Regularity of Weak Solution to Parabolic Fractional $p$-Laplacian

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Let $1 < p < \infty$ and $D$ is a bounded domain in $\mathbb{R}^n$. We consider the variational problem

$$\min\{ \int_D |\nabla u|^p \, dx : u \in W^{1,p}_0(D) \}.$$ 

Then the minimizer (so-called $p$-Laplacian) satisfies the equation

$$\text{div}( |\nabla u|^{p-2} \nabla u ) = 0 \text{ in } D.$$ 

And we also consider the corresponding parabolic $p$-Laplacian on $[a, b] \times D$:

$$u_t - \text{div}( |\nabla u|^{p-2} \nabla u ) = 0, \text{ for } (t, x) \in [a, b] \times D.$$
1.2. Some Classical Results for Local Case

For the $p-$Laplacian ($p > 1$):

$$\text{div}(|\nabla u|^{p-2}\nabla u) = 0.$$ 

**Ural’ceva (68)** proved that for $p > 2$, the weak solutions of $p$-Laplace Equation have Hölder continuous derivatives and **Evans (82)** gave another proof for this result. **Lewis (83)** and **DiBenedetto (83)** extended this result to the case $1 < p < 2$. **Uhlenbeck (77)** and **Tolksdoff (84)** proved $C^{1,\alpha}$ regularity for the $p$-Laplacian systems for some $\alpha > 0$. **Remark:** In general, the weak solutions to $p$-Laplacian do not have better regularity than $C^{1,\alpha}$. 
For the **parabolic $p$-Laplacian**:

$$u_t - \nabla \cdot (|\nabla u|^{p-2} \nabla u) = 0 \text{ in } [a, b] \times \Omega.$$

DiBenedetto (86) proved that any weak solution is locally Hölder continuous for $p > 2$. For the singular case $1 < p < 2$, Chen- DiBenedetto (92) proved that any essentially bounded weak solution is locally Hölder continuous.
In this section we consider the variational problem in non local form ($p > 1$ and $0 < s < 1$):

$$\min \left\{ \iint_{D \times D} \frac{|u(x) - u(y)|^p}{|x - y|^{n+ps}} \, dx \, dy : u \in W^{s,p}(D) \text{ and } u = g \text{ on } \mathbb{R}^n \setminus D \right\}$$

Then the minimizer $u$ is a fractional $p$-Laplacian and it satisfies the integral equation in weak sense:

$$\int_{D} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{n+ps}} \, dy = 0$$
2.1. Linear Case

For the simplest case $p = 2$, $0 < s < 1$, the equation would be reduced to

$$
\int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2s}} dy = 0, \quad u \in H^s
$$

It is equivalent to $(-\Delta)^s u(x) = 0$. By Caffarelli and Silvestre (2007) (An Extension Problem Related to the Fractional Laplacian, CPDE, 32 (8): 1245-1260), $u$ can be extended to $u^* : \mathbb{R}^{n+1} \to \mathbb{R}$ such that $u^*(x, 0) = u(x)$ and

$$
div(|y|^a \nabla u^*) = 0 \text{ in } \mathbb{R}^{n+1},
$$

where $a = 1 - 2s$. 
In Fabes, Kenig and Serapioni (1982) ( The local regularity of solutions of degenerate elliptic equations. CPDE 7(1): 77-116. ), they study the degenerate elliptic equation

\[ \text{div}(A(x)\nabla v) = 0 \]

where \(cw(x)|\xi|^2 \leq \xi^T A(x)\xi \leq Cw(x)|\xi|^2\) for any \(\xi \in \mathbb{R}^n\) and \(w(x) > 0\) is an \(A_2\)-weight function i.e. there exists a uniform constant \(C > 0\) such that

\[
\sup_B \frac{1}{|B|} \int_B w(x) dx \cdot \frac{1}{|B|} \int_B w(x)^{-1} dx \leq C.
\]

They prove that \(v\) is Hölder continuous.

