Decay rates for the damped wave equation on the torus

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BCAM Seminar
July, 5. 2012
Outline

The damped wave equation

Resolvent estimates and a priori bounds

Smooth damping

Rough damping
The damped wave equation

- $M$ a compact connected Riemannian manifold (or a bounded domain in $\mathbb{R}^n$), $\Delta$ the Laplace-Beltrami operator on $M$.

- Linear damped wave equation on $M$:

\[
\begin{aligned}
\left\{ \begin{array}{l}
p_t^2 u - \Delta u + b(x) \partial_t u = 0 \quad \text{in } \mathbb{R}^+ \times M, \\
(u, \partial_t u)|_{t=0} = (u_0, u_1) \quad \in H^1(M) \times L^2(M).
\end{array} \right.
\]

- Damping coefficient $b(x) \geq 0$:
  - either $b \in C^0(M)$, and $\omega := \{ b > 0 \}$,
  - or $b = 1_\omega$, $\omega$ open.

- Energy of a solution:

\[
E(u, t) = \frac{1}{2} (\| \partial_t u(t) \|^2_{L^2(M)} + \| \nabla u(t) \|^2_{L^2(M)}).
\]

- Dissipation identity

\[
\frac{d}{dt} E(u, t) = - \int_M b |\partial_t u|^2 dx \leq 0
\]

- If $\omega \neq \emptyset$, then $E(u, t) \to 0$ as $t \to +\infty$. 

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• Linear damped wave equation on $M$:

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\partial_t^2 u - \Delta u + b(x)\partial_t u = 0 & \text{in } \mathbb{R}^+ \times M, \\
(u, \partial_t u)|_{t=0} = (u_0, u_1) & \in H^1(M) \times L^2(M).
\end{cases} \quad \text{(DWE)}$$

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$$E(u, t) = \frac{1}{2} (\| \partial_t u(t) \|_{L^2(M)}^2 + \| \nabla u(t) \|_{L^2(M)}^2).$$

• Dissipation identity

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• If $\omega \neq \emptyset$, then $E(u, t) \to 0$ as $t \to +\infty$.

At which rate?
A first question: uniform decay

Definition
Uniform decay for (DWE) if \( \exists F(t) \xrightarrow{t \to \infty} 0 \) such that \( \forall (u_0, u_1) \in H^1 \times L^2, \)
\[ E(u, t) \leq F(t)E(u, 0). \]

Lemma
Uniform decay for (DWE) implies \( F(t) \leq Ce^{-\gamma t} \) for some \( C, \gamma > 0. \)

Definition (Rauch-Taylor '74, Bardos-Lebeau-Rauch '92)
\( \omega \) satisfies GCC in \( M \) if and only if every geodesic (ray of geometric optics) traveling at speed 1 in \( M \) meets \( \omega \) in finite time.

Theorem (Rauch Taylor '74, Bardos Lebeau Rauch '92, Burq Gérard '97)
\( \omega \) satisfies GCC \( \iff \) uniform decay for (DWE) (general case)
\( \iff \) (if \( b \in \mathcal{C}^0(M) \))
A first question: uniform decay

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Uniform decay for (DWE) if \( \exists F(t) \xrightarrow{t \to \infty} 0 \) such that \( \forall (u_0, u_1) \in H^1 \times L^2 \),
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Lemma
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What happens if GCC is not satisfied?
A first question: uniform decay (on the torus)

GCC is satisfied ($\Rightarrow$ uniform decay)
A first question: uniform decay (on the torus)

GCC is satisfied
(\implies \text{uniform decay})

GCC is NOT satisfied
(\implies \text{NO uniform decay})
A weaker notion: semi-uniform decay

**Definition**

(DWE) is (semi-uniformly) stable at rate $f(t)$, $f(t) \to 0$ as $t \to \infty$ if $\exists C > 0$ such that $\forall (u_0, u_1) \in H^2 \times H^1$,

$$E(u, t) \leq C(f(t))^2 \left( \|u_0\|_{H^2(M)}^2 + \|u_1\|_{H^1(M)}^2 \right), \quad \text{for all } t > 0.$$

**Theorem (Lebeau '96)**

- If $\omega \neq \emptyset$, then $f(t) = \frac{1}{\log(2+t)}$.
- This is optimal in general. Ex: $M = S^2$ and $\omega \cap N = \emptyset$, where $N$ is a neighborhood of an equator of $S^2$. 

