Multiplicative processes, fast convolution, and pricing

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BCAM - Bilbao, 3 February 2012
Overview

- Scaling in finance
- Does scaling play any role in option pricing?
  \[ P^p(x, t) = \frac{1}{t^H} G \left( \frac{x}{t^H} \right), \text{ what about } P^q(x, t)? \]
- Markovian processes
- Fast convolution algorithm
  computing \( n \)-dimensional integrals with \( m \)-points equally spaced grids in \( O(n \times m \times \log_2 m) \)
- A class of quadratic diffusion processes
The first pillar: scaling in finance


The collapse analysis furnishes the scaling function $g$ reported as the full line and the scaling exponent $D \simeq 1/2$, where

$$p_T(r) = \frac{1}{T^D} g \left( \frac{r}{T^D} \right).$$

Figure 1: Data collapse for $p_T(r)$ ($T$ measured in days) sampled from a record of about $2.7 \times 10^4$ DJI daily closures. The collapse analysis furnishes the scaling function $g$ reported as the full line and the scaling exponent $D \simeq 1/2$, where $p_T(r) = \frac{1}{T^D} g \left( \frac{r}{T^D} \right)$. Figure taken from F. Baldovin and A. Stella, *Proc. Natl. Acad. Sci.* 104 (2007) p. 19741.
The second pillar: Markovian description of data


The probability density $p(x, \tau)$ obeys the Fokker Planck equation

$$-	au \frac{\partial p}{\partial \tau} = -\frac{\partial}{\partial x} (D_1(x, \tau)p) + \frac{\partial^2}{\partial x^2} (D_2(x, \tau)p) ,$$

while the stochastic process is generated by the Langevin equation (under Itô prescription)

$$-\tau dx(\tau) = D_1(x, \tau)d\tau + \sqrt{\tau}D_2(x, \tau)dW(\tau) .$$

For the time series under consideration we have $D_1(x, \tau) = -\gamma x$, $\gamma = 0.93 \pm 0.02$ and $D_2(x, \tau) = \alpha \tau + \beta x^2$, with $\alpha = 0.016 \pm 0.002$ and $\beta = 0.11 \pm 0.02$.


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*Figure 2:* Comparison of the numerical solutions of the Fokker-Planck equation (solid lines) with the pdfs obtained directly from the data (open symbols). The scales $\tau$ are (from top to bottom): $\tau = 12h, 4h, 1h, 15\text{min}$ and $4\text{min}$.
Combining the two: scaling Markovian processes


- Scaling
  \[ X_t = t^H X_1, \]
  where equality holds in distribution.
  It is readily proved that \( \langle X_t^n \rangle = c_n t^{nH} \) and \( p(x, t) = t^{-H} F(x/t^H) \).

- A Markov process generated locally by a driftless SDE
  \[
  dX = \sqrt{D(X, t)} dW(t)
  \]

- Scaling implies \( D(X, t) = t^{2H-1} D(X/t^H) \)

- E.g. \( D(X/t^H) = D_0 (1 + \epsilon X^2/t^{2H}) \)

**Figure 3:** Log-log plot of the distribution of \( X_t \) using a quadratic diffusion coefficient showing the emergence of power-law tails.
Return dynamics under objective measure $\mathbb{P}$

Piecewise diffusion process can be defined by means of the driftless SDE

$$dX_t = \sigma \sqrt{1 + \frac{\epsilon |X_t|}{\sqrt{t}}} dW_t^\mathbb{P}, \quad \text{with} \quad X_0 = 0.$$ 

The closed-form solution reads

$$P^\mathbb{P}(x, t) = \frac{e^{-\alpha}}{2\sigma^{2\alpha}\epsilon^{\alpha}} \Gamma[\alpha, \alpha\sqrt{t}]$$

$$\times \exp \left[ -\frac{|x|}{\sigma^2 \epsilon \sqrt{t}} \right] \left( 1 + \frac{\epsilon |x|}{\sqrt{t}} \right)^{\alpha - 1}$$

with $\alpha = 1/(\sigma^2 \epsilon^2)$ and $\Gamma[a, z] = \int_z^\infty s^{a-1} e^{-s} \, ds$.

