligned}$$

Recalling that $h_k \to h$ weakly in $H^1(M,g_0)$ and strongly in L^2, we conclude strong convergence $h_k \to h$ in $H^1(M,g_0)$. The claim follows. □

By Lemma 2.2, when $\alpha = 0$ the functional $v \mapsto d^2E_f(u_f)(v,v)$ attains a minimum at $v = h$. It follows that

$$d^2E_f(u_f)(h,w) = 0 \quad \text{for all } w \in H^1(M,g_0);$$

that is, $h \in H^1(M,g_0)$ weakly solves the equation

$$-\Delta_{g_0} h = 2f e^{2u_f} h \quad \text{in} \quad (M,g_0). \tag{2.2}$$

In particular then h is smooth and classically solves (2.2).

Lemma 2.3. *Assume $\alpha = 0$ and let $h \in H^1(M,g_0)$ as determined in Lemma 2.2. Then*

$$d^4E_f(u_f)(h,h,h,h) = -8\int_M f e^{2u_f} h^4 < 0.$$

Proof. Note that $h \neq const$. Otherwise (2.2) would yield

$$\int_M f e^{2u_f} d\mu_{g_0} = 0$$

contrary to (1.2). Multiplying equation (2.2) by h^3 we get

$$2f e^{2u_f} h^4 = -h^3 \Delta_{g_0} h = -\frac{1}{4}\Delta_{g_0}(h^4) + 3|\nabla h|_{g_0}^2 h^2.$$

Upon integration this yields

$$d^4E_f(u_f)(h,h,h,h) = -8\int_M h^4 f e^{2u_f} d\mu_{g_0} = -12\int_M |\nabla h|_{g_0}^2 h^2 d\mu_{g_0} < 0,$$

as claimed. □

Proof of Proposition 2.1. Assume by contradiction that $\alpha = 0$ and let $h \in H^1(M,g_0)$ as determined in Lemma 2.2. Using the fact that $dE_f(u_f) = 0$ and the relation $d^2E_f(u_f)(h,h) = 0$ to expand

$$E_f(u_f + \varepsilon h) = E_f(u_f) + \frac{\varepsilon^3}{6} d^3E_f(u_f)(h,h,h) + O(\varepsilon^4),$$

and recalling that u_f is a relative minimizer, we see that $d^3E_f(u_f)(h,h,h) = 0$. But then the expansion to fourth order by Lemma 2.3 yields

$$E_f(u_f + \varepsilon h) = E_f(u_f) + \frac{\varepsilon^4}{24} d^4E_f(u_f)(h,h,h,h) + O(\varepsilon^5) < E_f(u_f)$$

for small $\varepsilon > 0$, and we arrive at the desired contradiction. □

Proposition 2.4. *Suppose that for some $f \in C^\infty(M,g_0)$ the functional E_f admits a relative minimizer $u_f \in H^1(M,g_0)$. Then there exists an open neighborhood U of f in $C^0(M,g_0)$ and a smooth map $U \ni \varphi \mapsto u_\varphi \in H^1(M,g_0)$ such that for every $\varphi \in U$ the function u_φ is a strict relative minimizer of E_φ.*

Proof. We define the function $F\colon C^0(M,g_0)\times H^1(M,g_0)\to H^{-1}(M,g_0)$ by letting

$$F(\varphi,u)=dE_\varphi(u).$$

Note that for any fixed $u\in H^1(M,g_0)$ the function $\varphi\mapsto F(\varphi,u)\in H^{-1}(M,g_0)$ is linear with differential $\partial_\varphi F(\,\cdot,u)$ represented by $e^{2u}\in L^2(M,g_0)$, and the map $u\mapsto e^{2u}$ is locally uniformly continuous in $H^1(M,g_0)$ by the Moser-Trudinger embedding. Similarly, the differential $\partial_u F(f,\,\cdot)$ with

$$\langle v\cdot\partial_u F(f,u),w\rangle_{H^{-1}\times H^1}=d^2E_f(u)(v,w)$$

is locally uniformly continuous in $H^1(M,g_0)$ with

$$\langle h\cdot\partial_u F(f,u_f),h\rangle_{H^{-1}\times H^1}=d^2E_f(u_f)(h,h)\ge c_0\|h\|^2_{H^1}$$

for all $h\in H^1(M,g_0)$ with a uniform constant $c_0>0$ by Proposition 2.1. Hence $\partial_u F(f,u_f)\colon H^1(M,g_0)\to H^{-1}(M,g_0)$ is continuously invertible. By the implicit function theorem there exist an open neighborhood $U\subset C^0(M,g_0)$ of f and an open neighborhood $V\subset H^1(M,g_0)$ of u_f together with a continuously differentiable map $G:U\ni\varphi\to u_\varphi\in V$ such that $F(\varphi,u_\varphi)=0$ for all $\varphi\in U$.

