Existence of a weak solution for a moving boundary fluid-structure interaction problem in blood flow

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Motivation

- Motivation for analysis of FSI problems come from industrial (for example aeroelasticity) and biological problems (biomechanics-blood flow).
- We are interested in modeling blood flow in compliant vessel.
- It is complicated due to multi-scale and multi-physics nature of the problem
Description of the geometry

- \( \eta \) denote the vertical displacement of the deformable boundary.
- Since domain is symmetric we consider only upper half of the domain.
- Fluid domain at time \( t \):
  \[
  \Omega_\eta(t) = \{(z, r) : 0 < z < L, 0 < r < R + \eta(t, z)\}
  \]
- \( \Gamma(t) = \{(z, R + \eta(t, z)) : 0 < z < 1\} \) is deformable boundary.
- Longitudinal displacement is neglected.
Fluid and structure equations

• The fluid flow is governed by the incompressible Navier-Stokes equations:

\[ \rho_f (\partial_t u + u \cdot \nabla u) = \nabla \cdot \sigma, \quad \nabla \cdot u = 0, \text{ in } \Omega_\eta(t), \quad t \in (0, T), \]

• \( \rho_f \) is fluid density.

• \( \sigma = -pI + 2\mu D(u) \) fluid stress tensor, \( D(u) = \frac{1}{2} (\nabla u + \nabla^\top u) \).

• The structure is modeled by a cylindrical linearly viscoelastic Koiter shell model

\[ \rho_s h \partial_t^2 \eta + C_0 \eta - C_1 \partial_x^2 \eta + C_2 \partial_x^4 \eta + \\
D_0 \partial_t \eta - D_1 \partial_x^2 \partial_t \eta + D_2 \partial_x^4 \partial_t \eta = f. \]

• It is physiologically reasonable structure model.

• We present purely elastic case, i.e. \( D_0 = D_1 = D_2 = 0 \) (mathematically harder case).
Coupling conditions

- The fluid and structure are coupled through the kinematic and dynamic coupling conditions.
- Continuity of velocity (no-slip):
  \[ u(t, z, R + \eta(t, z)) = \partial_t \eta(t, z) e_r, \]
- Balance of contact forces:
  \[ f = -\sqrt{1 + (\partial_z \eta)^2} \sigma n \cdot e_z, \text{ on } \Gamma(t), \ t \in (0, T). \]
- Term \( \sqrt{1 + (\partial_z \eta)^2} \) is Jacobian of transformation from Eulerian to Lagrangian coordinates.
Boundary and initial conditions

- On the bottom part of the boundary $\Gamma_b$, we have symmetry boundary conditions:
  \[ u_r = \partial_r u_z = 0, \quad \text{on } \Gamma_b. \]

- The flow is driven by a prescribed dynamic pressure drop at the inlet and outlet boundaries: \[ p + \frac{1}{2}|u|^2 = P_{in/out}(t), \quad u \times n = 0, \quad \text{on } \Gamma_{in/out}. \]

- The structure is clamped: \[ \eta(0) = \partial_z \eta(0) = \eta(L) = \partial_z \eta(L) = 0. \]

- The system is supplemented with initial conditions
  \[ u(0, .) = u_0, \quad \eta(0, .) = \eta_0, \quad \partial_t \eta(0, .) = \nu_0. \]
Full FSI problem

Find \( u = (u_z(t, z, r), u_r(t, z, r), \rho(t, z, r), \) and \( \eta(t, z) \) such that

\[
\begin{align*}
\rho_f (\partial_t u + (u \cdot \nabla) u) &= \nabla \cdot \sigma \\
\nabla \cdot u &= 0
\end{align*}
\] in \( \Omega_\eta(t), \ t \in (0, T), \)

\[
\begin{align*}
\rho_s h \partial_t^2 \eta + C_0 \eta - C_1 \partial_z^2 \eta + C_2 \partial_z^4 \eta &= -J \sigma n \cdot e_r \\
\end{align*}
\] on \((0, T) \times (0, L),\)

\[
\begin{align*}
u_r &= 0, \\
\partial_r u_z &= 0
\end{align*}
\] on \((0, T) \times \Gamma_b,\)

\[
\begin{align*}
p + \frac{\rho_f}{2} |u|^2 &= P_{in/out}(t), \\
u_r &= 0
\end{align*}
\] on \((0, T) \times \Gamma_{in/out},\)

\[
\begin{align*}
u(0, .) &= \mathbf{u}_0, \\
\eta(0, .) &= \eta_0, \\
\partial_t \eta(0, .) &= \nu_0
\end{align*}
\] at \( t = 0. \)
Description of the result

We prove existence of a weak solution of $2D - 1D$ FSI problem in blood flow.

- We use pressure inlet and outlet boundary conditions which introduce some technical difficulties.

- The most important novelty of this work is related to the method of proof. The proof is based on a semi-discrete, operator splitting Lie scheme, which was used by Guidoboni, Čanić et al. (’09) for a design of a stable, loosely coupled numerical scheme, called the kinematically coupled scheme. Therefore, in this work, we effectively prove that the kinematically coupled scheme converges to a weak solution of the underlying FSI problem.
History

Existence result for FSI problems in various settings - Conca, San Martín, Tucsnak ('99), Desjardins, Esteban ('99), San Martín, Starovoitov, Tucsnak ('02), Desjardins, Esteban, Grandmont, Le Tallec ('03), M. Boulakia ('03), Feireisl ('03), Takahashi ('03), Barbu, Grujić, Lasiecka, Tuffaha ('08), Houot, Munnier ('08), Galdi, Kyed ('09), Houot, San Martin, Tucsnak ('10), Guidoboni, Guidorzi, Padula ('12), …

Existence results for moving boundary problem for Navier-Stokes coupled with elasticity:

- **Strong solution**: Beirão da Veiga ('04), Lequeurre ('11, '12), Cheng and Shkoller ('10), Coutand and Shkoller. ('05, '06), Kukavica and Tuffaha ('12)

- **Weak solutions**: Chambolle, Desjardins, Esteban, and Grandmont ('05), Grandmont ('08)
Energy inequality

By formally taking solution \((u, \partial_t \eta)\) as test function in the weak formulation we get following energy estimate:

\[
\frac{d}{dt} E + D \leq C(P_{in/out}),
\]

where

\[
E = \frac{\rho_f}{2} \| u \|_{L^2(\Omega)}^2 + \frac{\rho_s h}{2} \| \eta_t \|_{L^2(\Gamma)}^2 + \frac{1}{2} \left( C_0 \| \eta \|_{L^2}^2 + C_1 \| \partial_z \eta \|_{L^2}^2 + C_2 \| \partial_z^2 \eta \|_{L^2}^2 \right),
\]

\[
D = \mu \| D(u) \|_{L^2(\Omega)}^2.
\]
Numerical schemes

- Roughly speaking, numerical schemes for FSI problems can be divided into: **monolithic** and **partitioned**.
- **Monolithic** schemes are stable, but computationally expensive and non-modular.
- Partitioned schemes can be divided into **Loosely coupled** and **Strongly coupled**.
- **Strongly coupled** require several sub-iterations between fluid and structure solver and therefore are computationally very expensive.
Loosely coupled schemes