A weaker notion: semi-uniform decay

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$$E(u, t) \leq C(f(t))^2 \left( \|u_0\|_{H^2(M)}^2 + \|u_1\|_{H^1(M)}^2 \right), \quad \text{for all } t > 0.$$  

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- If $\omega \neq \emptyset$, then $f(t) = \frac{1}{\log(2+t)}$.
- This is optimal in general. Ex: $M = \mathbb{S}^2$ and $\omega \cap N = \emptyset$, where $N$ is a neighborhood of an equator of $\mathbb{S}^2$.

Intermediate situations

Two extreme situations:
- Uniform decay $\Leftrightarrow \omega$ satisfies GCC
- Decay at rate $f(t) = \frac{1}{\log(2+t)} \Leftrightarrow \omega \neq \emptyset$.

Some intermediate situations:
- Christianson '10: energy decay at rate $e^{-C \sqrt{t}}$, $C > 0$ if the trapped set is a hyperbolic closed geodesic.
- Schenck '11: energy decay at rate $e^{-Ct}$ on a manifold with negative curvature, if the trapped set is a “small enough” neighborhood of a closed geodesic, and if the damping is “large enough”.

$\Rightarrow$ geodesic flow enjoys exponential instability properties around the trapped set.
Some intermediate situations:

- **Liu-Rao ’05**: $M$ is a square and $\omega$ contains a vertical strip. Trapped trajectories = family of parallel geodesics constituted by vertical lines.
  
  Energy decay at rate $\left(\frac{\log(t)}{t}\right)^{\frac{1}{2}}$.

- **Burq and Hitrik ’07**: $M$ is a partially rectangular domain and $\omega$ contains a neighborhood of the non-rectangular part. Energy decay at rate $\left(\frac{\log(t)}{t}\right)^{\frac{1}{2}}$.

- **Burq and Hitrik ’07**: if moreover
  
  - $b \in C^\infty$,
  - $b = b(x_1)$ (invariance in one direction),
  - Additional assumption on the vanishing rate of $b$.

  Then, the energy decays at rate $1/t^{1-\varepsilon}$.

$\Rightarrow$ Geodesic flow enjoys **linear instability** properties around the trapped set ("cylinder of periodic orbits").

$\Rightarrow$ On the torus, we expect similar polynomial decay.
Decay rates and resolvent estimates

\[(\text{DWE}) \iff \partial_t \begin{pmatrix} u \\ \partial_t u \end{pmatrix} = A \begin{pmatrix} u \\ \partial_t u \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ \Delta & -b \end{pmatrix} \]

\[\iff \begin{pmatrix} u \\ \partial_t u \end{pmatrix} = e^{tA} \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}.\]

**Lemma**

The spectrum of $A$ contains only isolated eigenvalues and we have

\[\text{Sp}(A) \setminus \{0\} \subset \left(-\|b\|_{L^\infty(M)}, 0\right) + i\mathbb{R}.\]

We set $P(s) = -\Delta - s^2 + is b$, $s \in \mathbb{R}$.