After shrinking U, if necessary, by continuity of G and (1.4) we also have

$$d^2E_\varphi(u_\varphi)(h,h)\ge\frac{c_0}{2}\|h\|^2_{H^1}$$

for all $\varphi\in U$, $h\in H^1$, and the critical points $G(\varphi)=u_\varphi$ of E_φ obtained in this manner will all be strict relative minimizers. □

3. Existence of a saddle-type critical point

We now give the proof of Theorem 1.7, which is our main result. Suppose that for some $\lambda\in]\lambda_1,\lambda_2[$ the functional E_λ admits a relative minimizer $u_\lambda\in H^1(M,g_0)$, which then is strict by Theorem 1.5. Hence there exists $\rho>0$ such that

$$E_\lambda(u_\lambda)=\inf_{||u-u_\lambda||_{H^1}<\rho}E_\lambda(u)<\beta_\lambda:=\inf_{\rho/2<||u-u_\lambda||_{H^1}<\rho}E_\lambda(u). \tag{3.1}$$

The value λ will be fixed throughout the following. Moreover, again by Theorem 1.5, we may also fix an open neighborhood $\Lambda\subset]-\infty,\lambda_2[$ of λ such that for each $\mu\in\Lambda$ there exists a strict relative minimizer $u_\mu\in H^1(M,g_0)$ of E_μ, smoothly depending on $\mu\in\Lambda$. Finally, recalling that for $\lambda>\lambda_1$ the functional E_λ is unbounded from below, we can also fix a function $v_\lambda\in H^1(M,g_0)$ such that

$$E_\lambda(v_\lambda)<E_\lambda(u_\lambda)$$

and hence

$$c_\lambda=\inf_{p\in P}\max_{t\in[0,1]}E_\lambda(p(t))\ge\beta_\lambda>E_\lambda(u_\lambda),$$

where

$$P=\{p\in C\left([0,1];H^1(M,g_0)\right):\ p(0)=u_\lambda,\,p(1)=v_\lambda\}. \tag{3.2}$$

Choosing a smaller neighborhood $\Lambda\subset]-\infty,\lambda_2[$ of λ, if necessary, by continuity we may then assume that there holds

$$E_\mu(v_\lambda)<E_\mu(u_\mu)\le\sup_{\nu\in\Lambda}E_\mu(u_\nu)<\beta_\mu:=\inf_{\rho/2<||u-u_\lambda||_{H^1}<\rho}E_\mu(u)\le c_\mu \tag{3.3}$$

for every $\mu\in\Lambda$, where

$$c_\mu:=\inf_{p\in P}\max_{t\in[0,1]}E_\mu(p(t)),\ \mu\in\Lambda\,. \tag{3.4}$$

Clearly, we may assume that $|\mu - \lambda| < 1$ for every $\mu \in \Lambda$.

With regard to the dependence of E and its derivative on μ we note the following lemma.

Lemma 3.1. *i) For any $m > 0$ there exists a constant $C = C(m)$ such that for every $\mu_1, \mu_2 \in \mathbb{R}$ and for every $u \in H^1(M, g_0)$ such that $||u||_{H^1(M)} \le m$ there holds*

$$||dE_{\mu_1}(u) - dE_{\mu_2}(u)|| \le C|\mu_1 - \mu_2|$$

ii) For each $\mu \in \mathbb{R}$, for each $u, v \in H^1(M, g_0)$ with $||v||_{H^1(M,g_0)} \le 1$, we have

$$E_\mu(u + v) \le E_\mu(u) + \langle dE_\mu(u), v\rangle + C\big(\int_M e^{8u}\, d\mu_{g_0}\big)^{1/4}$$

where $C = C(M, g_0)$ is a positive constant.