- Require only one fluid and only one structure solver in each time-step.
- Popular because of modularity and lower computational cost.
- Unconditionally unstable for some combination of physical and geometrical parameters (which are realistic in blood flow) (Causin, Gerbeau, Nobile ’05).
- Problem: the discrete energy does not mimic the energy of the continuous problem.
- For light structures, the miss-match is large, which causes stability issues.
ALE formulation on reference domain

- We want to rewrite problem in the reference configuration $\Omega = (0, L) \times (0, 1)$.
- Since we consider control domain, we cannot use Lagrangian coordinates.
- We use ALE mapping $A_\eta(t) : \Omega \rightarrow \Omega_\eta(t)$,

$$A_\eta(t)(\tilde{x}, \tilde{z}) = \left( \begin{array}{c} \tilde{x} \\ R + \eta(t, x) \tilde{z} \end{array} \right), \quad (\tilde{x}, \tilde{z}) \in \Omega.$$

- We have a problem on a fixed domain, but with coefficients that depend on the solution.
- Test functions still depend on the solution (because of divergence-free condition).
Lie (Marchuk-Yanenko) operator splitting scheme

- We consider initial value problem $\frac{d}{dt}\phi + A(\phi) = 0$, $\phi(0) = \phi_0$.
- We suppose that $A = A_1 + A_2$.
- Let $k = T/N$ be time-discretization step and $t_n = nk$. Then we define:

$$\frac{d}{dt}\phi_{n+\frac{i}{2}} + A_i(\phi_{n+\frac{i}{2}}) = 0 \quad \text{in} \quad (t_n, t_{n+1}),$$

$$\phi_{n+\frac{i}{2}}(t_n) = \phi^{n+\frac{i-1}{2}}, \quad n = 0, \ldots, N - 1, \quad i = 1, 2,$$

where $\phi^{n+\frac{i}{2}} = \phi_{n+\frac{i}{2}}(t_{n+1})$.

- To apply Lie scheme, we must rewrite original problem as first order problem. Therefore we introduce new unknown, structure velocity $v = \partial_t \eta$. 


First order ALE formulation

Find $u(t, \tilde{z}, \tilde{r}), p(t, \tilde{z}, \tilde{r}), \eta(t, \tilde{z})$, and $\nu(t, \tilde{z})$ such that:

$$
\rho_f \left( \partial_t u + \left( \left( u - \mathbf{w}^\eta \right) \cdot \nabla^\eta \right) u \right) = \nabla^\eta \cdot \sigma^\eta, \quad \nabla^\eta \cdot u = 0, \quad \in (0, T) \times \Omega,
$$

$$
u_r = 0, \quad \partial_r u_z = 0 \quad \on (0, T) \times \Gamma_b,
$$

$$p + \rho_f \frac{1}{2} |u|^2 = P_{in/out}(t), \quad u_r = 0 \quad \on (0, T) \times \Gamma_{in/out},
$$

$$u = \nu e_r, \quad \partial_t \eta = \nu, \quad \rho_s h \partial_t \nu + C_0 \eta - C_1 \partial_z^2 \eta + C_2 \partial_z^4 \eta = -J \sigma \mathbf{n} \cdot \mathbf{e}_z, \quad \on (0, T) \times (0, L),
$$

$$u(0, .) = u_0, \eta(0, .) = \eta_0, \nu(0, .) = \nu_0, \quad \at t = 0.$$
Semi-Discretization

- We use Semi-Discretization in time, i.e. we discretize only time variable $t$.
- Scheme is designed in such a way that we get semi-discrete energy inequalities (analogous to continuous case).
- That guarantees stability of the numerical scheme (which is not a case for classical loosely coupled schemes for FSI problems).
- In every subset scheme is implicit. It is crucial for stability!
Introduction

Kinematically coupled scheme

Existence proof

Stability of $\beta$-scheme

Numerical examples

**Step 1 - Elastodynamics**

Given $(u^n, \eta^n, v^n)$ from previous step, find $(u^{n+\frac{1}{2}}, v^{n+\frac{1}{2}}, \eta^{n+\frac{1}{2}})$ such that:

$$
\begin{align*}
    u^{n+\frac{1}{2}} &= u^n, \quad \text{in } \Omega \\
    \eta^{n+\frac{1}{2}} - \eta^n &= v^{n+\frac{1}{2}} \quad \text{on } (0, L) \\
    \frac{\rho_s h v^{n+\frac{1}{2}} - v^n}{\Delta t} + C_0 \eta^{n+\frac{1}{2}} - C_1 \partial_z^2 \eta^{n+\frac{1}{2}} + C_2 \partial_z^4 \eta^{n+\frac{1}{2}} &= 0 \quad \text{on } (0, L), \\
    \eta^{n+\frac{1}{2}}(0) &= \partial_z \eta^{n+\frac{1}{2}}(0) = \eta^{n+\frac{1}{2}}(L) = \partial_z \eta^{n+\frac{1}{2}}(L) = 0.
\end{align*}
$$

(1)
**Step 2 - Fluid + Structure inertia**

find \((\mathbf{u}^{n+1}, \mathbf{v}^{n+1}, \eta^{n+1})\) such that:

\[
\begin{align*}
\rho_f \left( \frac{\mathbf{u}^{n+1} - \mathbf{u}^{n+\frac{1}{2}}}{\Delta t} + \left( (\mathbf{u}^n - \mathbf{w}^{n+\frac{1}{2}}) \cdot \nabla \eta^n \right) \mathbf{u}^{n+1} \right) &= \nabla \eta^n \cdot \sigma^n (\mathbf{u}^{n+1}), \\
\nabla \eta^n \cdot \mathbf{u}^{n+1} &= 0 \quad \text{in } \Omega, \\
\mathbf{u}_r^{n+1} &= 0, \quad \partial_r \mathbf{u}_z^{n+1} = 0 \quad \text{on } \Gamma_b, \\
p^{n+1} + \rho_f \left| \mathbf{u}^{n+1} \right|^2 &= P_{\text{in/out}}(t), \quad \mathbf{u}_r^{n+1} = 0 \quad \text{on } \Gamma_{\text{in/out}}, \\
\mathbf{u}^{n+1} &= \mathbf{v}^{n+1} \mathbf{e}_r \quad \text{on } (0, L), \\
\eta^{n+1} &= \eta^{n+\frac{1}{2}} \quad \text{in } (0, L), \\
\rho_s h \left( \mathbf{v}^{n+1} - \mathbf{v}^{n+\frac{1}{2}} \right) &= J^{n+1} \sigma \mathbf{n} \cdot \mathbf{e}_r \quad \text{on } (0, L).
\end{align*}
\]
Few remarks

• $\nabla^n$ is gradient in ALE coordinates:

$$\nabla^n = \begin{pmatrix}
\partial_z - \tilde{r} \frac{\partial_z \eta}{R + \eta} \partial_{\tilde{r}} \\
\frac{1}{R + \eta} \partial_{\tilde{r}}
\end{pmatrix}.$$

• $\mathbf{w}_n = \partial_t \eta \tilde{r} \mathbf{e}_r$ is domain (ALE) velocity.