**Proposition (Lebeau '96, Batty-Duyckaerts '08, Borichev-Tomilov '10)**

For all $\alpha > 0$, following assertions are equivalent:

System (DWE) is stable at rate $\frac{1}{t^\alpha}$,

\[\| (is - A)^{-1} \|_{L(H^1 \times L^2)} \leq C|s|^{-\frac{1}{\alpha}}, \quad \forall s \in \mathbb{R}, |s| \geq s_0,\]

\[\| P(s)^{-1} \|_{L(L^2)} \leq Cs^{\frac{1}{\alpha} - 1}, \quad \forall s \geq s_0.\]
A priori upper bound

**Theorem**

*Suppose that there exists $T > 0$, $C > 0$ such that*

$$\|u_0\|^2_{L^2(M)} \leq C \int_0^T \|\sqrt{b} e^{it\Delta} u_0\|^2_{L^2(M)} dt, \quad \forall u_0 \in L^2(M),$$

*(Observability for Schrödinger). Then System (DWE) is stable at rate $\frac{1}{\sqrt{t}}$.*

For instance on the torus $\mathbb{T}^2 := (\mathbb{R}/2\pi\mathbb{Z})^2$: $\omega \neq \emptyset \implies$ Observability for Schrödinger (Jaffard '90) $\implies$ always decay at rate $\frac{1}{\sqrt{t}}$ (at least).

**Figure:** Torus $\mathbb{T}^2 := (\mathbb{R}/2\pi\mathbb{Z})^2$ and damping region $\omega = \{b > 0\}$. 
A priori upper bound

Proof.
Observability for Schrödinger in some time $T > 0$

$$\Uparrow \quad \text{(Burq-Zworski '04, Miller '05)}$$

$$\exists C > 0 \text{ s.t.}$$

$$\|u\|_{L^2}^2 \leq C \left( \|(-\Delta - s^2)u\|_{L^2}^2 + \|\sqrt{bu}\|_{L^2}^2 \right), \quad \forall s \in \mathbb{R}, u \in H^2$$

$$\leq C \left( \|(-\Delta - s^2 + isb - isb)u\|_{L^2}^2 + \|\sqrt{bu}\|_{L^2}^2 \right)$$

$$\leq C \left( \|P(s)u\|_{L^2}^2 + s^2 \|\sqrt{bu}\|_{L^2}^2 \right) \quad \forall s \geq s_0, u \in H^2.$$

Usual trick: $\text{Im} \left( P(s)u, u \right)_{L^2} = s \left( bu, u \right)_{L^2}$

$$\implies s \|\sqrt{bu}\|_{L^2}^2 \leq \|P(s)u\|_{L^2} \|u\|_{L^2}$$

$$\implies s^2 \|\sqrt{bu}\|_{L^2}^2 \leq \frac{C}{\varepsilon} s^2 \|P(s)u\|_{L^2}^2 + \varepsilon \|u\|_{L^2}^2.$$

$$\implies \|u\|_{L^2}^2 \leq Cs^2 \|P(s)u\|_{L^2}^2 \text{, i.e. polynomial stability at rate } \frac{1}{\sqrt{t}}. \quad \Box$$
A priori lower bound

Torus $\mathbb{T}^2 := (\mathbb{R}/2\pi\mathbb{Z})^2$, when GCC is “strongly violated”.

**Theorem**

*Suppose that there exists $(x_0, \xi_0) \in T^*\mathbb{T}^2$, $\xi_0 \neq 0$, such that*

$$\bar{\omega} \cap \{x_0 + \tau\xi_0, \tau \in \mathbb{R}\} = \emptyset.$$  

*Then there exist $C > 0$ and $(s_n)_{n \in \mathbb{N}}, s_n \to +\infty$ such that*

$$\|P(s_n)^{-1}\|_{L(L^2)} \geq C.$$  

**NO GCC $\implies$ decay at rate at most $1/t$.**
A priori lower bound (simple quasimodes)

Proof (simple quasimodes).