Proof. i) Pick $v \in H^1(M, g_0)$ such that $||v||_{H^1(M,g_0)} \le 1$ and compute

$$\begin{aligned}\langle dE_{\mu_1}(u) - dE_{\mu_2}(u), v\rangle &= (\mu_2 - \mu_1)\int_M e^{2u} v\, d\mu_{g_0}\\ &\le |\mu_2 - \mu_1|\big(\int_M e^{4u}\, d\mu_{g_0}\big)^{1/2}||v||_{L^2(M,g_0)} \le |\mu_2 - \mu_1|\big(\int_M e^{4u}\, d\mu_{g_0}\big)^{1/2}.\end{aligned}$$

The claim follows from the Moser-Trudinger inequality.

ii) By Taylor's expansion, for every $x \in M$ there exists $\theta(x) \in]0, 1[$ such that

$$\begin{aligned}E_\mu(u + v) - E_\mu(u) - \langle dE_\mu(u), v\rangle &= \frac{1}{2}\int_M |\nabla v|^2_{g_0}\, d\mu_{g_0} - \int_M f_\mu e^{2(u+\theta v)} v^2\, d\mu_{g_0}\\ &\le \frac{1}{2}||v||^2_{H^1(M,g_0)} + ||f_\mu||_\infty \int_M e^{2(u+\theta v)} v^2\, d\mu_{g_0}\ .\end{aligned}$$

By Hölder's inequality and Sobolev's embedding we get

$$\begin{aligned}\int_M e^{2(u+\theta v)} v^2\, d\mu_{g_0} &\le \big(\int_M e^{4(u+\theta v)}\, d\mu_{g_0}\big)^{1/2}||v||^2_{L^4(M)}\\ &\le C\big(\int_M e^{8u}\, d\mu_{g_0} \cdot \int_M e^{8|v|}\, d\mu_{g_0}\big)^{1/4},\end{aligned}$$

and again our claim follows from the Moser-Trudinger inequality. □

Note that there holds

$$\frac{d}{d\mu}E_\mu(u) = -\frac{1}{2}\int_M e^{2u}\, d\mu_{g_0} < 0 \tag{3.5}$$

for every $u \in H^1(M, g_0)$ and every $\mu \in \mathbb{R}$. It follows that the function

$$\Lambda \ni \mu \mapsto c_\mu$$

is non-increasing in μ, and therefore differentiable at almost every $\mu \in \Lambda$. We now have the following result.

Proposition 3.2. *Suppose the map $\Lambda \ni \mu \mapsto c_\mu$ is differentiable at some $\mu > \lambda$. Then there exists a sequence $(p_n)_{n\in\mathbb{N}}$ in P and a corresponding sequence of points $u_n = p_n(t_n) \in H^1(M, g_0)$, $n \in \mathbb{N}$, such that*

$$E_\mu(u_n) \to c_\mu,\ \sup_{0\le t\le 1} E_\mu(p_n(t)) \to c_\mu,\ dE_\mu(u_n) \to 0 \text{ in } H^{-1} \text{ as } n \to \infty, \tag{3.6}$$

and with (u_n) satisfying, in addition, the "entropy bound"

$$\int_M e^{2u_n}\, d\mu_{g_0} = 2\Big|\frac{d}{d\mu}E_\mu(u_n)\Big| \le C = C(\mu), \text{ uniformly in } n. \tag{3.7}$$

Remark 3.3. Observe that the energy bound (3.6) together with (3.7) imply a uniform bound

$$||u||^2_{H^1} + \int_M e^{2u}\, d\mu_{g_0} \le C = C(\mu) \tag{3.8}$$

for all $u = u_n$ as above. Indeed, from (3.7) and Jensen's inequality we find the uniform bound

$$2\int_M u\, d\mu_{g_0} \le \log\Big(\int_M e^{2u}\, d\mu_{g_0}\Big) \le C; \tag{3.9}$$

hence, recalling that $k_0 < 0$, we also obtain the estimate

$$\begin{aligned} ||\nabla u||^2_{L^2} &= 2E_\mu(u) - k_0\int_M u\, d\mu_{g_0} + \int_M (f+\mu)e^{2u}\, d\mu_{g_0} \\ &\le 2E_\mu(u) + C \le 2c_\mu + 2(\mu_n - \mu) + C \le C \end{aligned} \tag{3.10}$$

for all such $u = u_n$, with uniform constants C possibly depending on μ. In addition, since $k_0 < 0$, from (3.10) we also obtain a uniform lower bound for the average of u, which together with (3.9) implies (3.8).

