Introduction to Set Oriented Numerics

Roberto Castelli

BCAM

Bilbao, 1st February 2011
Main Goal: Study the long term behavior of complex Dynamical Systems

Set Oriented Numerics

- Computation of several short term trajectories instead of single long term trajectory
- Approximation of Global structure
  - Invariant Sets: global attractors, Invariant manifolds
  - Invariant measures, almost invariant sets
  - Transport operators
  - Multiobjective optimization (Pareto set)
Invariant Sets
  Relative Global Attractor
  Invariant manifold

GAIO Implementation

Application: detecting connecting orbit

Bibliography
Invariant Sets
  Relative Global Attractor
  Invariant manifold

GAIO Implementation

Application: detecting connecting orbit

Bibliography
**Invariant Sets**

Relative Global Attractor

Invariant manifold

**GAIO Implementation**

Application: detecting connecting orbit

**Bibliography**
Consider

\[ x_{k+1} = f(x_k), \quad k = 0, 1, 2, \ldots \]

\( f : \mathbb{R}^n \to \mathbb{R}^n \) diffeomorphism.

**Invariant set**

\( A \subset \mathbb{R}^n \) is Invariant if \( f(A) = A \)

**Definition Relative Global attractor**

Let \( Q \subset \mathbb{R}^n \) be a compact set. Define **global attractor relative to** \( Q \) by

\[ A_Q = \bigcap_{j \geq 0} f^j(Q) \]

**Properties**

\( A_Q \subset Q \)

\( f^{-1}(A_Q) \subset A_Q \) but not necessarily \( f(A_Q) \subset A_Q \).
Computation of Relative Global Attractor

The Subdivision-Selection Algorithm

From the initial set $\mathcal{B}_0 = Q$, define a sequence $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3$ of finite collections of compact subsets of $\mathbb{R}^n$, such that

- $\text{diam}(\mathcal{B}_k) = \max_{B \in \mathcal{B}_k} \text{diam}(B) \to 0, \quad k \to \infty$
- $\mathcal{B}_k$ approaches $A_Q$

Inductively $\mathcal{B}_k$ is obtained from $\mathcal{B}_{k-1}$ in two steps

1. **Subdivision:** define a new collection $\tilde{\mathcal{B}}_k$ such that
   
   $$\bigcup_{B \in \tilde{\mathcal{B}}_k} B = \bigcup_{B \in \mathcal{B}_{k-1}} B$$

   $$\text{diam}(\tilde{\mathcal{B}}_k) \leq \theta_k \text{diam}(\mathcal{B}_{k-1}), \quad \theta_k \in (0, 1)$$

2. **Selection:** define $\mathcal{B}_k$ as

   $$\mathcal{B}_k = \left\{ B \in \tilde{\mathcal{B}}_k : \exists \tilde{B} \in \tilde{\mathcal{B}}_k \text{ such that } f^{-1}(B) \cap \tilde{B} \neq \emptyset \right\}$$
Computation of Relative Global Attractor

The Subdivision-Selection Algorithm

From the initial set $\mathcal{B}_0 = Q$, define a sequence $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3$ of finite collections of compact subsets of $\mathbb{R}^n$, such that

1. $\text{diam}(\mathcal{B}_k) = \max_{B \in \mathcal{B}_k} \text{diam}(B) \to 0, \quad k \to \infty$

2. $\mathcal{B}_k$ approaches $A_Q$

3. Inductively $\mathcal{B}_k$ is obtained from $\mathcal{B}_{k-1}$ in two steps

   1. **Subdivision:** define a new collection $\tilde{\mathcal{B}}_k$ such that

      $\bigcup_{B \in \tilde{\mathcal{B}}_k} B = \bigcup_{B \in \mathcal{B}_{k-1}} B$

      $\text{diam}(\tilde{\mathcal{B}}_k) \leq \theta_k \text{diam}(\mathcal{B}_{k-1}), \quad \theta_k \in (0, 1)$

   2. **Selection:** define $\mathcal{B}_k$ as

      $\mathcal{B}_k = \{ B \in \tilde{\mathcal{B}}_k : \exists B \in \tilde{\mathcal{B}}_k \text{ such that } f^{-1}(B) \cap \tilde{B} \neq \emptyset \}$
The Subdivision-Selection Algorithm

From the initial set $B_0 = Q$, define a sequence $B_1, B_2, B_3$ of finite collections of compact subsets of $\mathbb{R}^n$, such that

1. $\operatorname{diam}(B_k) = \max_{B \in B_k} \operatorname{diam}(B) \to 0$, $k \to \infty$
2. $B_k$ approaches $A_Q$

Inductively $B_k$ is obtained from $B_{k-1}$ in two steps

1. **Subdivision**: define a new collection $\tilde{B}_k$ such that
   \[
   \bigcup_{B \in \tilde{B}_k} B = \bigcup_{B \in B_{k-1}} B
   \]
   \[
   \operatorname{diam}(\tilde{B}_k) \leq \theta_k \operatorname{diam}(B_{k-1}), \quad \theta_k \in (0, 1)
   \]