We set \( \varphi_n(x_1, x_2) = \chi(x_1) e^{inx_2} \) and \( s_n = n \).

\[
P(s_n) \varphi_n = -\Delta (\chi(x_1)e^{inx_2}) - n^2 \chi(x_1)e^{inx_2} + inb\chi(x_1)e^{inx_2} = \chi''(x_1)e^{inx_2}.
\]

Hence \( \|P(s_n)\varphi_n\|_{L^2} \sim cte \sim \|\varphi_n\|_{L^2} \) and \( \|P(s_n)^{-1}\|_{L(L^2)} \geq C \).
A priori bounds: conclusion

As soon as GCC is (strongly) not satisfied, we have

\[ 1 \lesssim \| P(s)^{-1} \|_{L^2(L^2)} \lesssim s \]

Best decay rate \(\longrightarrow\) between \(1/\sqrt{t}\) and \(1/t\).
As soon as GCC is (strongly) not satisfied, we have

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Best decay rate \( \longrightarrow \) between \( 1/\sqrt{t} \) and \( 1/t \).

Depending on what?
Smooth damping coefficients

Theorem
Suppose that \( \omega \neq \emptyset \), that \( \sqrt{b} \in C^\infty(\mathbb{T}^2) \), and that there exist \( \varepsilon \in (0, \varepsilon_0) \) and \( C_\varepsilon > 0 \) such that

\[
|\nabla b(x)| \leq C_\varepsilon b^{1-\varepsilon}(x), \quad \text{for } x \in \mathbb{T}^2.
\]

(1)

Then, there exist \( C > 0 \) and \( s_0 \geq 0 \) such that for all \( s \geq s_0 \),

\[
\|P(s)^{-1}\|_{\mathcal{L}(L^2(\mathbb{T}^2))} \leq Cs^\delta, \quad \text{with } \delta = 8\varepsilon
\]

Hence, in this situation, (DWE) is stable at rate \( \frac{1}{t^{1+\delta}} \).
Smooth damping coefficients

Theorem

Suppose that $\omega \neq \emptyset$, that $\sqrt{b} \in C^\infty(\mathbb{T}^2)$, and that there exist $\varepsilon \in (0, \varepsilon_0)$ and $C_\varepsilon > 0$ such that

$$|\nabla b(x)| \leq C_\varepsilon b^{1-\varepsilon}(x), \quad \text{for } x \in \mathbb{T}^2. \quad (1)$$

Then, there exist $C > 0$ and $s_0 \geq 0$ such that for all $s \geq s_0$,

$$\|P(s)^{-1}\|_{L(L^2(\mathbb{T}^2))} \leq Cs^\delta, \quad \text{with } \delta = 8\varepsilon$$

Hence, in this situation, (DWE) is stable at rate $\frac{1}{t^{1+\delta}}$.

- generalizes Burq-Hitrik ’07 in the case of non-invariant damping function $b$ with several trapped directions.
- (1) = local assumption in a neighborhood of $\partial \omega$.
- Ex: $b \sim e^{-\frac{1}{x^\gamma}}$, then $b' \sim \log(\frac{1}{b})^{\gamma+1} b$ on $\omega$; (1) is satisfied for all $\varepsilon > 0$.
- The a priori lower bound $1/t$ is sharp in any geometric situation!
Smooth damping coefficients: idea of the proof

Prove $\|u\|_{L^2(T^2)} \leq Cs^\delta \|(-\Delta - s^2 + isb)u\|_{L^2(T^2)}$ for all $s \geq s_0$, $u \in H^2(T^2)$.

† with $h = 1/s$

Prove $\|u\|_{L^2(T^2)} \leq \frac{C}{h^{2+\delta}} \|(-h^2\Delta - 1 + ihb)u\|_{L^2(T^2)}$ for all $h \leq h_0$, $u \in H^2(T^2)$.