Proof of Proposition 3.2. Let $\mu \in \Lambda$ be a point of differentiability of c_μ. For a sequence of numbers $\mu_n \in \Lambda$ with $\mu_n \downarrow \mu$ as $n \to \infty$ let (p_n) be a sequence of paths $p_n \in P$ such that

$$\max_{t\in[0,1]} E_\mu(p_n(t)) \le c_\mu + (\mu_n - \mu),\ n \in \mathbb{N}.$$

For any point $u = p_n(t_n),\ t_n \in [0,1]$, with

$$E_{\mu_n}(u) \ge c_{\mu_n} - (\mu_n - \mu) \tag{3.11}$$

then by (3.5) we have

$$c_{\mu_n} - (\mu_n - \mu) \le E_{\mu_n}(u) \le E_\mu(u) \le \max_{t\in[0,1]} E_\mu(p_n(t)) \le c_\mu + (\mu_n - \mu). \tag{3.12}$$

Letting $\alpha = -c'_\mu + 1 > 0$, for sufficiently large $n_0 \in \mathbb{N}$ and any $n \ge n_0$ we have

$$c_{\mu_n} \ge c_\mu - \alpha(\mu_n - \mu).$$

Thus from (3.12) we obtain the estimate

$$0 \le \frac{E_\mu(u) - E_{\mu_n}(u)}{\mu_n - \mu} = \frac{1}{2}\int_M e^{2u}\, d\mu_{g_0} \le \alpha + 2,\ n \ge n_0, \tag{3.13}$$

for all such (p_n) and u, and hence by Remark 3.3 also the uniform bound

$$||u||^2_{H^1} + \int_M e^{2u}\, d\mu_{g_0} \le C_1 \tag{3.14}$$

with a uniform constant $C_1 = C_1(\mu)$. Note that n_0 is independent of the choice of (p_n).

Now assume by contradiction that there exists $\delta > 0$ such that $||dE_\mu(u)|| \ge 2\delta$ for sufficiently large n for every (p_n) and $u = u_n = p_n(t_n) \in H^1(M, g_0)$ as above.

By (3.14) we then also have the uniform bound $\int_M e^{8u}\,d\mu_{g_0} < m$ for some number $m > 0$, and Lemma 3.1 implies

$$\begin{aligned}
&\langle dE_{\mu_n}(u), dE_\mu(u)\rangle = ||dE_\mu(u)||^2 - \langle dE_\mu(u) - dE_{\mu_n}(u), dE_\mu(u)\rangle \\
&\ge \frac{1}{2}||dE_\mu(u)||^2 - \frac{1}{2}||dE_\mu(u) - dE_{\mu_n}(u)||^2 \ge \frac{1}{2}||dE_\mu(u)||^2 - C|\mu - \mu_n|^2 \\
&\ge 2\delta^2 - C|\mu - \mu_n|^2 \ge \delta^2
\end{aligned} \tag{3.15}$$

for any such (p_n) and u, if $n \ge n_1$ for some sufficiently large $n_1 \ge n_0$.

Fix now such a sequence of paths (p_n). Choose a function $\phi \in C^\infty(\mathbb{R})$ such that $0 \le \phi \le 1$ and with $\phi(s) = 1$ for $s \ge -1/2$, $\phi(s) = 0$ for $s \le -1$. For $n \in N$, $w \in H^1(M, g_0)$ let

$$\phi_n(w) \equiv \phi\Big(\frac{E_{\mu_n}(w) - c_{\mu_n}}{\mu_n - \mu}\Big).$$

Note that for $u = p_n(t_n)$ there holds $\phi_n(u) = 0$ unless u satisfies (3.11).