• In numerical simulations, one can use the ALE transformation $A_{\eta^n}$ to “transform” the problem back to domain $\Omega_{\eta^n}$ and solve it there, thereby avoiding the un-necessary calculation of the transformed gradient. The ALE velocity is the only extra term that needs to be included with that approach.

• Step 2 can be viewed stationary Navier-Stokes-like problem on a fixed domain, coupled with the structure inertia through the Robin-type boundary condition. This is crucial for stability of the scheme.
Weak solution on moving domain

1. \( u \in L^\infty(L^2) \cap L^2(H^1), \eta \in L^\infty(H^2) \cap W^{1,\infty}(L^2), \)

2. \( \nabla \cdot u = 0, u(t, z, R + \eta(t, z)) = \partial_t \eta(t, z)e_r, \)

3. for every \((q, \psi)\) such that \( q(t, z, R + \eta(t, z)) = \psi(t, z)e_r, \nabla \cdot q = 0 \) following equality holds:

\[
\rho_f \int_0^T \int_{\Omega_\eta(t)} \left( -u \cdot \partial_t q + \frac{1}{2}(u \cdot \nabla)u \cdot q - \frac{1}{2}(u \cdot \nabla)q \cdot u \right)
\]

\[
+2\mu \int_0^T \int_{\Omega_\eta(t)} D(u) : D(q) - \frac{\rho_f}{2} \int_0^T \int_0^L (\partial_t \eta)^2 \psi - \rho s h_s \int_0^T \int_0^L \partial_t \eta \partial_t \psi
\]

\[
+ \int_0^T \int_0^L \left( C_0 \eta \psi + C_1 \partial_z \eta \partial_z \psi + C_2 \partial_z^2 \eta \partial_z^2 \psi \right) = \pm \int_0^T P_{\text{in}/\text{out}} \int_{\Gamma_{\text{in}/\text{out}}} q_z
\]

\[
+ \rho_f \int_{\Omega_{\eta_0}} u_0 \cdot q(0) + \rho s h \int_0^L v_0 \psi(0).
\]
Weak solution of the reference domain

We say that \((u, \eta)\) is weak solution of

1. \(u \in L^\infty(L^2) \cap L^2(H^1), \eta \in L^\infty(H^2) \cap W^{1,\infty}(L^2),\)
2. \(\nabla^\eta \cdot u = 0, \ u(t, z, 1) = \partial_t \eta(t, z)e_r,\)
3. for every \((q, \psi)\) such that \(q(t, z, 1) = \psi(t, z)e_r, \ \nabla^\eta \cdot v = 0\) following equality holds:

\[-\rho_f \left( \int_0^T \int_\Omega (R + \eta)u \cdot \partial_t q + \frac{1}{2} \int_0^T \int_\Omega (R + \eta) ((u - w^\eta) \cdot \nabla^\eta)u \cdot q \right) \]

\[-((u - w^\eta) \cdot \nabla^\eta)q \cdot u)) + 2\mu \int_0^T \int_\Omega (R + \eta)D^\eta(u) : D^\eta(q) \]

\[-\rho_f \int_0^T \int_\Omega (\partial_t \eta)u \cdot q - \rho_s h \int_0^T \int_0^L \partial_t \eta \partial_t \psi + \int_0^T \int_0^L (C_0 \eta \psi + C_1 \partial_z \eta \partial_z \psi \]

\(+ C_2 \partial^2_z \eta \partial^2_z \psi) = \pm R \int_0^T P_{in/out} \int_{\Gamma_{in/out}} q_z + \rho_f \int_{\Omega_{\eta_0}} u_0 \cdot q(0) + \rho_s h \int_0^L v_0 \psi(0)\]
Weak formulation of Step 1

\[ u^{n + \frac{1}{2}} = u^n. \]

Then we define \((v^{n + \frac{1}{2}}, \eta^{n + \frac{1}{2}}) \in H_0^2(0, L) \times H_0^2(0, L)\) as a solution of the following problem, written in weak form:

\[
\int_0^L \frac{\eta^{n + \frac{1}{2}} - \eta^n}{\Delta t} \phi = \int_0^L v^{n + \frac{1}{2}} \phi, \quad \phi \in L^2(0, L),
\]

\[
\rho_s h \int_0^L \frac{v^{n + \frac{1}{2}} - v^n}{\Delta t} \psi
\]

\[
+ \int_0^T \int_0^L (C_0 \eta^{n + \frac{1}{2}} \psi + C_1 \partial_z \eta^{n + \frac{1}{2}} \partial_z \psi + C_2 \partial^2_z \eta^{n + \frac{1}{2}} \partial^2_z \psi) = 0, \quad \psi \in H_0^2(0, L).
\]
Weak formulation of Step 2

for every \((q, \psi)\), such that \(q(t, z, 1) = \psi(t, z)\), \(\nabla \eta^n \cdot q = 0\)

\[
\rho_f \int_\Omega (R + \eta^n) \left( \frac{u^{n+1} - u^{n+\frac{1}{2}}}{\Delta t} \right) \cdot q + \frac{1}{2} \left[ (u^n - v^{n+\frac{1}{2}} \text{re}_r) \cdot \nabla \eta^n \right] u^{n+1} \cdot q
\]

\[
- \frac{1}{2} \left[ (u^n - v^{n+\frac{1}{2}} \text{re}_r) \cdot \nabla \eta^n \right] q \cdot u^{n+1} + \frac{\rho_f}{2} \int_\Omega v^{n+\frac{1}{2}} u^{n+1} \cdot q
\]

\[
+ 2\mu \int_\Omega (R + \eta^n) \mathbf{D} \eta^n (u) : \mathbf{D} \eta^n (q)
\]

\[
+ \rho_s h_s \int_0^L \frac{v^{n+1} - v^{n+\frac{1}{2}}}{\Delta t} \psi = R \left( P_{in}^n \int_{\Gamma_{in}} q_z - P_{out}^n \int_{\Gamma_{out}} q_z \right),
\]

with \(\nabla \eta^n \cdot u^{n+1} = 0\), \(u_{|\Gamma}^{n+1} = v^{n+1} \text{e}_r\), \(\eta^{n+1} = \eta^{n+\frac{1}{2}}\).
Proposition

Let $\Delta t > 0$, and assume that $\eta^n$ are such that $R + \eta^n \geq R_{\text{min}} > 0$, $n = 0, \ldots, N$. Then, the fluid sub-problem defined has a unique weak solution $(u^{n+1}, v^{n+1})$.