$=:P_h$

We suppose that this is false. There exists $0 < h_n \to 0$ and $u_n \in H^2(T^2)$ such that

$$\begin{cases}
\|u_n\|_{L^2(T^2)} = 1, \\
h_n^{-2-\delta} \|P_h u_n\|_{L^2(T^2)} \to 0.
\end{cases}$$
Smooth damping coefficients: idea of the proof

- Skip the index $n$

\[
\begin{align*}
    h & \to 0^+ , \\
    \|u_h\|_{L^2} & = 1 , \\
    \|P_h u_h\|_{L^2} & = o(h^{2+\delta}) , \\
    \|\sqrt{b}u_h\|_{L^2} & = o(h^{\frac{1+\delta}{2}}). \quad \text{(Bonus)}
\end{align*}
\]

$\Rightarrow$ usual trick $h\|\sqrt{b}u_h\|_{L^2}^2 = \text{Im}(P_h u_h, u_h)_{L^2} = o(h^{2+\delta})$.

- Semiclassical measure associated to $(h, u_h)$: Up to a subsequence, there exists $\mu \in M^+(T^*\mathbb{T}^2)$ such that

\[
\begin{align*}
    \langle \text{Op}_h(a) u_h, u_h \rangle_{L^2(\mathbb{T}^2)} \to \langle \mu, a \rangle \quad \text{for all } a = a(x, \xi) \in \mathcal{C}_c^\infty(T^*\mathbb{T}^2).
\end{align*}
\]

- characterizes the defect of convergence to zero for $(u_h)$. 
Smooth damping coefficients: idea of the proof

Properties of the sequence:

\[
\begin{aligned}
  &h \to 0^+, \\
  &\|u_h\|_{L^2} = 1, \\
  &\|P_h u_h\|_{L^2} = o(h^{2+\delta}), \\
  &\|\sqrt{b} u_h\|_{L^2} = o(h^{1+\delta}). \quad \text{(Bonus)}
\end{aligned}
\]

First properties of the semiclassical measure:

Lemma
We have

1. \(\text{supp}(\mu) \subset \mathbb{T}^2 \times \{||\xi||^2 = 1\} = S^*\mathbb{T}^2\),
2. \(\mu(T^*\mathbb{T}^2) = 1\),
3. \(\mu(x + \tau \xi, \xi) = \mu(x, \xi)\)’, for all \(\tau \in \mathbb{R}\),
4. \(\langle \mu, b \rangle = 0\).

Remark: Here, we only use \(\|P_h u_h\|_{L^2} = o(h^1)\)!
Smooth damping coefficients: idea of the proof

Properties of the sequence:

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\begin{aligned}
    & h \to 0^+, \\
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\end{aligned}
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First properties of the semiclassical measure:

**Lemma**

We have

1. \(\text{supp}(\mu) \subset \mathbb{T}^2 \times \{|\xi|^2 = 1\} = S^*\mathbb{T}^2,\)

2. \(\mu(T^*\mathbb{T}^2) = 1,\)

3. "\(\mu(x + \tau \xi, \xi) = \mu(x, \xi)\)”, for all \(\tau \in \mathbb{R},\)

4. \(\langle \mu, b \rangle = 0.\)

Remark: Here, we only use \(\|P_hu_h\|_{L^2} = o(h^1)!\)

**Goal:** prove that \(\mu \equiv 0 \leadsto \text{obtain a contradiction with } \mu(T^*\mathbb{T}^2) = 1.\)
Smooth damping coefficients: idea of the proof

Lemma

\[ \mu = \sum_{\Gamma \text{ rational direction}} \mu|_{T^2 \times \Gamma} \quad \text{where } \mu|_{T^2 \times \Gamma} \in \mathcal{M}^+(T^*T^2) \text{ is invariant.} \]

- “\( \Gamma \) rational direction” if \( \Gamma = \mathbb{R}\xi_0 \) for \( \xi_0 \in \mathbb{R}^2 \setminus \{0\} \) such that \( k \cdot \xi_0 = 0 \) for some \( k \in \mathbb{Z}^2 \setminus \{0\} \).
  \[ \implies \Gamma = \mathbb{R}\xi_0 \text{ is periodic in } T^2. \]