Identifying $dE_\mu(w) \in H^{-1}$ with a vector in $H^1(M, g_0)$ through the inner product, we define new comparison paths $\tilde{p}_n$ by letting

$$\tilde{p}_n(t) := p_n(t) - \sqrt{\mu_n - \mu}\;\phi_n(p_n(t))\frac{dE_\mu(p_n(t))}{||dE_\mu(p_n(t))||},\ 0 \le t \le 1,\ n \ge n_1.$$

Writing again $u = p_n(t_n)$ and likewise $\tilde{u} = \tilde{p}_n(t_n)$ for brevity and recalling that for $n \ge n_1$ we have $|\mu - \mu_n| \le 1$, we find $||u - \tilde{u}||_{H^1} \le 1$. Hence for any $u = p_n(t_n)$ satisfying (3.11) by the second part of Lemma 3.1 and (3.14) with constants $C = C(\mu)$ independent of $u = p_n(t_n)$ for sufficiently large $n \ge n_1$ on account of (3.15) we obtain

$$\begin{aligned}
E_{\mu_n}(\tilde{u}) &\le E_{\mu_n}(u) - \frac{\sqrt{\mu_n - \mu}\,\phi_n(u)}{||dE_\mu(u)||}\langle dE_{\mu_n}(u), dE_\mu(u)\rangle + C(\mu_n - \mu)\phi_n^2(u) \\
&\le E_{\mu_n}(u) - \frac{1}{2}\sqrt{\mu_n - \mu}\,\phi_n(u)||dE_\mu(u)|| + C(\mu_n - \mu)\phi_n^2(u) \\
&\le E_{\mu_n}(u) - \delta\sqrt{\mu_n - \mu}\,\phi_n(u) + C(\mu_n - \mu)\phi_n^2(u) \\
&\le E_{\mu_n}(u) - \frac{1}{2}\delta\sqrt{\mu_n - \mu}\,\phi_n(u).
\end{aligned}$$

It follows that

$$c_{\mu_n} \le \max_{t\in[0,1]} E_{\mu_n}(\tilde{p}_n(t)) \le \max_{t\in[0,1]} \big(E_{\mu_n}(p_n(t)) - \frac{\delta}{2}\sqrt{\mu_n - \mu}\,\phi_n(p_n(t))\big).$$

Since the maximum in the last inequality can only be achieved at points t where $E_{\mu_n}(p_n(t)) \ge c_{\mu_n} - (\mu_n - \mu)/2$ and hence $\phi_n(p_n(t)) = 1$, we find

$$\begin{aligned}
c_{\mu_n} &\le \max_{t\in[0,1]} E_{\mu_n}(p_n(t)) - \frac{\delta}{2}\sqrt{\mu_n - \mu} \\
&\le \max_{t\in[0,1]} E_\mu(p_n(t)) - \frac{\delta}{2}\sqrt{\mu_n - \mu} \\
&\le c_\mu + (\mu_n - \mu) - \frac{\delta}{2}\sqrt{\mu_n - \mu} \\
&\le c_{\mu_n} + \alpha(\mu_n - \mu) - \frac{\delta}{2}\sqrt{\mu_n - \mu} < c_{\mu_n}
\end{aligned}$$

for $n \ge n_1$, a contradiction. □

Proposition 3.4. *Let μ be a point of differentiability for the map c_μ. Then the functional E_μ admits a critical point u^μ such that $E_\mu(u^\mu) = c_\mu$, and u^μ is not a relative minimizer of E_μ.*

Proof. Let μ be a point of differentiability for the map c_μ. Then Proposition 3.2 guarantees the existence of a sequence $(p_n)_{n\in\mathbb{N}}$ in P and a corresponding sequence of points $u_n = p_n(t_n) \in H^1(M, g_0)$, $n \in \mathbb{N}$, satisfying (3.6) and (3.7), hence by Remark 3.3 also (3.8). Passing to a subsequence, if necessary, we may then assume that $u_n \rightharpoonup u^\mu$ weakly in $H^1(M, g_0)$ as $n \to \infty$ for some $u^\mu \in H^1(M, g_0)$. Recalling that the map $H^1(M, g_0) \ni \varphi \mapsto e^{2\varphi} \in L^2(M, g_0)$ is compact, we also may assume that $e^{2u_n} \to e^{2u^\mu}$ in $L^2(M, g_0)$.