- Proposition is proved with usage of Lax-Milgram Lemma.
- Energy is preserved:

$$\frac{\rho f}{2} \left( \int_{\Omega} (R + \eta^n)|u^{n+1}|^2 + \Delta t v^{n+1/2} |u^{n+1}|^2 \right) = \frac{\rho f}{2} \int_{\Omega} (R + \eta^{n+1})|u^{n+1}|^2.$$
Lemma
(The Uniform Energy Bounds) Let $\Delta t > 0$ and $N = T / \Delta t > 0$. There exists a constant $C > 0$ independent of $\Delta t$ (and $N$), such that the following estimates hold:

1. $E_{n+1}^{n+\frac{1}{2}} \leq C, E_{n}^{n+1} \leq C$, for all $n = 0, ..., N - 1$,

2. $\sum_{j=1}^{N} D_{N}^{j} \leq C$,

3. $\sum_{n=0}^{N-1} \left( \int_{\Omega} (R + \eta^{n})|u^{n+1} - u^{n}|^{2} + \|v^{n+1} - v^{n+\frac{1}{2}}\|^{2}_{L^{2}(0,L)} + \|v^{n+\frac{1}{2}} - v^{n}\|^{2}_{L^{2}(0,L)} \right) \leq C$,

4. $\sum_{n=0}^{N-1} \left( C_{0}\|\eta^{n+1} - \eta^{n}\|^{2}_{L^{2}(0,L)} + C_{1}\|\partial_{z}(\eta^{n+1} - \eta^{n})\|^{2}_{L^{2}(0,L)} + C_{2}\|\partial_{z}^{2}(\eta^{n+1} - \eta^{n})\|^{2}_{L^{2}(0,L)} \right) \leq C$.  

Discrete energy inequality
Discrete energy inequality II

\[ E_{N}^{n+\frac{1}{2}}, E_{N}^{n+1}, \text{and } D_{N}^{i} \text{ are discrete kinetic energy and dissipation:} \]

\[ E_{N}^{n+\frac{i}{2}} = \frac{1}{2} \left( \rho_{f} \int_{\Omega} (R + \eta^{n}) |u_{N}^{n+\frac{i}{2}}|^{2} + \rho_{s} h_{s} \|v_{N}^{n+\frac{i}{2}}\|_{L^{2}(0,L)}^{2} \right. \]

\[ + C_{0} \|\eta_{N}^{n+\frac{i}{2}}\|_{L^{2}(0,L)}^{2} + C_{1} \|\partial_{z}\eta_{N}^{n+\frac{i}{2}}\|_{L^{2}(0,L)}^{2} + C_{2} \|\partial_{z}^{2}\eta_{N}^{n+\frac{i}{2}}\|_{L^{2}(0,L)}^{2} \right), \]

\[ D_{N}^{n+1} = \Delta t \mu \int_{\Omega} (R + \eta^{n}) |D^{n}(u_{N}^{n+1})|^{2}, \quad n = 0, \ldots, N, \quad i = 0, 1. \]

\[ C \text{ depends only on the parameters in the problem, on the kinetic energy of the initial data } E_{0}, \text{ and on the energy norm of the inlet and outlet data } \|P_{in/out}\|_{L^{2}(0,T)}^{2}. \]
Approximate solutions

- We define approximate solutions as piece-wise constant in time:

\[ u_N(t, \cdot) = u^n_N, \; \eta_N(t, \cdot) = \eta^n_N, \; v_N(t, \cdot) = v^n_N, \; v^*_N(t, \cdot) = v^{n-\frac{1}{2}}_N, \]

\[ t \in ((n-1)\Delta t, n\Delta t], \; n = 1 \ldots N. \]

- We show that approximate solutions are well defined for small \( T \), i.e. \( R + \eta_N > 0 \).

- Discrete energy inequality implies:
  1. Sequence \((\eta_N)_{n \in \mathbb{N}}\) is uniformly bounded in \( L^\infty(0, T; H^2_0(0, L)) \).
  2. Sequence \((v_N)_{n \in \mathbb{N}}\) is uniformly bounded in \( L^\infty(0, T; L^2(0, L)) \).
  3. Sequence \((v^*_N)_{n \in \mathbb{N}}\) is uniformly bounded in \( L^\infty(0, T; L^2(0, L)) \).
  4. Sequence \((u_N)_{n \in \mathbb{N}}\) is uniformly bounded in \( L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)) \).
Weak and weak* limits

Lemma

(Weak and weak* convergences) There exist subsequences \((\eta_N)_{N \in \mathbb{N}}, (\nu_N)_{N \in \mathbb{N}}, (\nu^*_N)_{N \in \mathbb{N}}, \) and \((u_N)_{N \in \mathbb{N}},\) and the functions
\[
\eta \in L^\infty(0, T; H^2_0(0, L)), \quad \nu \in L^\infty(0, T; L^2(0, L)), \quad \nu^* \in L^\infty(0, T; L^2(0, L)), \quad \text{and} \quad u \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)),
\]
such that
\[
\begin{align*}
\eta_N & \rightharpoonup^* \eta \quad \text{in} \quad L^\infty(0, T; H^2_0(0, L)), \\
\nu_N & \rightharpoonup^* \nu \quad \text{in} \quad L^\infty(0, T; L^2(0, L)), \\
\nu^*_N & \rightharpoonup^* \nu^* \quad \text{in} \quad L^\infty(0, T; L^2(0, L)), \\
u_N & \rightharpoonup \nu \quad \text{weakly} \quad \text{in} \quad L^2(0, T; H^1(\Omega)), \\
u_N & \rightharpoonup \nu \quad \text{weakly} \quad \text{in} \quad L^2(0, T; H^1(\Omega)).
\end{align*}
\]
Furthermore,
\[
\nu = \nu^*.
\]
Strong convergence

- Since problem is non-linear, we need strong convergence to show that obtained limits are weak solution of considered problem.
- Again we use discrete energy inequality, more precisely:

\[
\sum_{n=0}^{N-1} \left( \int_{\Omega} (R + \eta^n) |u^{n+1} - u^n|^2 + \| v^{n+1} - v^{n+\frac{1}{2}} \|^2_{L^2(0,L)} \\
+ \| v^{n+\frac{1}{2}} - v^n \|^2_{L^2(0,L)} \right) \leq C.
\]