Following a delta hedge strategy we can derive a Black&Scholes and Merton like PDE

$$\frac{\partial O}{\partial t} + r S_t \frac{\partial O}{\partial S_t} + \sigma^2 S_t^2 \sqrt{t} + \epsilon |\ln S_t - \ln S_0| \frac{\partial^2 O}{\partial S_t^2} - rO = 0$$

with final time condition $O(S_T, T) = (S_T - K)^+$. The fair price of a call option reads

$$O(S_0, t_0) = e^{-r(T-t_0)} \int_{-\infty}^{+\infty} dS_T (S_T - K)^+ G^Q(S_T, T; S_0, t_0)$$

where $G^Q$ is the Green function solving the Fokker-Planck equation.

$$dX_t = \left[ r - \frac{\sigma^2}{2} \left( 1 + \epsilon \frac{|X_t|}{\sqrt{t}} \right) \right] dt + \sigma \sqrt{1 + \epsilon \frac{|X_t|}{\sqrt{t}}} dW_t^Q, \quad X_{t_0} = 0$$

Risk neutrality removes scaling properties! ...What is the empirical evidence?
A numerical emergency exit

• Consider the generic process $X_{\tau}$, whose dynamics is described by the following general SDE

$$dX_{\tau} = M_X(X_{\tau}, \tau)d\tau + D_X(X_{\tau}, \tau)dW_{\tau}, \quad X_{\tau=0} = X_0.$$ 

• Introduce the Lamperti transform

$$Z_{\tau}(X_{\tau}, \tau) = \int_{X_0}^{X_{\tau}} \frac{d\hat{X}}{D_X(\hat{X}, \tau)}.$$ 

• Apply Itô Lemma to $Z_{\tau}(X_{\tau}, \tau)$, the dynamics of the $Z_{\tau}$ process becomes

$$dZ_{\tau} = M_Z(Z_{\tau}, \tau)d\tau + dW_{\tau}, \quad Z_{\tau=0} = 0,$$

with

$$M_Z(Z_{\tau}, \tau) = \frac{\tilde{M}_X(X(Z_{\tau}), \tau)}{\tilde{D}_X(X(Z_{\tau}), \tau)} + \frac{\partial}{\partial \tau} \int_{X_0}^{X(Z_{\tau})} \frac{d\hat{X}}{D_X(\hat{X}, \tau)} - \frac{1}{2} \frac{\partial}{\partial X} \tilde{D}_X(X(Z_{\tau}), \tau).$$
Fast convolution algorithm


The transition probability for a generic $\tau > 0$ can be written as a finite high dimensional integral

$$p(z^n|z^0) \simeq \int_{z^n} \int_{z^{n-1}} \cdots \int_{z^1} \prod_{i=1}^{n-1} dz_i \pi(z^n|z^{n-1})\pi(z^{n-1}|z^{n-2}) \cdots \pi(z^1|z^0),$$

where $\pi$ is the short time transition PDF which could be chosen equal to a Normal density

$$\pi(z^{i+1}|z^i) = \frac{1}{\sqrt{2\pi\Delta\tau}} \exp \left[ -\frac{(z^{i+1} - z^i - M_Z(z^i, \tau^i)\Delta\tau)^2}{2\Delta\tau} \right].$$

By means of the change of variables $\xi^i = z^i + M_Z(z^i, \tau^i)\Delta\tau$, the new transition becomes symmetric when exchanging the role of $z^{i+1}$ and $\xi^i$

$$\pi(z^{i+1}|z^i(\xi^i)) = \pi \left( (z^{i+1} - \xi^i)^2 \right)$$
Fast convolution algorithm

For every one-dimensional integration we have

\[
p(z^{i+1}|z^0) = \int_{z^i} dz^i \pi \left( (z^{i+1} - \xi^i(z^i))^2 \right) p(z^i|z^0) \simeq \Delta z \sum_{k=0}^{2m-1} C_{jk} P_k^i
\]

Computational complexity \(O(m \log_2(m))\): \( CP^i = \text{Re } \left[ F^{-1} \left( F(C^t_{1\text{st\ row}}) \cdot F(P^i) \right) \right] \)
Figure 5: Objective probability density functions of $Z_{\tau}$ at time $\tau = 1$, comparison between analytical expressions (dashed and dotted lines), Monte Carlo histograms (symbols), and output of FCA (solid lines) for $\sigma^2 = 1$ and $\epsilon = 0.5, 1, 2$. Log-linear curves in the Right Panel have been shifted for readability.
Figure 6: Risk neutral probability density functions of $Z_\tau$ at time $\tau = 1$, comparison between Monte Carlo histograms (bars and symbols), and output of FCA (solid lines) for $\sigma^2 = 1$ and $\epsilon = 0.5, 1, 2$. Log-linear curves in the Right Panel have been shifted for readability.
**Quadratic diffusion:** The microscopic equation


\[dX_t = \frac{aX_t + b}{g(t)} dt + \sqrt{\frac{cX_t^2 + dX_t + e}{g(t)}} dW_t,\]