2. **Selection**: define $B_k$ as
   \[
   B_k = \{ B \in \tilde{B}_k : \exists \tilde{B} \in \tilde{B}_k \text{ such that } f^{-1}(B) \cap \tilde{B} \neq \emptyset \} \]
Example: Hénon map

Consider the Hénon map

\[
\begin{align*}
    x_{k+1} &= 1 - ax_k^2 + y_k/5 \\
    y_{k+1} &= 5bx_k
\end{align*}
\]

\(a = 1, b = 0.54\)

Covering of the Relative Global Attractor of \(Q = [-2, 2] \times [-8, 8]\) at different subdivision steps.
Convergence Result

The abstract subdivision algorithm converges to the relative global attractor $A_Q$. Denote with $Q_k = \bigcup_{B \in B_k} B$, $k > 0$

**Theorem** Dellnitz and Hohmann (1997)

Let $A_Q$ be a global attractor relative to the compact set $Q$ and $B_0$ a finite collection of closed subsets with $Q_0 = \bigcup_{B \in B_0} B = Q$ Then

$$\lim_{k \to \infty} h(A_Q, Q_k) = 0$$

where $h(B, C)$ denotes the Hausdorff distance between two compact sets $B, C \subset \mathbb{R}^n$
Invariant Sets

Relative Global Attractor

Invariant manifold

GAIO Implementation

Application: detecting connecting orbit

Bibliography
Let $p$ be a **hyperbolic fixed point** for $f$. Define **Local Stable Manifold** of $p$

$$W^s_\varepsilon(p) = \{x : \| f^k(x) - p \| < \varepsilon, \text{for every } k \in \mathbb{N} \}$$

**Local unstable**: $k \mapsto (-k)$

The sets

$$W^u(p) = \bigcup_{k \in \mathbb{N}} f^k(W^u_\varepsilon(p)), \quad W^s(p) = \bigcup_{k \in \mathbb{N}} f^{-k}(W^s_\varepsilon(p))$$

are the **global (un)-stable manifolds of $p$**.
Let $p$ be a hyperbolic fixed point for $f$. Define Local Stable Manifold of $p$

$$W^s_\varepsilon(p) = \{x : \| f^k(x) - p \| < \varepsilon, \text{ for every } k \in \mathbb{N} \}$$

(Local unstable: $k \mapsto (-k)$)

The sets

$$W^u(p) = \bigcup_{k \in \mathbb{N}} f^k(W^u_\varepsilon(p)), \quad W^s(p) = \bigcup_{k \in \mathbb{N}} f^{-k}(W^s_\varepsilon(p))$$

are the global (un)-stable manifolds of $p$. 

Computing the unstable manifold

Let \( p \) a fixed point for \( f \) and \( U_\varepsilon(p) \) a \( \varepsilon \)-neighborhood of \( p \).  

**Remark**: if \( \varepsilon \) is small enough, then \( W^u_\varepsilon(p) \) is the global attractor relative to \( U_\varepsilon(p) \):

\[
W^u_\varepsilon(p) = \bigcap_{k \geq 0} f^k(U_\varepsilon(p))
\]
Computing the unstable manifold

Initialization - Continuation Algorithm

**Aim:** Compute the unstable manifold of a point \( p \) into a (large) compact set \( Q \), \( (\ p \in Q) \)

1. Given \( Q \), compute \( \mathcal{P}_0, \mathcal{P}_1, \ldots, \mathcal{P}_l \) nested sequence of fine partitions of \( Q \). Select the element \( C \in \mathcal{P}_l \) such that \( p \in C \) and \( A_C = W^u_{loc}(p) \cap C \)

2. Initialization Starting from \( \mathcal{B}_0 = \{C\} \), refine the approximation of \( W^u_{loc}(p) \cap C \) by subdivision, yielding \( \mathcal{B}_{k}^{(0)} \subset \mathcal{P}_{l+k} \)

3. Continuation From \( \mathcal{B}_{k}^{(j-1)} \) compute

\[
\mathcal{B}_k^{(j)} = \{B \in \mathcal{P}_{l+k} : f(B') \cap B \neq \emptyset, \text{for some } B' \in \mathcal{B}_k^{(j-1)}\}\]
Computing the unstable manifold

**Initialization - Continuation Algorithm**

**Aim:** Compute the unstable manifold of a point \( p \) into a (large) compact set \( Q, \ ( p \in Q) \)

1. Given \( Q \), compute \( P_0, P_1, \ldots P_l \) nested sequence of fine partitions of \( Q \). Select the element \( C \in P_l \) such that \( p \in C \) and \( A_C = W^u_{loc}(p) \cap C \)

2. Initialization Starting from \( B_0 = \{C\} \), refine the approximation of \( W^u_{loc}(p) \cap C \) by subdivision, yielding \( B^{(0)}_k \subset P_{l+k} \)

