Domain decomposition Fourier finite element method for the simulation of 3D marine CSEM measurements

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Outline

Model Setup

EM modeling approaches
  3D FE
  2.5D FE
  Fourier-FE

Numerical results
  Accuracy
  Comparison of CPU time

Conclusions
CSEM method: Model setup

source -> receivers

Reservoir

air
seawater
Maxwell’s equations (frequency domain) are:

\[ \nabla \times \mathbf{E} = i \omega \mu \mathbf{H} \]
\[ \nabla \times \mathbf{H} = (\sigma - i \omega \epsilon) \mathbf{E} + \mathbf{J} \]
\[ \nabla \cdot (\mathbf{\epsilon E}) = 0 \]
\[ \nabla \cdot (\mu \mathbf{H}) = 0 \]
3D FE approach

\[ \nabla \times \left( \mu^{-1} \nabla \times E \right) - i\omega \tilde{\sigma} E = i\omega J \]

\[ \tilde{\sigma} = \sigma - i\omega \epsilon \]

\[ (n \times E) \mid_{\Gamma = \partial\Omega} = 0 \]
3D FE approach

\[
\int_{\Omega} (\nabla \times \mathbf{F})^* \mu^{-1} (\nabla \times \mathbf{E}) d\Omega - i\omega \int_{\Omega} \mathbf{F}^* \bar{\sigma} \mathbf{E} d\Omega = i\omega \int_{\Omega} \mathbf{F}^* \mathbf{J} d\Omega
\]

\[
\mathbf{F} \in H_\Gamma(\text{curl}; \Omega) = \{ \mathbf{F} \in H(\text{curl}; \Omega) : (\mathbf{n} \times \mathbf{F}) |_{\Gamma} = \mathbf{0} \}
\]

\[
H(\text{curl}; \Omega) = \left\{ \mathbf{F} \in (L^2(\Omega))^3 : \text{curl} \mathbf{F} \in (L^2(\Omega))^3 \right\}
\]
3D FE approach

- Discretize entire domain, $\Omega$ into a number of finite elements, e.g. tetrahedral, or hexahedra, etc.
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- Memory requirement is $O(N^{4/3})$. 
2.5D FE approach

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- Sources and receivers are 3D.
- Then, 3D problem $\Rightarrow$ a sequence of independent 2D problems.
- Domain, $\Omega \Rightarrow \Omega_{2D}$. 
2.5D FE approach

- Discretize entire domain, $\Omega_{2D}$ into a number of finite elements, e.g. triangular, or quadrilateral.
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- Limitation: Because of the 2D geology assumption, the 2.5D marine CSEM modeling can not always be relied upon for a consistent treatment of the real environment.
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- Our aim: reduce computational work to $\mathcal{O}(N_1^{1.5} + N_2^2)$ to remove the above limitation.
Fourier FE approach: Computational domains

\[ \text{Subdomain}, \Omega_1 \]
\[ \sigma_{2D}(x, z), \mu_{2D}(x, z) \]

\[ \text{Subdomain}, \Omega_2 \]
\[ \sigma(x, y, z), \mu(x, y, z) \]
Fourier FE approach: Basis functions

- $e^{i r y}$: Fourier basis function.
- $\Phi(x, z)$: 2D coupled $H(\text{curl}; \Omega_1)$ and $H^1(\Omega_1)$ basis function.
- $\Psi(x, y, z)$: 3D $H(\text{curl}; \Omega_2)$ basis function.
Fourier FE approach: Basis functions

- $e^{i\gamma y}$: Fourier basis function.
- $\Phi(x, z)$: 2D coupled $H(\text{curl}; \Omega_1)$ and $H^1(\Omega_1)$ basis function.
- $\psi(x, y, z)$: 3D $H(\text{curl}; \Omega_2)$ basis function.
- $e^{i\gamma y} \Phi(x, z)$: For $\Omega_1$.
- $\psi(x, y, z)$: For $\Omega_2$. 
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- $e^{i ry} \Phi(x, z)$: For $\Omega_1$.
- $\Psi(x, y, z)$: For $\Omega_2$.

$$E(x, y, z) = \sum_{m=-M}^{M} \sum_{n=1}^{N} E_{1,mn} \Phi_n(x, z) e^{irmy} + \sum_{k=1}^{K} E_{2,k} \Psi_k(x, y, z).$$
Fourier FE approach: Linear system

- $\Omega_1$: Triangular finite elements.
- $\Omega_2$: Prismatic finite elements.
Fourier FE approach: Linear system

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\[
\begin{bmatrix}
A_{\Omega_1} & A_{\Omega_1 \cap \Omega_2} \\
A^*_{\Omega_1 \cap \Omega_2} & A_{\Omega_2}
\end{bmatrix} \begin{bmatrix}
E_1 \\
E_2
\end{bmatrix} = \begin{bmatrix}
b_1 \\
0
\end{bmatrix}.
\]
Fourier FE approach: Computational complexity

**Table:** FLOPS estimates for solving a FE system of linear equations using a direct solver.

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- The cost corresponding to $\Omega_1$: $O(N_1^{1.5})$.
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- The total cost with FFE: FLOPS = $O(N_1^{1.5} + N_2^2)$. 
Numerical results: Accuracy: 1D model
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Numerical results: Accuracy: 3D model
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\[ |E_x| (V/m) \]

\[ |E_z| (V/m) \]

- 9 modes
- 3D FE
- IE
Numerical results: Accuracy: 3D complex model
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- $|E_x| (V/m)$ vs $x (m)$
- $|E_z| (V/m)$ vs $x (m)$
Numerical results: Comparison of CPU time

![Graph showing comparison of CPU time for 2.5D FE, 3D FFE, and 3D FE methods. The x-axis represents the number of unknowns, and the y-axis represents CPU time (s). The graph shows that CPU time increases with the number of unknowns for all methods, with 3D FE showing the highest CPU time.]
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- FFE reduces the computational complexity of traditional 3D simulators.