\(W_t\) is the standard Brownian motion, \(a, b, c, d,\) and \(e\) are real constants, \(1/g(t)\) is a non negative smooth function of the time over \(D.\)

Applying Itô’s Lemma to \(f(X_t) = X_t^n\) and taking expectation, we obtain the linear ordinary differential equation (ODE) satisfied by the \(n\)-th order moment \(\mu_n(t) = \langle X_t^n \rangle\)

\[g(t) \frac{d}{dt} \mu_n(t) = F_n \mu_n(t) + A_n \mu_{n-1}(t) + B_n \mu_{n-2}(t),\]

with \(A_n, B_n,\) and \(F_n\) functions of the model parameters.

Previous equation simplifies introducing the monotonously increasing function

\[\tau(t) = \int_{t_0}^t 1/g(s) ds.\]
$g(t)$ affects the time scaling of the process

$$\mu_n(t) = \sum_{j=0}^{n} c_j e^{F_n-j\tau(t)}$$

- For a constant $g(t) = 1$, we have $\tau = t - t_0$ and the $n$-th order moment is characterized by the superposition of $n$ exponentials with time constants $\{1/|F_n|, \ldots, 1/|F_1|\}$.
- When $g(t) = t$, we have terms of the form
  $$e^{F_n-j\tau(t)} = t^{F_n-j} t_0^{-F_n-j},$$
  producing a power law time scaling of the moments.
- More generally, for $g(t) = t^\beta$ ($\beta \neq 1$) the time dependence turns out to be a stretched exponential with stretching exponent $1 - \beta$:
  $$e^{F_n-j\tau(t)} = e^{F_n-j^{\frac{1}{1-\beta}} \left(t^{1-\beta} - t_0^{1-\beta}\right)}.$$
Scaling: analytical vs Monte Carlo

Figure 7: (Left) Scaling of the moments for $a = b = 9.5 \times 10^{-2}$ and $c = d = e = 8.3 \times 10^{-2}$. (Right) Lowest order moments for $a = -20$, $b = d = e = 0.1$, $c = 4.5$, with $g = t^\beta$, $\beta = 2$ and $\hat{p}_t(x) = \delta(x)$. In the inset the last converging moment is compared to the first diverging one.
Approaching the stationary state

G.B. and Sofia Cazzaniga, Multiplicative noise, fast convolution, and pricing; arXiv:1107.1451v1 [q-fin.CP]

Figure 8: PDF of $Z_{\tau}$ for increasing values of $\tau(t) = \int_{t_{0}}^{t} ds/g(s)$, $a = -20$, $b = d = e = 0.1$, and $c = 4.5$; solid line corresponds to the analytical stationary solution, dashed ones to FCA, while bars in the Left Panel and symbols in the Right Panel to MC simulation.
Minimal model of financial stylized facts


\[
\begin{align*}
\mathrm{d}X_t &= \sqrt{c} \ Y_t \mathrm{d}W_t, \quad X_0 = 0 \\
\mathrm{d}Y_t &= (aY_t + b)\mathrm{d}t + \sqrt{c}Y_t \mathrm{d}W_{2,t}, \quad Y_0 = y_0
\end{align*}
\]

$Y_t$ relaxes toward an Inverse Gamma distribution

\[
p_{IG}(\sigma; \alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{e^{-\beta/\sigma}}{\sigma^{\alpha+1}}
\]

with \textbf{shape} $\alpha = 1 - \frac{2a}{c}$ and \textbf{scale} $\beta = \frac{2b}{c}$. This model reproduces the \textbf{excess of kurtosis} and the \textbf{skewed} nature of daily financial returns.

**Figure 9:** In log-linear scale, probability distributions for S&P500 returns (1970-2010), shifted for readability.
Local volatility or stochastic volatility?