- We multiply last inequality by \( \Delta t \) and take into account definition of approximate solutions to get:

\[
\| \tau_{\Delta t} u_N - u_N \|^2_{L^2((0,T) \times \Omega)} + \| \tau_{\Delta t} v_N - v_N \|^2_{L^2((0,T) \times (0,L))} \leq C \Delta t,
\]

where \( \tau_h f(t,.):= f(t-h,.) \), \( h \in \mathbb{R} \), is translation in time.
Strong convergence II

- Remember that $\Delta t = \frac{T}{N}$. Therefore, previous inequality is not uniform in $N$.
- However, using the fact that previous equality is valid for every $\Delta t$ (i.e. $N$), one can show following theorem:

**Theorem**

Sequences $(v_N)_{N \in \mathbb{N}}, (u_N)_{N \in \mathbb{N}}$ are relatively compact in $L^2(0, T; L^2(0, L))$ and $L^2(0, T; L^2(\Omega))$ respectively.
Strong convergence of $\eta_N$

- First we remember that \[ \frac{\eta^{n+1} - \eta^n}{\Delta t} = \frac{\eta^{n+1/2} - \eta^n}{\Delta t} = v^{n+\frac{1}{2}}. \]

- We now use the following result on continuous embeddings:

\[ L^\infty(0, T; H^2_0(0, L)) \cap W^{1,\infty}(0, T; L^2(0, L)) \hookrightarrow C^{0,1-\alpha}([0, T]; H^{2\alpha}(0, L)), \]

for $0 < \alpha < 1$.

- Finally, we use Arzelà-Ascoli Theorem to get:

\[ \eta_N \rightarrow \eta \text{ in } L^\infty(0, T; H^s(0, L)), 0 < s < 2. \]

- Specially, for $s > \frac{3}{2}$:

\[ \eta_N \rightarrow \eta \text{ in } L^\infty(0, T; C^1[0, L]). \]
Convergence of approximate solution

Let us now summarize obtained convergence results:

\[
\begin{align*}
\eta_N & \rightarrow \eta \text{ weakly}^{*} \text{ in } L^\infty (0, T; H^2_0 (0, L)) \\
v_n & \rightarrow v \text{ weakly}^{*} \text{ in } L^\infty (0, T; L^2 (0, L)) \\
u_N & \rightarrow u \text{ weakly}^{*} \text{ in } L^\infty (0, T; L^2 (\Omega)) \\
u_N & \rightarrow u \text{ weakly in } L^2 (0, T; H^1 (\Omega)) \\
u_N & \rightarrow u \text{ in } L^2 (0, T; L^2 (\Omega)), \\
u_N & \rightarrow v \text{ in } L^2 (0, T; L^2 (0, L)), \\
\eta_N & \rightarrow \eta \text{ in } L^\infty (0, T; H^s (0, L)), \ 0 \leq s < 2.
\end{align*}
\]
Test functions

- Before passing to the limit, we need to deal with one more difficulty, namely, test functions depends on $N$!
- More precisely, test function in step 2 must satisfy transformed divergence free condition: $\nabla \eta^n \cdot q = 0$.
- Therefore one can not pass to the limit directly.
- We construct dense subset $\mathcal{X}$ of test functions on original domain such that every test function $q \in \mathcal{X}$ can be transform to reference domain via ALE mapping $A_{\eta_N}$, for $N \geq N_0(v)$.
- Therefore we use test functions $q_N(t,.) := q(t,.) \circ A_{\tau_{\Delta t}\eta_N}(t)$.
- We use fact that $\eta_N \rightarrow \eta$ in $C([0, T]; C^1[0, L])$ and $\eta_N(x) \geq r_{min}$, $N \in \mathbb{N}$ (this can be ensured by taking $T$ small enough).
Passing to the limit

$$\rho_f \int_0^T \int_\Omega \left( R + \tau \Delta t \eta_N \right) \left( \partial_t \tilde{u}_N \cdot q_N + \frac{1}{2} \left( \tau \Delta t u_N - w_N \right) \cdot \nabla^{\tau \Delta t \eta_N} u_N \cdot q_N \right)$$

$$- \frac{1}{2} \left( \tau \Delta t u_N - w_N \right) \cdot \nabla^{\tau \Delta t \eta_N} q_N \cdot u_N + \frac{\rho_f}{2} \int_\Omega v^*_N u_N \cdot q_N$$

$$+ \int_\Omega \left( R + \tau \Delta t \eta_N \right) 2 \mu D^{\tau \Delta t \eta_N} (u_N) : D^{\tau \Delta t \eta_N} (q_N) + \rho_s h_s \int_0^T \int_0^L \partial_t \tilde{v}_N \psi$$

$$+ \int_0^T \int_0^L \left( C_0 \eta_N \psi + C_1 \partial_z \eta_N \partial_z \psi + C_2 \partial_z^2 \eta_N \partial_z^2 \psi \right)$$

$$= R \left( \int_0^T P_{in}^N dt \int_0^R q_z(t, 0, r) dr - \int_0^T P_{out}^N dt \int_0^R q_z(t, L, r) dr \right),$$
Passing to the limit II

• Using the convergence results obtained for the approximate functions, and for the test functions \( q_N \), we can pass to the limit directly in all the terms except in the term that contains \( \partial_t \tilde{u}_N \).

• To deal with this term we notice that, since \( q_N \) are smooth on sub-intervals \( (j\Delta t, (j + 1)\Delta t) \), we can use integration by parts on these sub-intervals.

• After some calculation and using obtained strong convergence properties we get:

\[
\int_0^T \int_\Omega \left( R + \tau_{\Delta t} \eta_N \right) \partial_t \tilde{u}_N \cdot q_N \to - \int_0^T \int_\Omega \left( R + \eta \right) u \cdot \partial_t \tilde{q} - \int_0^T \int_\Omega \partial_t \eta u \cdot \tilde{q} - \int_\Omega \left( R + \eta_0 \right) u_0 \cdot \tilde{q}(0),
\]

where \( \tilde{q} = q \circ A_{\eta} \).
Main theorem

Theorem
Let $\varrho_f$, $\varrho_s$, $\mu$, $h_s$, $C_i > 0$, $D_i \geq 0$, $i = 1, 2, 3$. Suppose that the initial data $v_0 \in L^2(0, L)$, $u_0 \in L^2(\Omega_{\eta_0})$, and $\eta_0 \in H^2_0(0, L)$ is such that $(R + \eta_0(z)) > 0$, $z \in [0, L]$. Furthermore, let $P_{in}$, $P_{out} \in L^2_{\text{loc}}(0, \infty)$.