- If \( \Gamma \) is an irrational direction
  \[ \implies \Gamma = \mathbb{R}\xi_0 \text{ is dense in } T^2. \]
  \[ \mu|_{T^2 \times \Gamma} \text{ is invariant and vanishes on } \omega \implies \mu|_{T^2 \times \Gamma} \equiv 0. \]
Smooth damping coefficients: idea of the proof

Lemma

\[ \mu = \sum_{\Gamma \text{ rational direction}} \mu|_{T^2 \times \Gamma} \quad \text{where } \mu|_{T^2 \times \Gamma} \in \mathcal{M}^+ \left( T^* T^2 \right) \text{ is invariant.} \]

- “\( \Gamma \) rational direction” if \( \Gamma = \mathbb{R} \xi_0 \) for \( \xi_0 \in \mathbb{R}^2 \setminus \{0\} \) such that \( k \cdot \xi_0 = 0 \) for some \( k \in \mathbb{Z}^2 \setminus \{0\} \).
  \[ \Rightarrow \Gamma = \mathbb{R} \xi_0 \text{ is periodic in } T^2. \]

- If \( \Gamma \) is an irrational direction
  \[ \Rightarrow \Gamma = \mathbb{R} \xi_0 \text{ is dense in } T^2. \]
  \[ \mu|_{T^2 \times \Gamma} \text{ is invariant and vanishes on } \omega \Rightarrow \mu|_{T^2 \times \Gamma} \equiv 0. \]
Smooth damping coefficients: idea of the proof

We fix $\Gamma$, and want to prove that $\mu_{\Gamma} := \mu|_{T^2 \times \Gamma}$ vanishes. Take for instance $\Gamma = \mathbb{R}\xi_2$, $\xi_2 = (0, 1)$.

THREE steps to prove that $\mu_{\Gamma} \equiv 0$. 
**Smooth damping coefficients: idea of the proof**

**STEP 1:** Understand the possible concentration rate of the sequence $u_h$ towards $\Gamma$:

**Lemma**

For all $0 < \alpha \leq \frac{3+\delta}{4}$, we have $\langle \mu_\Gamma, a \rangle = \lim_{h \to 0} (\text{Op}_h (a(x, \xi) \chi(\xi_1/h^\alpha)) \ u_h, u_h)_{L^2}$

Idea of proof: consider 2-microlocal semiclassical measures (Miller '97, Fermanian-Kammerer '05) at scale $\alpha$: $\nu_\alpha$:

$\langle \nu_\alpha, a(x_1, \xi, \infty \xi_1) \rangle = \lim_{h \to 0} \left( \text{Op}_h \left( a \left( x_1, \xi, \xi_1 \frac{h^\alpha}{h} \right) \left( 1 - \chi(\xi_1/h^\alpha) \right) \right) \ u_h, u_h \right)_{L^2}$

- $\langle \nu_\alpha, \langle b \rangle_\Gamma \rangle = 0$, ($\langle b \rangle_\Gamma$ average of $b$ in the direction $\Gamma$).
- Transverse propagation law: $\partial_{x_1} \nu_\alpha = 0$ if $0 < \alpha \leq \frac{3+\delta}{4}$.
- Hence $\nu_\alpha = 0$ for $0 < \alpha \leq \frac{3+\delta}{4}$. 
Smooth damping coefficients: idea of the proof

STEP 2: construction of a particular cutoff function:

Proposition

Set \( w_h = \text{Op}_h (\chi(\xi_1/h^\alpha)) u_h \). For \( \delta = 8\varepsilon, \varepsilon < \varepsilon_0 \), there exists \( \chi_h \in \mathcal{C}_\infty^\infty \) valued in \([0, 1]\), such that

1. \( \chi_h = \chi_h(x_1) \) does not depend \( x_2 \),
2. \( b \leq c_0 h \) on \( \text{supp}(\chi_h) \),
3. \( \| (1 - \chi_h) w_h \|_{L^2(\mathbb{T}^2)} = o(1) \).