Return dynamics under risk-neutral measure $\mathbb{Q}$

\[
dS_t = rS_t dt + \sigma S_t d\Omega_t, \quad S_{t_0} = S_0 \\
d\Omega_t = \Sigma(\Omega_t, t)dW_t^Q, \quad \Omega_{t_0} = \Omega_0
\]

with

\[
\Sigma(\Omega_t, t) = \begin{cases} 
A^{-\frac{\alpha}{2}} P(\Omega_t, t)^{-\frac{\alpha}{2}} & t > 0 \\
0 & t = 0
\end{cases}
\text{and } P(\Omega_t, t) = \frac{1}{N_t} (1 + \alpha \beta_t \Omega_t^2)^{-\frac{1}{\alpha}}
\]

with $\beta_t = [(1 - \alpha)(2 - \alpha)t]^{-\frac{2}{\alpha}}$, $N_t = A/\sqrt{\beta_t}$, $A = \sqrt{\frac{\pi}{\alpha}} \Gamma \left( \frac{1}{\alpha} - \frac{1}{2} \right) / \Gamma \left( \frac{1}{\alpha} \right)$, $\alpha \in (0, \frac{1}{2})$ and $t_0 \geq 0$. See Slide 4: \[d\Omega_t = \left( \frac{A}{N_t} \right)^{-\frac{\alpha}{2}} \sqrt{1 + \alpha \beta_t \Omega_t^2} dW_t\]
Fast convolution vs Monte Carlo

Figure 10: Risk neutral probability for $\alpha = 0.1$, $\Omega_0 = 0$, $t_0 = 0.2$, and $T - t_0 = 0.5$. 
Pricing and implied volatility surfaces

The log-return $X_T$ satisfies the equation

$$X_T = X_t + r(T-t) - \frac{1}{2}\sigma^2 \int_t^T \Sigma^2(s, \Omega_s) \, ds + \sigma(\Omega_T - \Omega_t),$$

for $T > t \geq t_0$. Switching to the integral time $\tau = \ln t/t_0$ for $t_0 > 0$, the Lamperti transform of $\Omega_\tau$ is readily computed

$$Z_\tau = \frac{1}{\sqrt{c}} \left[ \text{asinh} \left( C_{\alpha, t_0, \tau} \Omega_\tau \right) - \text{asinh} \left( C_{\alpha, t_0, \tau} \Omega_0 \right) \right],$$

where $C_{\alpha, t_0, \tau} = \sqrt{\alpha \beta t_0} \, e^{-\tau/(2-\alpha)}$.

According to pricing theory the price of a Plain Vanilla option is proportional to

$$\mathbb{E}^Q \left[ \left( e^{r(T-t_0) + \sigma[\Omega(Z'_\tau(T)) - \Omega(Z_0)]} - \frac{(2-\alpha)\sigma^2}{4} [e(T) - e(t_0)] - \frac{\sigma^2}{2} \int_0^T \Omega(Z_{\tau'})^2 \, d\tau' - e^k \right)^+ \mid S_0, Z_0 \right].$$
To compute previous expectation we define the set of ancillary variables \{U^1, \ldots, U^n\} satisfying the recursive relation
\[
U^{i+1} = U^i + \Delta \tau \Omega(Z^{i+1})^2,
\]
with \(U^1 = \Delta \tau \Omega(Z^1)^2\), and we provide the relation
\[
p^Q_{UZ}(u^i, z^{i+1}) = \int_{z^i} dz^i p^Q_{UZ}(u^i, z^i) \pi^Q(z^{i+1}|z^i).
\]

Figure 11: Risk neutral probability for \(\alpha = 0.1, \Omega_0 = 0, t_0 = 0.2\) and \(T - t_0 = 0.5\).
Figure 12: FCA implied volatility surfaces, $\alpha = 0.1$, $t_0 = 0.2$, $\Omega_0 = 0$ (Left), and $\Omega_0 = 0.5$ (Right); dashed lines for $T - t_0 = 0.5$ correspond to 95% Confidence Level from MC simulation.
Figure 13: FCA implied volatility surfaces, $\alpha = 0.4$, $t_0 = 0.2$, $\Omega_0 = 0$ (Left), and $\Omega_0 = 0.5$ (Right); dashed lines for $T - t_0 = 0.5$ correspond to 95% Confidence Level from MC simulation.