Then there exist $T > 0$ and a weak solution of $(u, \eta)$ of Considered FSI problem has at least one weak solution on $(0, T)$, which satisfies the following energy estimate:

$$E(t) + \int_0^t D(\tau) d\tau \leq E_0 + C\left(\|P_{in}\|_{L^2(0, T)}^2 + \|P_{out}\|_{L^2(0, T)}^2\), \quad t \in [0, T],$$

where $C$ depends only on the coefficients, and $E(t)$ and $D(t)$ are given by
Main theorem II

\[ E(t) = \frac{\rho_f}{2} \|u\|_{L^2(\Omega_\eta(t))}^2 + \frac{\rho_s h}{2} \|\partial_t \eta\|_{L^2(0,L)}^2 \]

\[ + \frac{1}{2} (C_0 \|\eta\|_{L^2(0,L)}^2 + C_1 \|\partial_z \eta\|_{L^2(0,L)}^2 + C_2 \|\partial_z^2 \eta\|_{L^2(0,L)}^2), \]

\[ D(t) = \mu \|D(u)\|_{L^2(\Omega_\eta(t))}^2 \]

Furthermore, one of the following is true:

1. \( T = \infty \),

2. \( \lim_{t \to T} \min_{z \in [0,L]} (R + \eta(z)) = 0. \)
Kinematically coupled $\beta$-scheme

- Recently, M. Bukač, S. Čanić, et. al. introduced modification of the kinematically coupled scheme, called ‘kinematically coupled $\beta$-scheme’
- It is stable and has increased consistency and accuracy - comparable to the monolithic scheme
- Retains advantages of loosely coupled schemes - modularity and lower computational cost.
- $\beta$ is parameter used to distribute the fluid pressure between the fluid and structure sub-problems, $0 \leq \beta \leq 1$
Scheme description

• Since pressure is dominant term in fluid force, idea is to load the structure with pressure from fluid step.

\[ \sigma n = \sigma n \cdot e_r + \beta p n \cdot e_r - \beta p n \cdot e_r, \]

\[ (I) \]

\[ (II) \]

• Part I is taken into account in fluid step, where pressure term is taken explicitly from previous step.

\[ Part I = (\sigma^{n+1} + \beta p^n)n \cdot e_r. \]

• Pressure in Part II is taken from fluid step.

• By analysis of simplified problem, we show that
Stability result

Theorem

Provided that $0 \leq \beta \leq 1$, the kinematically coupled $\beta$-scheme is unconditionally stable.

- Joined work with S. Čanić (UH) and M. Bukač (UH & U. Pittsburg)
- Proof by analysis of simplified FSI problem introduced by Causin, Gerbeau, Nobile’05
- This simplified problem retains the main difficulties associated with the “added mass effect” which is responsible for the loss of stability in “classical” loosely coupled schemes applied to blood flow.
Simplified problem

\begin{align*}
\rho_f \frac{\partial u}{\partial t} + \nabla p &= 0 & \text{in } \Omega \times (0, T), \\
\nabla \cdot u &= 0 & \text{in } \Omega \times (0, T), \\
u \cdot n &= 0 & \text{on } \Gamma_b \times (0, T), \\
p &= p_{\text{in/out}}(t) & \text{on } \Gamma_{\text{in/out}} \times (0, T), \\
u \cdot n &= \frac{\partial \eta}{\partial t} & \text{on } \Gamma \times (0, T), \\
p &= \rho_{sh} \frac{\partial^2 \eta}{\partial t^2} + C_0 \eta - C_1 \frac{\partial^2 \eta}{\partial z^2} & \text{on } \Gamma \times (0, T).
\end{align*}

This is linear problem on the fixed domain!
THE SPLITTING

KINEMATICALLY COUPLED SCHEME

**FLUID**

\[
\begin{align*}
\rho_f \frac{\partial u}{\partial t} + \nabla p &= 0 & \text{in } \Omega \times (t_n, t_{n+1}) \\
\nabla \cdot u &= 0 & \text{in } \Omega \times (t_n, t_{n+1}) \\
u \cdot n &= 0 & \text{on } \Gamma_b \times (t_n, t_{n+1}) \\
p &= p_{\text{in/out}} & \text{on } \Gamma_{\text{in/out}} \times (t_n, t_{n+1}) \\
p - \beta p^n &= \rho_s h \frac{\partial u}{\partial t} & \text{on } \Gamma \times (t_n, t_{n+1}) \\
\text{Solution: } u(t_{n+1}) &= u^{n+1}, p(t_{n+1}) &= p^{n+1}
\end{align*}
\]

**STRUCTURE**

\[
\begin{align*}
\frac{\partial^2 \eta}{\partial t^2} + C_0 \eta - C_1 \frac{\partial^2 \eta}{\partial z^2} &= \beta p^{n+1} & \text{on } \Gamma \times (t_n, t_{n+1}) \\
u_r &= \frac{\partial \eta}{\partial t} & \text{on } \Gamma \times (t_n, t_{n+1})
\end{align*}
\]

CLASSICAL LOOSELY COUPLED

**FLUID**

\[
\begin{align*}
\rho_f \frac{\partial u}{\partial t} + \nabla p &= 0 & \text{in } \Omega \times (t_n, t_{n+1}) \\
\nabla \cdot u &= 0 & \text{in } \Omega \times (t_n, t_{n+1}) \\
u \cdot n &= 0 & \text{on } \Gamma_b \times (t_n, t_{n+1}) \\
p &= p_{\text{in/out}} & \text{on } \Gamma_{\text{in/out}} \times (t_n, t_{n+1}) \\
u_r &= \frac{\partial \eta^n}{\partial t} & \text{on } \Gamma \times (t_n, t_{n+1}) \\
\text{Solution: } u(t_{n+1}) &= u^{n+1}, p(t_{n+1}) &= p^{n+1}
\end{align*}
\]

**STRUCTURE**

\[
\begin{align*}
\frac{\partial^2 \eta}{\partial t^2} + C_0 \eta - C_1 \frac{\partial^2 \eta}{\partial z^2} &= p^{n+1} & \text{on } \Gamma \times (t_n, t_{n+1})
\end{align*}
\]
WRITTEN IN TERMS OF THE PRESSURE

**KINEMATICALLY COUPLED SCHEME**

**FLUID**
\[ -\Delta p = 0 \quad \text{in} \quad \Omega \times (t_n, t_{n+1}) \]
\[ \frac{\partial p}{\partial n} = 0 \quad \text{on} \quad \Gamma_b \times (t_n, t_{n+1}) \]
\[ p = p_{in/out} \quad \text{on} \quad \Gamma_{in/out} \times (t_n, t_{n+1}) \]
\[ p + \frac{\rho_s h}{\rho_f} \frac{\partial p}{\partial n} = \beta p^n \quad \text{on} \quad \Omega \times (t_n, t_{n+1}) \]

**STRUCTURE**
\[ \frac{\partial^2 \eta}{\partial t^2} + C_0 \eta - C_1 \frac{\partial \eta}{\partial t} = \beta p^{n+1} \bigg|_\Gamma \quad \text{on} \quad \Gamma \times (t_n, t_{n+1}) \]
\[ u_r \bigg|_\Gamma = \frac{\partial \eta}{\partial t} \quad \text{on} \quad \Gamma \times (t_n, t_{n+1}) \]

