Dynamics of delay differential equations with distributed delays

Kiss, Gábor

BCAM - Basque Center for Applied Mathematics
Bilbao
Spain

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Outline

Delay differential equation

Stability charts

An experiment

Future work
Applications

- Population dynamics
- Infections diseases
- Neuronal dynamics
- Car traffic dynamics
- Laser dynamics
Wright’s equation

\[ \dot{x}(t) = -\alpha x(t-1)\{1 + x(t)\} \quad (\alpha > 0). \tag{1} \]

E. M. Wright.
A non-linear difference-differential equation. 

J.-P. Lessard.
Recent advances about the uniqueness of the slowly oscillating periodic solutions of Wright’s equation. 
Wright’s equation

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M. C. Mackey and L. Glass.
Oscillation and chaos in physiological control systems.

\[
\dot{x}(t) = \beta \frac{x(t-\tau)}{1 + x^n(t-\tau)} - \gamma x, \quad \gamma, \beta, n > 0.
\]  

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Domain-decomposition method for the global dynamics of delay differential equations with unimodal feedback.
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Domain-decomposition method for the global dynamics of delay differential equations with unimodal feedback. 
W. Gurney, S. Blythe, and R. Nisbet.
Nicholson’s blowflies revisited.

**Nicholson’s blowflies equation**: it is of the form

\[
\dot{x}(t) = -\gamma x(t) + px(t - \tau)e^{-ax(t-\tau)}
\]  

(3)

A. Nicholson.
The self-adjustment of populations to change.

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*Nicholson’s blowflies equation*; it is of the form

$$\dot{x}(t) = -\gamma x(t) + px(t - \tau)e^{-ax(t-\tau)}$$  \hspace{1cm} (3)

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An outline of the dynamics of animal populations.  
*Insect ecology and population management: readings in theory, technique, and strategy, 2:3, 1972.*
\[
\dot{x}(t) = -ax(t) - bg(x(t - \tau)), \quad a, b \in \mathbb{R}, \quad \tau \geq 0
\] (4)

\[
\dot{x}(t) = -ax(t) + g \left( \int_0^h x(t - \tau) d\mu(\tau) \right).
\] (5)

Phase space: the Banach \( C = C([0, h], \mathbb{R}) \) space of continuous functions mapping the interval \([0, h]\) into \( \mathbb{R} \), with the supremum norm. Here \( a \in \mathbb{R}, \; g \in C^1 \) and the integral is of Stieltjes-type.
\[ \dot{x}(t) = -ax(t) - bg(x(t - \tau)), \quad a, b \in \mathbb{R}, \quad \tau \geq 0 \]  

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Typically, only one time lag has been introduced in modeling using differential–delay equations, but for better models and for mathematical interest it is desirable to study equations in which two or more or more time–lags may appear.

R. D. Nussbaum.
Differential-delay equations with two time lags.
The integral is of Stieltjes-type, \( \mu : \mathbb{R} \to \mathbb{R} \) is a non-decreasing and right-continuous function satisfying

(A1) \( \mu(\tau) = 1, \text{ if } \tau \geq h \)

and

(A2) \( \mu(\tau) = 0, \text{ if } \tau < 0, \)

where \( a, b \in \mathbb{R} \ h \geq 0 \). (A1) and (A2) together with monotonicity of function \( \mu \) imply that

\[
\int_{0}^{h} d\mu(\tau) = 1. \tag{6}
\]

In (8) \( E \) is the \( E = \int_{0}^{h} \tau d\mu(\tau) \) average delay.
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BCAM and University of Szeged
If $\hat{x}_t$ is an equilibrium point of (5) then $y_t^{\psi} = D_2F(t, \hat{x})\psi$ is the unique solution of the linear variational equation which, when (5) is considered, is of form

$$\dot{y}(t) = -ay(t) - b \int_0^h y(t + \tau)d\mu(\tau)$$

(7)

where $b = -g'(x)$, $\hat{x} \in C$, $x \in \mathbb{R}$.
\[
\dot{x}(t) = -ax(t) - bx(t - E) \tag{8}
\]

and

\[
\dot{x}(t) = -ax(t) - b \int_{0}^{h} x(t - \tau) d\mu(\tau). \tag{9}
\]
Theorem

The zero solution $x \equiv 0$ of $\dot{x}(t) = -ax(t) - bx(t - E)$ is asymptotically stable if

$$E < \frac{\arccos \left( -\frac{a}{b} \right)}{\sqrt{b^2 - a^2}}, \ b > |a|.$$
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Theorem (Krisztin)