Obviously, \(|y|^a \in A_2\), then \(u^* \in C^\alpha\) and so is \(u\).
For the case of measurable coefficients, i.e. the integral kernel $K(x, y)$ satisfies:

$$\lambda |x - y|^{-(n+2s)} \leq K(x, y) \leq \Lambda |x - y|^{-(n+2s)}$$

for two uniform positive constants $0 < \lambda \leq \Lambda < \infty$ and for any $x \neq y$, $K(x, y) = K(y, x)$. The equation is

$$\int_{\mathbb{R}^n} K(x, y)(u(x) - u(y)) dy = 0$$

**Silvestre (2006)** and **Kassmann (2009)** proved that any bounded weak solution is Hölder continuous.


The equation is
\[
\int_{\mathbb{R}^n} K(x, y)|u(x) - u(y)|^{p-2}(u(x) - u(y))dy = 0
\]
where the integral kernel \(K(x, y)\) satisfies:
\[
\lambda |x - y|^{-(n+ps)} \leq K(x, y) \leq \Lambda |x - y|^{-(n+ps)}
\]
for two uniform positive constants \(0 < \lambda \leq \Lambda < \infty\) and for any \(x \neq y\),
\[
K(x, y) = K(y, x)
\]
Theorem A. Any bounded weak solution is locally Hölder continuous for $p > 2$ and $ps < p - 1$.

The proof depends on the following oscillation lemma:

Lemma 2.1. Assume that $p > 2$, $0 < ps < p - 1$. If $u$ is a solution to the fractional $p$-Laplacian and also satisfies the following conditions:

\[
\begin{align*}
|u(x)| & \leq 1 \quad x \in \mathbb{R}^n \\
|\{x \in B_1 : u(x) \leq 0\}| & > \mu > 0
\end{align*}
\]

where $\mu$ are two positive constants. Then $u \leq 1 - \gamma$ in $B_{1/2}$ for some $\gamma > 0$.

Remark: Proof of Lemma 2.1. is similar to that of Lemma 3.3 and we will give the details there.
Now we discuss parabolic fractional $p$-Laplacian. Let us assume that $n \geq 2$, $0 < s < 1$ and $p > 1$. The measurable function $K : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ satisfies that

$$K(t, x, y) = K(t, y, x) \text{ for any } x \neq y$$

and the following: there are two positive constants $\lambda$ and $\Lambda$ such that

$$\frac{\lambda}{|x - y|^{n + ps}} \leq K(t, x, y) \leq \frac{\Lambda}{|x - y|^{n + ps}}, \quad \forall \ x, \ y \in \mathbb{R}^n.$$

Usually, the parabolic fractional $p$-Laplacian is like the following form:

$$u_t(x, t) + \int_{\mathbb{R}^n} K(t, x, y)|u(x) - u(y)|^{p-2}(u(x) - u(y))dy = 0 \quad (\ast)$$
In the following we will introduce the weak solution of (\(*\)):

**Definition** We say \( w \in C([a, b]; L^2(\mathbb{R}^n)) \cap L^p([a, b]; W^{s,p}(\mathbb{R}^n)) \) to be a weak solution to the parabolic fractional \( p \)-Laplacian if and only if \( \forall \ \eta \in C_0^\infty(\mathbb{R}^n) \) and \( t \in [a, b] \), the following holds:

\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} K(t, x, y)|w(x) - w(y)|^{p-2}(w(x) - w(y))\eta(x)\,dx\,dy
= - \int_{\mathbb{R}^n} w_t(x)\eta(x)\,dx.
\]
3.1. Linear Equation

For the special case: $0 < s < 1$ and $p = 2$, then the equation becomes

$$u_t(x, t) + \int_{\mathbb{R}^n} K(t, x, y)(u(x) - u(y))dy = 0, \quad (*)$$

where $\frac{\lambda}{|x - y|^{n+2s}} \leq K(t, x, y) \leq \frac{\Lambda}{|x - y|^{n+2s}}$ and $K(t, x, y) = K(t, y, x)$ for $x \neq y$.