**CLASSICAL LOOSELY COUPLED**

**FLUID**
\[ -\Delta p = 0 \quad \text{in} \quad \Omega \times (t_n, t_{n+1}) \]
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\[ p = p_{in/out} \quad \text{on} \quad \Gamma_{in/out} \times (t_n, t_{n+1}) \]
\[ \frac{\partial p}{\partial n} = -\rho_f \frac{\partial^2 \eta}{\partial t^2} \quad \text{on} \quad \Gamma \times (t_n, t_{n+1}) \]

**STRUCTURE**
\[ \frac{\partial^2 \eta}{\partial t^2} + C_0 \eta - C_1 \frac{\partial \eta}{\partial t} = p^{n+1} \bigg|_\Gamma \quad \text{on} \quad \Gamma \times (t_n, t_{n+1}) \]

Robin

Neumann
THE SOLUTION OPERATORS

KINEMATICALLY COUPLED SCHEME

**FLUID**

\[ R : H^{-1/2}(\Gamma) \rightarrow Q = \left\{ H^1(\Omega) \mid p_{\text{in/out}} = 0 \right\} \]

\[ \mathcal{R}(\beta \pmb{p}^n) = p_h \]

\[ p = p_{\text{in/out}} \quad p = p_h + p_{\text{ext}} \quad \text{on} \quad \Gamma_b \times (t_n, t_{n+1}) \]

\[ p + \frac{\rho_f h}{\rho_f} \frac{\partial p}{\partial n} = \beta \pmb{p}^n \quad \text{on} \quad \Gamma \times (t_n, t_{n+1}) \]

**STRUCTURE**

\[ \frac{\partial^2 \eta}{\partial t^2} + C_0 \eta - C_1 \frac{\partial \eta}{\partial t} = \beta p^{n+1} \quad \text{on} \quad \Gamma \times (t_n, t_{n+1}) \]

\[ u_r \bigg|_\Gamma = \frac{\partial \eta}{\partial t} \quad \text{on} \quad \Gamma \times (t_n, t_{n+1}) \]

**CLASSICAL LOOSELY COUPLED**

**FLUID**

\[ \mathcal{N} : H^{-1/2}(\Gamma) \rightarrow Q = \left\{ H^1(\Omega) \mid p_{\text{in/out}} = 0 \right\} \]

\[ \mathcal{N}(\rho_f \frac{\partial^2 \eta^n}{\partial t^2}) = p_h \]

\[ p = p_{\text{in/out}} \quad p = p_h + p_{\text{ext}} \quad \text{on} \quad \Gamma_b \times (t_n, t_{n+1}) \]

\[ \frac{\partial p}{\partial n} = -\rho_f \frac{\partial \eta}{\partial t} \quad \text{on} \quad \Gamma \times (t_n, t_{n+1}) \]

**STRUCTURE**

\[ \frac{\partial^2 \eta}{\partial t^2} + C_0 \eta - C_1 \frac{\partial \eta}{\partial t} = p^{n+1} \quad \text{on} \quad \Gamma \times (t_n, t_{n+1}) \]
WRITTEN IN TERMS OF THE PRESSURE

KINEMATICALLY COUPLED SCHEME

**FLUID**

\[
\mathcal{R} : H^{-1/2}(\Gamma) \rightarrow Q = \{H^1(\Omega) \mid p_{\text{in/out}} = 0\}
\]

\[
\mathcal{R}(\beta p^n_h) = p_h
\]

\[
p = p_{\text{in/out}} \quad p = p_h + p_{\text{ext}} \quad \text{on } \Gamma_b \times (t_n, t_{n+1})
\]

\[
p + \frac{\rho_s h}{\rho_f} \frac{\partial p}{\partial n} = \beta p^n_h \quad \text{on } \Gamma \times (t_n, t_{n+1})
\]

Robin

\[
\frac{\partial u_r}{\partial \alpha} = -\frac{1}{\rho_f} \frac{\partial p}{\partial n} \quad \text{on } \Gamma \times (t_n, t_{n+1})
\]

**FSI PROBLEM**

\[
\frac{\partial^2 \eta}{\partial \alpha^2} + L(\eta) = \beta(p_{\text{ext}}^{n+1} + \mathcal{R}(\beta p^n_h)) \quad \text{on } \Gamma \times (t_n, t_{n+1})
\]

\[
\frac{\partial u_r}{\partial \alpha} = \frac{\partial \eta}{\partial \alpha} \quad \text{on } \Gamma \times (t_n, t_{n+1})
\]

CLASSICAL LOOSELY COUPLED

**FLUID**

\[
\mathcal{N} : H^{-1/2}(\Gamma) \rightarrow Q = \{H^1(\Omega) \mid p_{\text{in/out}} = 0\}
\]

\[
\mathcal{N}(\frac{\partial \eta^n}{\partial \alpha^2}) = p_h
\]

\[
p = p_{\text{in/out}} \quad p = p_h + p_{\text{ext}} \quad \text{on } \Gamma_b \times (t_n, t_{n+1})
\]

\[
\frac{\partial p}{\partial n} = -\rho_f \frac{\partial \eta^n}{\partial \alpha^2} \quad \text{on } \Gamma \times (t_n, t_{n+1})
\]

Neumann

**FSI PROBLEM**

\[
\frac{\partial^2 \eta}{\partial \alpha^2} + L(\eta) = p_{\text{ext}}^{n+1} + \mathcal{N}(\rho_f \frac{\partial \eta^n}{\partial \alpha^2}) \quad \text{on } \Gamma \times (t_n, t_{n+1})
\]
**STABILITY ANALYSIS: CRUCIAL STEP**

**KINEMATICALLY COUPLED SCHEME**

**FLUID + STRUCTURE**

\[ \rho_s h \frac{\partial^2 \eta}{\partial t^2} + L(\eta) = \beta (p_{ext}^{n+1} + \mathcal{R}(\mathcal{P}^n)) \bigg|_\Gamma \]

Iterative procedure: given in terms of inlet/outlet pressure, initial pressure, and operator \( \mathcal{R} \) on \( \Gamma \).