Thus, with error $o(1) \to 0$ as $n \to \infty$ we obtain

$$\begin{aligned} o(1) &= \langle dE_\mu(u_n), u_n - u^\mu \rangle \\ &= \int_M (\nabla u_n, \nabla u_n - \nabla u^\mu)_{g_0}\, d\mu_{g_0} \\ &\quad + k_0 \int_M (u_n - u^\mu)\, d\mu_{g_0} - \int_M f_\mu e^{2u_n}(u_n - u^\mu)\, d\mu_{g_0} \\ &= \|\nabla u_n - \nabla u^\mu\|^2_{L^2} + o(1), \end{aligned}$$

that is, $u_n \to u^\mu$ strongly in $H^1(M, g_0)$ as $n \to \infty$. But then we also have convergence $E_\mu(u_n) \to E_\mu(u^\mu)$ and $dE_\mu(u_n) \to dE_\mu(u^\mu)$ as $n \to \infty$, and u^μ is a critical point for E_μ at level $E_\mu(u^\mu) = c_\mu$.

Finally, u^μ cannot be a relative minimizer of E_μ; otherwise an estimate similar to (3.1) would give a contradiction to our choice of (p_n) with $\sup_{0\le t\le 1} E_\mu(p_n(t)) \to c_\mu$ as $n \to \infty$ and the fact that $u_n = p_n(t_n)$ for some $t_n \in [0, 1]$, $n \in \mathbb{N}$. □

4. Compactness

Finally, we show the existence of a critical point u^λ of E_λ not of minimum type for any $\lambda \in]\lambda_1, \lambda_2[$ such that E_λ admits a relative minimizer u_λ. The function u^λ will then be distinct from u_λ, and we will have achieved the proof of Theorem 1.7.

By Proposition 3.4, for any λ as above there is a sequence of numbers $\mu_n \downarrow \lambda$ and critical points u_n of E_{μ_n} with $E_{\mu_n}(u_n) = c_{\mu_n}$ and not of minimum type for every $n \in \mathbb{N}$. Our aim is to show that (u_n) is relatively compact. First note the following estimate.

Lemma 4.1. *Let $f \in C^\infty(M)$ and suppose $u \in H^1(M, g_0)$ is a critical point for the functional E_f. Then with a constant $C(f)$ depending only on $\|f\|_{C^1}$ and on (M, g_0) we have*

$$\int_M f^4 e^{2u}\, d\mu_{g_0} \le C(f) \tag{4.1}$$

Proof. Rearranging terms in (1.3) we obtain

(4.2)
$$2\int_M |\nabla u|^2_{g_0} e^{-2u}\, d\mu_{g_0} \le k_0 \int_M e^{-2u}\, d\mu_{g_0} - \int_M f\, d\mu_{g_0} < -\int_M f\, d\mu_{g_0} \le C_1(f).$$

Next, multiply (1.1) by f^3 and integrate by parts to find

$$\begin{aligned}\int_M f^4 e^{2u}\,d\mu_{g_0} &= 3\int_M (\nabla u, \nabla f)_{g_0} f^2\,d\mu_{g_0} + k_0\int_M f^3\,d\mu_{g_0}\\ &\le C_2(f)\int_M |\nabla u|_{g_0} f^2\,d\mu_{g_0} + C_2(f).\end{aligned} \tag{4.3}$$

But by using Young's inequality $2ab \le \delta a^2 + \delta^{-1}b^2$, in view of (4.1) we can bound

$$\begin{aligned}C_2(f)\int_M |\nabla u|_{g_0} f^2\,d\mu_{g_0} &\le \frac{1}{2}\int_M f^4 e^{2u}\,d\mu_{g_0} + C_3(f)\int_M |\nabla u|^2_{g_0} e^{-2u}\,d\mu_{g_0}\\ &\le \frac{1}{2}\int_M f^4 e^{2u}\,d\mu_{g_0} + C_4(f).\end{aligned}$$

Our claim then follows from (4.3). $\square$

Via Jensen's inequality – applied with the probabilty measure $\frac{f^2}{\|f\|^2_{L^2}}\,d\mu_{g_0}$ – from (4.1) we conclude the bound

$$\int_M f_+^2 u\,d\mu_{g_0} \le \int_M f^2 u\,d\mu_{g_0} \le \|f\|^2_{L^2}\log\Big(\frac{\int_M f^2 e^u\,d\mu_{g_0}}{\|f\|^2_{L^2}}\Big) \le C(f), \tag{4.4}$$

where for any $s \in \mathbb{R}$ we let $s_+ = \max\{s, 0\}$.