The zero solution $x \equiv 0$ of $\dot{x}(t) = -b \int_{0}^{h} x(t - \tau) d\mu(\tau)$, is asymptotically stable if

$$E = \int_{0}^{h} \tau d\mu(\tau) < \frac{\pi}{2b}.$$
The corresponding function and equation related to

\[ \dot{x}(t) = -ax(t) - b \int_{0}^{h} x(t - \tau)d\mu(\tau), \]  \hspace{1cm} (10)

are

\[ h(\lambda) : \mathbb{C} \rightarrow \mathbb{C}, \lambda \mapsto \lambda + a + b \int_{0}^{h} e^{-\lambda\tau} d\mu(\tau) \]  \hspace{1cm} (11)

and

\[ \lambda + a + b \int_{0}^{h} e^{-\lambda\tau} d\mu(\tau) = 0, \lambda \in \mathbb{C}. \]  \hspace{1cm} (12)
**Definition**

Let $\mu : \mathbb{R} \to \mathbb{R}$ be a monotonically nondecreasing function with expected value $E$. We say that $\mu$ is *symmetric about its expectation* if

$$\mu(E - x) = 1 - \mu(E + x - 0).$$  \hspace{1cm} (13)

**Lemma**

Let $\mu : \mathbb{R} \to \mathbb{R}$ be symmetric about its expectation $E > 0$ in

$$\dot{x}(t) = -ax(t) - b \int_{0}^{h} x(t - \tau) d\mu(\tau).$$  \hspace{1cm} (14)

Then $\Gamma_{0} < |\Gamma_{k,l}^{+}|$ and $\Gamma_{0} < |\Gamma_{k,m}^{-}|$ on $\tilde{I}_{k}^{+}$ and $\tilde{I}_{k}^{-}$, respectively, for $1 \leq l \leq i$, $1 \leq m \leq j$. 
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Then $\Gamma_0 \prec |\Gamma^+_k, l|$ and $\Gamma_0 \prec |\Gamma^-_k, m|$ on $\tilde{I}^+_k$ and $\tilde{I}^-_k$, respectively, for $1 \leq l \leq i$, $1 \leq m \leq j$. 

Dynamics of delay differential equations with distributed delays

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Theorem

Let us fix a number $E > 0$, furthermore, let $b > |a|$, and consider

$$\dot{x}(t) = -ax(t) - bx(t - E). \quad (15)$$

Let us suppose that the trivial solution $x \equiv 0$ of equation (15) is asymptotically stable for a given pair of parameters $a, b$. Then the trivial solution $x \equiv 0$ of equation

$$\dot{x}(t) = -ax(t) - b \int_{0}^{h} x(t - \tau)d\mu(\tau), \quad (16)$$

given with an arbitrary distribution function that is symmetric about the fixed expectation $E$ is asymptotically stable.
G. Kiss and B. Krauskopf
Stability implications of delay distribution for first-order and second-order systems.
**Theorem**

Let $\mu$ be symmetric about its expected value $E$. Then, the trivial solution $x = 0$ of

$$
\dot{x}(t) = -ax(t) - b \int_0^h x(t - \tau) d\mu(\tau),
$$

is asymptotically stable if

$$
E < \frac{\arccos\left(\frac{-a}{b}\right)}{\sqrt{b^2 - a^2}}, \text{ where } b > |a|.
$$
Does delay distribution always increase the stability region?

\[ \ddot{x}(t) = -\dot{x}(t) - ax(t) - bx(t - 1), \quad (18) \]

and

\[ \ddot{x}(t) = -\dot{x}(t) - ax(t) - b \left( \frac{1}{2}x(t - \tau_1) + \frac{1}{2}x(t - \tau_2) \right), \quad (19) \]

where \( \tau_1 = 0.55 \) and \( \tau_1 = 1.45 \), so that we have a mean of \( E = 1 \).
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Dynamics of delay differential equations with distributed delays

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Is stability preserving order dependent?

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Stabilizing effect of delay distribution for a class of second-order systems without instantaneous feedback.
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Equations with two delays

\[ \dot{x}(t) = -ax(t) - b \left( 0.5x(t - 1.65) + 0.5x(t - 0.35) \right) \]
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\[ \dot{x}(t) = -ax(t) - b \left( 0.5x(t - 1.65) + 0.5x(t - 0.35) \right) \{1 + x(t)\}, \]

\[ a = .48, \ b = 4.85 \]
Equations with two delays

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\[ \dot{x}(t) = -ax(t) - b(0.5x(t - 1.65) + 0.5x(t - 0.35)) \{1 + x(t)\}, \]

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\(a = 0, \ b = 4.85\)
Theorem

Let $a = 0$. Then

$$\dot{x}(t) = -ax(t) - b \left(0.5x(t - 1.65) + 0.5x(t - 0.35)\right) \{1 + x(t)\},$$

has at least three nontrivial coexisting periodic solutions at the parameter value $b = 6.8$. 
Computation needed

- Periodic solutions to the Van der Pole’s oscillator

\[ \ddot{x}(t) - \varepsilon \dot{x}(t)(1 - x^2(t)) + x(t - \tau) - kx(t) = 0. \quad (20) \]

- R. D. Nussbaum
  Periodic solutions of some nonlinear autonomous functional differential equation.

- Compute the stability of periodic solutions
- Compute invariant tori in infinite dimension
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