L. Caffarelli, C. Chan and A. Vasseur recently proved that any weak solution to $(*)$ with initial value in $L^2(\mathbb{R}^n)$ is uniformly bounded and Hölder continuous. (see Regularity theory for parabolic nonlinear integral operators, J. Amer. Math. Soc. 24(3), 849-869, 2011)
3.2. Parabolic Fractional $p$-Laplacian

We assume that $n \geq 2$, $0 < ps < p - 1$ and $p > 2$. The measurable function $K : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ is symmetric in $x, y$ for any $x \neq y$ and satisfies the following estimate: there are two positive constants $\lambda$ and $\Lambda$ such that

$$\frac{\lambda}{|x - y|^{n + ps}} \leq K(t, x, y) \leq \frac{\Lambda}{|x - y|^{n + ps}}, \quad \forall \ x, \ y \in \mathbb{R}^n$$

Usually, the parabolic fractional $p$-Laplacian is like the following form:

$$u_t(x, t) + \int_{\mathbb{R}^n} K(x, y)|u(x) - u(y)|^{p-2}(u(x) - u(y))dy = 0.$$
**Theorem B.** Any weak solution to the parabolic fractional \( p \)-Laplacian above with the initial data in \( L^p(\mathbb{R}^n) \) is locally Hölder continuous.

For the proof, there are two main steps to go:

(1) if the initial data for \( w \) is bounded in \( L^p(\mathbb{R}^n) \), then it is essentially bounded.

(2) we need to give some De Giorgi oscillation lemma for \( w \).

Firstly, we have

**Lemma 3.1** Any weak solution

\( w \in C([a, b]; L^2(\mathbb{R}^n)) \cap L^p([a, b]; W^{s,p}(\mathbb{R}^n)) \) to the parabolic fractional \( p \)-Laplacian is uniformly bounded on \([t_0, b] \times \mathbb{R}^n\) for any \( a < t_0 < b \). if the initial data for \( w \) is bounded in \( L^p(\mathbb{R}^n) \).
For simplicity, we may assume that $a = -2$ and $b = 0$. Then $w \in C([-2, 0]; L^2(\mathbb{R}^n)) \cap L^p([-2, 0]; W^{s,p}(\mathbb{R}^n))$ and Lemma 3.1. can be reduced to the following Lemma:

**Lemma 3.2** There is a constant $\epsilon_0 \in (0, 1)$ depending only on $n$, $p$, $s$, $\lambda$ and $\Lambda$ such that any weak solution $w : [-2, 0] \times \mathbb{R}^n \to \mathbb{R}$ of parabolic fractional $p$-Laplacian, the following is true:

If

$$\int_{\mathbb{R}^n} |w(-2, x)|^p dx \leq \epsilon_0,$$

then

$$w(t, x) \leq 1/2 \quad \text{on} \quad [-1, 0] \times \mathbb{R}^n.$$
Outline of Proof: Let $T_k = (-1 - 2^{-k})$, $\lambda_k = 1/2 - 2^{-k-1})$ and $w_k = (w - \lambda_k)_+$. We define

$$U_k \triangleq \sup_{0 \leq t \leq T_k} \int_{\mathbb{R}^n} w_k^2(x, t) \, dx + \int_0^{T_k} \int_{\mathbb{R}^n \times \mathbb{R}^n} K(x, y) |w_k(t, x) - w_k(t, y)|^p \, dt \, dxdy.$$

Then by Sobolev imbedding theorem and interpolation inequality, we can get

$$U_k \leq C^k U_{k-1}^{1+\alpha}$$

where $C > 1$ and $0 < \alpha < 1$ are two uniform constants.
Therefore there exists some positive constant $\delta_0$, such that if $U_1 \leq \delta_0$, then
\[ \lim_{k \to 0} U_k = 0. \]

And
\[ U_1 \leq 2^{p-2} \int_{\mathbb{R}^n} |w(x, -2)|^p \, dx \]

Thus if we let $\epsilon_0 = \delta_0 / 2^{p-2}$ and $\int_{\mathbb{R}^n} |w(x, -2)|^p \, dx \leq \epsilon_0$, then
\[ w(t, x) \leq 1/2 \quad \text{on} \quad [-1, 0] \times \mathbb{R}^n. \]
Now we will give the De Giorgi oscillation lemma:

**Lemma 3.3.** Let $w$ be a weak solution and $w \leq 1$ on $[-2, 0] \times \mathbb{R}^n$. If the following condition holds:

$$|\{w < 0\} \cap \left([-2, -1] \times B_1\right)| \geq \mu > 0,$$

where $\mu$ is a constant. Then there is a small number $\theta > 0$ such that

$$w \leq 1 - \theta \quad \text{in} \quad [-1, 0] \times B_1.$$
Proof. We define the function \( m : [-2, 0] \rightarrow \mathbb{R} \) as

\[
m(t) = \int_{-2}^{t} c_0 \{ x : w(t, x) < 0 \} \cap B_1 e^{-C_1(t-s)} ds
\]

where \( c_0 \) and \( C_1 \) are two positive constants which will be determined later. In fact, \( c_0 \) is chosen to be very small, and \( C_1 \) is very large, then \( m(t) \) can be very small:

\[
m(t) \leq \frac{2\omega_n c_0}{C_1}, \quad \forall \ t \in [-2, 0]
\]

where \( \omega_n \) is the volume of the unit ball in \( \mathbb{R}^n \).
For $t \in [-1, 0]$,

$$m(t) \geq c_0 e^{-2C_1} \left| \{ w < 0 \} \cap ([ -2, -1] \times B_1) \right| \geq c_0 e^{-2C_1} \mu.$$  

And we let $\eta : \mathbb{R}^n \to \mathbb{R}$ be defined as follows: $\eta \in C_0^\infty(\mathbb{R}^n)$ and

$$\eta(x) = \begin{cases} 
1 & \text{for } |x| \leq 1 \\
0 & \text{for } |x| \geq 2 \\
\text{non increasing in } |x| & \text{for all } x \in \mathbb{R}^n 
\end{cases}$$  

We will prove the following claim:

$$w(t, x) \leq 1 - m(t)\eta(x) \text{ on } [-2, 0] \times \mathbb{R}^n.$$
This claim implies Lemma 3.3:
In fact, if the claim holds true, then

\[ w(t,x) \leq 1 - m(t) \] on \([-1,0] \times B_1 \).

And for \( t \in [-1,0] \),

\[ m(t) \geq c_0 e^{-2C_1} |\{ w < 0 \} \cap \{ [-2,-1] \times B_1 \}| \geq c_0 e^{-2C_1} \mu, \]

thus

\[ w(t,x) \leq 1 - c_0 e^{-2C_1} \mu \] on \([-1,0] \times B_1 \).
Next, **we will prove this claim by contradiction.** If the claim was not true, then there is some point \((t, x) \in [-2, 0] \times \mathbb{R}^n\), such that

\[
w(t, x) + m(t)\eta(x) > 1
\]

Now let \((t_0, x_0)\) be the point which realizes the maximum of \(w(t, x) + m(t)\eta(x)\) on \([-2, 0] \times \mathbb{R}^n\). Then

\[
w_t(t_0, x_0) \geq -m'(t_0)\eta(x_0).
\]

In the following, we let \(w(x) = w(t_0, x)\). If \(w(y) \leq 0\), then obviously

\[
w(x_0) + m(t_0)\eta(x_0) - w(y) - m(t_0)\eta(y) \geq 1 - m(t_0) \geq 1 - \frac{2\omega_n c_0}{C_1}.
\]
Thus

\[
\int_{\mathbb{R}^n} K(x_0, y)|w(x_0) - w(y)|^{p-2}(w(x_0) - w(y))dy \\
\geq -|m(t_0)|^{p-1} \int_{\mathbb{R}^n} K(x_0, y)|\eta(x_0) - \eta(y)|^{p-2}(\eta(x_0) - \eta(y))dy \\
+ C \cdot (1 - \frac{2\omega_n c_0}{C_1})^{p-1}|\{y \in B_1 : w(t_0, y) \leq 0\}|
\]

Since \(ps < p - 1\), then

\[
|\int_{\mathbb{R}^n} K(x_0, y)|\eta(x_0) - \eta(y)|^{p-2}(\eta(x_0) - \eta(y))dy| \leq C_2
\]

where \(C_2\) is the constant which depends only on \(n, p, s, \lambda, \Lambda\).
Hence, we have

\[ m'(t_0) + C_2|m(t_0)|^{p-1} \geq C \cdot (1 - \frac{2\omega_n c_0}{C_1})^{p-1} \{|{y \in B_1 : w(t_0, y) \leq 0}|\}. \]

which implies that

\[
c_0 \{|{y \in B_1 : w(t_0, y) \leq 0}| - C_1 m(t_0) + C_2|m(t_0)|^{p-1} \geq C \cdot (1 - \frac{2\omega_n c_0}{C_1})^{p-1} |\{|y \in B_1 : w(t_0, y) \leq 0|\}|.
\]
If we choose the constants $c_0$ and $C_1$ in the following way:

\[
\begin{align*}
4\omega_n c_0 &< C_1 \\
c_0 e^{-2C_1} &< \frac{1}{4\mu} \\
2c_0 &< C(1 - \frac{2\omega_n c_0}{C_1})^{p-1} \\
C_1 &\geq 2C_2
\end{align*}
\]