Given any non-constant $f \in C^\infty(M)$, for any $\lambda \in]\lambda_1, \lambda_2[$ such that E_λ admits a relative minimizer and any sequence $\mu_n \downarrow \lambda$ as above, upon replacing f in (4.4) by $f_{\mu_n} \ge f_\lambda$ we then obtain the uniform bound

$$\int_M f^2_{\lambda+} u_n\,d\mu_{g_0} \le C(f),\ n \in \mathbb{N}. \tag{4.5}$$

Note that $f_{\lambda+} \not\equiv 0$ by choice of λ. Normalizing the measure $f^2_{\lambda+}\,d\mu_{g_0}$, finally, from (4.5) we arrive at the uniform bound

$$\bar{u}_n^{(f_\lambda)} := \int_M f^2_{\lambda+} u_n\,d\mu_{g_0}/\|f_{\lambda+}\|^2_{L^2} \le C(f) \tag{4.6}$$

for the f_λ-average of u_n, $n \in \mathbb{N}$. From (4.6) now we can deduce boundedness of (u_n).

Lemma 4.2. *For u_n as above there exists a uniform constant $C > 0$ such that*

$$\|u_n\|_{H^1} \le C,\ n \in \mathbb{N}. \tag{4.7}$$

Proof. By a variant of the Poincare inequality with a uniform constant $C = C(f_\lambda)$ we have

$$|\bar{u} - \bar{u}^{(f_\lambda)}| \le C\|\nabla u\|_{L^2} \tag{4.8}$$

for all $u \in H^1(M, g_0)$, where $\bar{u} = \int_M u\,d\mu_{g_0}$ is the average of u. In view of the Gauss-Bonnet Theorem for $u = u_n$ we have

$$\begin{aligned}E_{\mu_n}(u) = \frac{1}{2}\int_M \big(|\nabla u|^2_{g_0} + 2k_0 u - f e^{2u}\big)\,d\mu_{g_0} &= \frac{1}{2}\|\nabla u\|^2_{L^2} + k_0\bar{u} - \pi\chi(M)\\ &\ge \frac{1}{2}\|\nabla u\|^2_{L^2} + k_0\bar{u}^{(f_\lambda)} + k_0|\bar{u} - \bar{u}^{(f_\lambda)}| - \pi\chi(M)\\ &\ge \frac{1}{2}\|\nabla u\|^2_{L^2} - C\|\nabla u\|_{L^2} + k_0\bar{u}^{(f_\lambda)} - C.\end{aligned}$$

Also using (4.6) and (4.8) to bound

$$k_0\bar{u}^{(f_\lambda)} \geq |k_0||\bar{u}^{(f_\lambda)}| - C \geq |k_0||\bar{u}| - C|\bar{u} - \bar{u}^{(f_\lambda)}| - C \geq |k_0||\bar{u}| - C\|\nabla u\|_{L^2} - C$$

we find

$$E_{\mu_n}(u) \geq \frac{1}{2}\|\nabla u\|^2_{L^2} + |k_0||\bar{u}| - C\|\nabla u\|_{L^2} - C \geq \frac{1}{4}\|\nabla u\|^2_{L^2} + |k_0||\bar{u}| - C,$$

and our claim follows from the uniform bound $E_{\mu_n}(u_n) = c_{\mu_n} \leq c_\lambda$, $n \in \mathbb{N}$. □

Proof of Theorem 1.7 (completed). The same argument as in the proof of Proposition 3.4 now yields convergence of a subsequence $u_n \to u^\lambda$ in $H^1(M, g_0)$ as $n \to \infty$, and by continuity there holds $dE_\lambda(u^\lambda) = 0$ and $E_\lambda(u^\lambda) = \lim_{n\to\infty} c_{\mu_n} \geq \beta_\lambda > E_\lambda(u_\lambda)$; thus, $u^\lambda \neq u_\lambda$.

Moreover, u^λ cannot be a relative minimizer of E_λ; otherwise, by Theorem 1.5 the function u^λ would be a strict relative minimizer of E_λ in the sense of (1.4), and by continuity also u_n would be a strict relative minimizer of E_{μ_n}, contrary to assumption. □

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