• If \( b \) is invariant in one direction, \( \chi_h = \chi(\frac{b}{c_0 h}) \) (Burq-Hitrik ’07).
• Assumptions on \( b \) used here, together with the \( o(h^{2+\delta}) \).
Smooth damping coefficients: idea of the proof

**STEP 3**: possible concentration rate for the sequence $w_h$ towards $\Gamma$:

**Lemma**
we have $\langle \mu_\Gamma, a \rangle = \lim_{R \to +\infty} \lim_{h \to 0} \langle \text{Op}_h (a(x, \xi)\chi(\xi_1/Rh^1)) w_h, w_h \rangle_{L^2}$.

Idea of proof: consider 2-microlocal semiclassical measures $\tilde{\nu}_1$ (Fermanian-Kammerer '00, Anantharaman-Macia '11) at scale 1 associated to $w_h$:

- $\langle \tilde{\nu}_1, \langle b \rangle_\Gamma \rangle = 0$, ($\langle b \rangle_\Gamma$ average of $b$ in the direction $\Gamma$).
- Transverse propagation law: $\partial_{x_1} \tilde{\nu}_1 = 0$ (uses $\chi_h$ in an essential way).
- Hence $\tilde{\nu}_1 = 0$.

Consequences (Anantharaman-Macia '11):

- $\mu_\Gamma = 0$,
- $\implies \mu = 0$, (this holds for any $\Gamma$),
- $\implies$ contradiction with $\mu(T^*\mathbb{T}^2) = 1$. 
Rough damping: a particular case

Numerical study of the case

Figure: $b(x_1, x_2) = \kappa \mathbb{1}_{(0, \sigma)}(x_1)$ characteristic function of a strip

... because the spectrum has a particular shape (see Asch-Lebeau '03)
Rough damping: a particular case

Numerical study of the case

\[ b = \kappa \quad \text{in} \quad \omega \]

\[ b = 0 \quad \text{in} \quad \sigma \]

Figure: \( b(x_1, x_2) = \kappa \mathbb{1}_{(0, \sigma)}(x_1) \) characteristic function of a strip

... because the spectrum has a particular shape (see Asch-Lebeau '03) and because we failed to prove something...
Proposition

For $\alpha > 0$, the following assertions are equivalent:

- System (DWE) is stable at rate $\frac{1}{t^{\alpha}}$,

- $\|(is - A)^{-1}\|_{L(H^1 \times L^2)} \leq C|s|^{\frac{1}{\alpha}}$ for all $s \in \mathbb{R}$, $|s| \geq s_0$ (Batty Duyckaerts ’08, Borichev Tomilov ’10),

- $\|(z - A)^{-1}\|_{L(H^1 \times L^2)} \leq C|\text{Im}(z)|^{\frac{1}{\alpha}}$ for all $z \in \mathbb{C}$, satisfying $|z| \geq s_0$ and $|\text{Re}(z)| \leq \frac{1}{C|\text{Im}(z)|^{\frac{1}{\alpha}}}$.  

Consequence: Decay at rate $\frac{1}{t^{\alpha}} \implies$ No spectrum in

$$C(\alpha, K) := \left\{ z \in \mathbb{C}, 0 \geq \text{Re}(z) \geq \frac{1}{K|\text{Im}(z)|^{\frac{1}{\alpha}}} \right\},$$

for some $K > 0$.  

Rough damping: some simulations
Rough damping: some simulations

Figure: Full spectrum of the operator $A_h$

Discretization $N = 50$, damping $b(x_1, x_2) = 21_{(0,1/2)}(x_1)$. 
Figure: Branch of the spectrum of the operator $A_h$ closest to the positive imaginary axes

Discretization $N = 50$, damping $b(x_1, x_2) = 2\mathbb{1}_{(0,1/2)}(x_1)$.