Then $m(t_0) = 0$ and $w(t_0, x_0) > 1$ which contradicts the assumption of Lemma 3.1:

\[
w(t, x) \leq 1 \quad \text{on} \quad [-2, 0] \times \mathbb{R}^n.
\]
From Lemma 3.3, we will have the following:

**Corollary 3.4.** Let $w$ be a weak solution to the parabolic fractional $p$-Laplacian and $|w| \leq 1$ on $[-1, 0] \times \mathbb{R}^n$. Then

$$\max_{Q_{1/4}} |w| \leq 1 - \theta$$

where $\theta$ is the same constant as that in Lemma 3.1. and

$$Q_r = [-r^{ps}, 0] \times B_r, \quad \forall \ r > 0.$$
Now we prove that any weak solution to (1) is Hölder continuous. Firstly, we have the following lemma:

**Lemma 3.5.** Let \( w : [-2, 0] \times \mathbb{R}^n \rightarrow \mathbb{R} \) be a weak solution to the parabolic fractional \( p \)-Laplacian and without loss of generality, we assume that \(|w| \leq 1\) on \([-1, 0] \times B_1(0)\). Let \( \omega = \sup_{Q_1} w - \inf_{Q_1} w \).

Then for any natural number \( n \),

\[
\sup_{K^n} w - \inf_{K^n} w \leq (1 - \theta)^n \omega
\]

where \( K^n = (-\left(\frac{1}{4}\right)^ps, 0] \times B_{r_n}(0) \), \( r_n = \left(\frac{1}{4}\right)^n(1 - \theta)\left(\frac{n(p-2)}{ps}\right)^\frac{1}{n(p-2)} \) and \( \theta \) is from Lemma 3.2.
Proof. We prove it by induction. Obviously,

\[ K^1 = (-(1/4)^{ps}, 0] \times B_{r_1}(0) \]
\[ \subset (-(1/4)^{ps}, 0] \times B_{1/4}(0) \]

Hence by Lemma 3.3, we have \( \sup_{K^1} w - \inf_{K^1} w \leq (1 - \theta)\omega \).

So the statement is true for \( n = 1 \). Now we assume that the statement is true for \( n = m \) where \( m \) is some positive integer. We define \( \overline{w} \) as follows:

\[ \overline{w}(t, x) = \frac{w((1/4)^{mps} t, r_m x)}{(1 - \theta)^m}. \]

By our assumption, we have that \( \sup_{Q_1} \overline{w} - \inf_{Q_1} \overline{w} \leq \omega \).
And by the argument for $n = 1$, we get

$$
\sup_{K^1} \bar{w} - \inf_{K^1} \bar{w} \leq (1 - \theta)\omega.
$$

Let us go back to $w$, then we have

$$
\sup_{K^{m+1}} w - \inf_{K^{m+1}} w \leq (1 - \theta)^{m+1}\omega.
$$

Hence the statement is also true for $n = m + 1$. Therefore for any natural number $n$,

$$
\sup_{K^n} w - \inf_{K^n} w \leq (1 - \theta)^n\omega.
$$
From Lemma 3.5, we also can get the following proposition:

**Proposition 3.6.** Let \( w : [-2, 0] \times \mathbb{R}^n \rightarrow \mathbb{R} \) be a weak solution and without loss of generality, we assume that \( |w| \leq 1 \) on \([-1, 0] \times B_1(0)\). Then there exist constants \( \gamma > 1 \) and \( \alpha > 0 \) such that for any \( 0 < \rho < 1 \), we have

\[
\sup_{Q_\rho} w - \inf_{Q_\rho} w \leq \gamma \omega \rho^\alpha.
\]
1. Any higher regularity for the weak solution?

2. Can we remove the conditions $ps < p - 1$ and $p > 2$?