**CLASSICAL LOOSELY COUPLED**

**FLUID + STRUCTURE**

\[ \rho_s h \frac{\partial^2 \eta}{\partial t^2} + L(\eta) = p_{ext}^{n+1} + N(-\rho_f \frac{\partial^2 \eta^n}{\partial t^2}) \bigg|_\Gamma \]

\[ \rho_s h \frac{\partial^2 \eta}{\partial t^2} + \rho_f N \frac{\partial^2 \eta^n}{\partial t^2} + L(\eta) = p_{ext}^{n+1} \]

**EXPLICIT INERTIA TERM**

(ADDED MASS EFFECT BY THE FLUID!)
STABILITY ANALYSIS: CRUCIAL STEP

KINEMATICALLY COUPLED SCHEME

<table>
<thead>
<tr>
<th>FLUID + STRUCTURE</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \rho_s h \frac{\partial^2 \eta}{\partial t^2} + L(\eta) = \beta(p_{\text{ext}}^{n+1} + R(\beta p^n)) \big</td>
</tr>
</tbody>
</table>

CLASSICAL LOOSELY COUPLED

<table>
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<td>( \rho_s h \frac{\partial^2 \eta}{\partial t^2} + \rho_f N \frac{\partial^2 \eta^n}{\partial t^2} + L(\eta) = p_{\text{ext}}^{n+1} )</td>
</tr>
</tbody>
</table>

PROPOSITION: Operators \( R \) and \( N \) are both continuous, compact, self-adjoint, and positive on \( L^2(\Gamma) \).

Thus, there exists a complete orthogonal basis of eigenfunctions. Denote the corresponding eigenvalues by \( \lambda_i \) and \( \mu_i \), respectively.

IN THE EIGENVECTOR BASIS:

\[
\begin{align*}
\rho_s h \frac{\partial^2 \eta_j}{\partial t^2} + L(\eta_j) &= \beta((p_{\text{ext}}^{n+1})_j + \lambda_j \beta(p^n)_j) \big|_\Gamma, \\
&\quad j = 1, \ldots, n \\
\rho_s h \frac{\partial^2 \eta_j}{\partial t^2} + \rho_f \mu_j \frac{\partial^2 \eta^n_j}{\partial t^2} + L(\eta_j) &= (p_{\text{ext}}^{n+1})_j, \\
&\quad j = 1, \ldots, n
\end{align*}
\]
STABILITY ANALYSIS: CRUCIAL STEP

KINEMATICALLY COUPLED SCHEME

FLUID + STRUCTURE

\[ \rho_s h \frac{\partial^2 \eta}{\partial t^2} + L(\eta) = \beta (p_{ext}^{n+1} + \mathcal{R}(\beta p^n)) \big|_\Gamma \]

CLASSICAL LOOSELY COUPLED

FLUID + STRUCTURE

\[ \rho_s h \frac{\partial^2 \eta}{\partial t^2} + \rho_f N_\Gamma \frac{\partial^2 \eta}{\partial t^2} + L(\eta) = p_{ext}^{n+1} \]

PROPOSITION: Operators \( \mathcal{R}_\Gamma \) and \( N_\Gamma \) are both continuous, compact, self-adjoint, and positive on \( L^2(\Gamma') \).

Thus, there exists a complete orthogonal basis sequence of eigenfunctions. Denote the corresponding eigenvalues by \( \lambda_j \) and \( \mu_j \).

DISCRETIZATION

\[ \rho_s h \frac{\partial \eta_j^{n+1}}{\partial t^2} + L(\eta_j^{n+1}) = \beta ((p_{ext}^{n+1})_j + \lambda_j \beta (p^n)_j) \big|_{\Gamma}, \]

\[ \rho_s h \frac{\partial \eta_j^{n+1}}{\partial t^2} + \rho_f \mu_j \frac{\partial \eta_j^n}{\partial t^2} + L(\eta_j^n) = (p_{ext}^{n+1})_j, \]

\( j = 1, ..., n \)
STABILITY ANALYSIS: CRUCIAL STEP

KINEMATICALLY COUPLED SCHEME

STABLE FOR ALL $\Delta t$ IF THE RIGHT HAND-SIDE IS WELL DEFINED!

$$\rho_s h \frac{\partial^2 \eta_j^{n+1}}{\partial t^2} + L(\eta_j^{n+1}) = \beta((p_{ext}^{n+1})_j + \lambda_j \beta(p^n)_j)_I,$$

$\quad j = 1,\ldots,n$

CLASSICAL LOOSELY COUPLED

UNSTABLE FOR ALL $\Delta t$ IF

$$\frac{\rho_s h}{\rho_f \mu_{max}} \approx \frac{\rho_s h \pi^2 R}{\rho_f 2L} < 1$$

$$\rho_s h \frac{\partial^2 \eta_j^{n+1}}{\partial t^2} + \rho_f \mu_j \frac{\partial^2 \eta_j^n}{\partial t^2} + L(\eta_j^n) = (p_{ext}^{n+1})_j,$$

$\quad j = 1,\ldots,n$

THE RIGHT HAND-SIDE OF
\[ \rho_s h \frac{\partial^2 \eta_j}{\partial t^2} + L(\eta_j^{n+1}) = \beta((p_{ext}^{n+1})_j + \lambda_j \beta(p^n)_j) \bigg|_{\Gamma}, \]

\[ \beta((p_{ext}^{n+1})_j + \lambda_j \beta(p^n)_j) = \beta((p_{ext}^{n+1})_j + \sum_{i=1}^{n} (-1)^i \beta(\lambda_j \beta)^i (p_{ext}^{n+1-i})_j + (-1)^{n+1} \beta(\lambda_j \beta)^{n+1} (p_0)_j \]

As \( n \to \infty \), the series that defines the right hand-side converges if \( |\lambda_j \beta| < 1 \)

A calculation of \( \lambda_j \) shows that \( \lambda_{\text{max}} < 1 \). Thus, the right hand-side converges if

\[ 0 \leq \beta \leq 1. \]

Theorem:
The kinematically coupled scheme is unconditionally stable for all \( 0 \leq \beta \leq 1 \).
NUMERICAL RESULTS

COMPARISON WITH FSI BENCHMARK PROBLEM

THE BENCHMARK PROBLEM OF Formaggia, Gerbeau, Nobile, and Quarteroni*
(radial displacement only)

\[ \rho_s h \frac{\partial^2 \eta_r}{\partial t^2} - kGh \frac{\partial^2 \eta_r}{\partial z^2} + \frac{Eh}{1 - \sigma^2} \frac{\eta_r}{R^2} - \gamma \frac{\partial^3 \eta_r}{\partial z^2 \partial t} = f. \]

\[ p_{in}(t) = \begin{cases} 
\frac{p_{max}}{2} [1 - \cos \left(\frac{2\pi t}{t_{\max}}\right)] & \text{if } t \leq t_{\max} \\
0 & \text{if } t > t_{\max}
\end{cases}, \quad p_{out}(t) = 0 \forall t \in (0, T) \]


Ref. solution: \( \Delta t = 10^{-6} \) for all methods.
Introduction

Kinematically coupled scheme

Existence proof

Stability of $\beta$-scheme

Numerical examples
Work in progress 1: THICK WALL STRUCTURE
Work in progress 2: THICK-THIN WALL STRUCTURE MODELING FSI WITH MULTI-LAYERED ARTERIAL WALLS
References


Thank you for your attention!