We expect $-\text{Re}(z) \sim \frac{C}{\text{Im}(z)\beta}$ on this branch:

$$\log(-\text{Re}(z)) \sim -\frac{1}{\beta} \log(\text{Im}(z)) + C$$
Rough damping: some simulations

\begin{figure}
\centering
\includegraphics[width=\textwidth]{log_Re_z_vs_log_Im_z.png}
\caption{log(− Re(z)) as a function of log(Im(z)) for \( z \in \text{Sp}(A_h) \), \( z \) on the branch selected in the previous slide.}
\end{figure}

Discretization \( N = 50 \), damping \( b(x_1, x_2) = 2 \mathbb{1}_{(0,1/2)}(x_1) \).
Dependance with respect to the mesh size

- damping function: $b(x_1, x_2) = 21_{(0,1/2)}(x_1)$,
- change the number of discretization points $N$.

<table>
<thead>
<tr>
<th>Discretization size $N$</th>
<th>20</th>
<th>30</th>
<th>40</th>
<th>50</th>
</tr>
</thead>
<tbody>
<tr>
<td>Slope $-1/\beta$</td>
<td>-1.411</td>
<td>-1.389</td>
<td>-1.385</td>
<td>-1.386</td>
</tr>
</tbody>
</table>

Consequence: Decay at rate $\frac{1}{t^\alpha} \implies \alpha \leq \frac{1}{1.38} < 1$. 
Dependance with respect to the thickness of the damping region

- discretization: $N = 40$,
- change the support of the damping function $b(x_1, x_2) = 21_{(0, \sigma)}(x_1)$, $\sigma \in (0, 1)$.

<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-1/\beta$</td>
<td>−1.526</td>
<td>−1.476</td>
<td>−1.395</td>
<td>−1.385</td>
<td>−1.374</td>
<td>−1.370</td>
<td>−1.362</td>
</tr>
</tbody>
</table>

- Consequence: Decay at rate $\frac{1}{t^\alpha} \implies \alpha \leq \frac{1}{1.53} < 1$.
- monotonicity not proved...
Dependance with respect to the thickness of the damping region

Figure: Full spectrum of the operator $A_h$

Discretization $N = 40$, damping $b(x_1, x_2) = 21_{(0,0.3)}(x_1)$. 
Dependance with respect to the size of the damping function

- discretization fixed $N = 40$,
- support of the damping coefficient fixed,
- change its value, i.e. $b(x_1, x_2) = \kappa 1_{(0,1/2)}(x_1), \kappa \geq 0$.

<table>
<thead>
<tr>
<th>$\kappa$</th>
<th>1</th>
<th>2</th>
<th>4</th>
<th>8</th>
<th>16</th>
<th>32</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-1/\beta$</td>
<td>$-1.338$</td>
<td>$-1.385$</td>
<td>$-1.406$</td>
<td>$-1.474$</td>
<td>$-1.532$</td>
<td>$-1.637$</td>
</tr>
</tbody>
</table>

- Consequence: Decay at rate $\frac{1}{t^\alpha} \implies \alpha \leq \frac{1}{1.63} < 1$...
- monotonicity not proved (not expected)...

Dependance with respect to the size of the damping function

Figure: Full spectrum of the operator $A_h$

Discretization $N = 40$, damping $b(x_1, x_2) = 161_{(0,1/2)}(x_1)$. 
Conclusion and open problems

Conclusion:

- Decay rates on the torus seem to depend only on the vanishing rate of $b$, and not on the number of trapped directions!
- The \textit{a priori} lower bound $\frac{1}{t}$ is sharp in any geometrical situation (almost reached for smooth $b$)!
- The \textit{a priori} lower bound $\frac{1}{t}$ does not seem to be reached for $b = \kappa 1_{(0, \sigma)}(x_1)$ (Nonenmacher proved that it is not reached)!

Some open problems:

- Is the \textit{a priori} bound $\frac{1}{\sqrt{t}}$ is sharp on the torus for some rough damping coefficients?
- (Even if $b$ is invariant in one direction) what is the precise link between the vanishing rate of $b$ and the decay rate for (DWE)?
- What happens in $\mathbb{T}^d$, $d \geq 3$?