3. Continuation From \( B^{(j-1)}_k \) compute

\[
B^{(j)}_k = \{ B \in P_{l+k} : f(B') \cap B \neq \emptyset, \text{for some } B' \in B^{(j-1)}_k \}\]
Computing the unstable manifold

**Initialization - Continuation Algorithm**

**Aim:** Compute the unstable manifold of a point \( p \) into a (large) compact set \( Q \), ( \( p \in Q \) )

1. Given \( Q \), compute \( P_0, P_1, \ldots P_l \) nested sequence of fine partitions of \( Q \). Select the element \( C \in P_l \) such that \( p \in C \) and \( A_C = W_{loc}^u(p) \cap C \)

2. **Initialization** Starting from \( B_0 = \{ C \} \), refine the approximation of \( W_{loc}^u(p) \cap C \) by subdivision, yielding \( B_k^{(0)} \subset P_{l+k} \)

3. **Continuation** From \( B_k^{(j-1)} \) compute

\[
B_k^{(j)} = \{ B \in P_{l+k} : f(B') \cap B \neq \emptyset, \text{ for some } B' \in B_k^{(j-1)} \}
\]
Computing the unstable manifold

**Initialization - Continuation Algorithm**

**Aim:** Compute the unstable manifold of a point \( p \) into a (large) compact set \( Q, \ (p \in Q) \)

1. Given \( Q \), compute \( \mathcal{P}_0, \mathcal{P}_1, \ldots \mathcal{P}_l \) nested sequence of fine partitions of \( Q \). Select the element \( C \in \mathcal{P}_l \) such that \( p \in C \) and \( A_C = W^u_{loc}(p) \cap C \)

2. **Initialization** Starting from \( \mathcal{B}_0 = \{C\} \), refine the approximation of \( W^u_{loc}(p) \cap C \) by subdivision, yielding \( \mathcal{B}^{(0)}_k \subset \mathcal{P}_{l+k} \)

3. **Continuation** From \( \mathcal{B}^{(j-1)}_k \) compute

\[
\mathcal{B}^{(j)}_k = \{B \in \mathcal{P}_{l+k} : f(B') \cap B \neq \emptyset, \text{for some } B' \in \mathcal{B}^{(j-1)}_k \}
\]
Lorenz system

\[
\begin{align*}
\dot{x} &= \sigma(y - x) \\
\dot{y} &= \rho x - y - xz \\
\dot{z} &= -\beta z + xy
\end{align*}
\]

(0, 0, 0) is fixed point
\(\sigma = 10, \ \rho = 28, \ \beta = 8/3\)

Covering of the two-dimensional stable manifold of the origin
Left: \(l = 9, k = 6, j = 4\) initial box \(Q = [-70, 70] \times [-70, 70] \times [-80, 80]\),
Right: \(l = 21, k = 0, j = 10\), initial box \(Q = [-120, 120] \times [-120, 120] \times [-160, 160]\)
Stable manifold Lorenz system
Chua Circuit, \[
\begin{aligned}
\dot{x} &= \alpha(y - m_0 x - \frac{1}{3} m_1 x^3) \\
\dot{y} &= x - y + z \\
\dot{z} &= -\beta y
\end{aligned}
\]

Fixed points \( P_\pm = \left( \pm \sqrt{-3 \frac{m_0}{m_1}}, 0, \mp \sqrt{-3 \frac{m_0}{m_1}} \right) \)

\[
\begin{align*}
\alpha &= 18, \quad \beta = 33 \\
m_0 &= -0.2, \quad m_1 = 0.01
\end{align*}
\]

Figure: (a) Approximation of the relative global attractor for the Chua system (b) Union of the coverings of the unstable manifold of the fixed points \( P_\pm \).
OUTLINE

Invariant Sets
  Relative Global Attractor
  Invariant manifold

GAIO Implementation

Application: detecting connecting orbit

Bibliography
GAIO Implementation

GAIO: Global Analysis Invariant Object

Boxes: Generalized rectangle $B(c, r) \subset \mathbb{R}^n$

$$B(c, r) = \{ y \in \mathbb{R}^n : |y_i - c_i| \leq r_i, \; i : 1 \ldots n \}$$

identified by a centre and vector of radii, $c, r \in \mathbb{R}^n$.

Subdivision: by bisection along one of the coordinate direction: from $B(c, r)$ to $B_1(c_+, r^1), \; B_2(c_-, r^1)$

$$r^1_i = \begin{cases} 
    r_i & i \neq j \\
    r_i/2 & i = j 
\end{cases}$$

$$c_i^\pm = \begin{cases} 
    c_i & i \neq j \\
    c_i \pm r_i/2 & i = j 
\end{cases}$$
Storage of boxes: Binary tree

In Fig: representation of three subdivision steps in three dimension. The spatial coordinates of a collection are completely determined by the tree structure of the boxes and the initial rectangle $B_0$. 
Box Map and Box Intersection

The Subdivision and the Continuation algorithm are based on the theoretical selection step

\[ \mathcal{B}_k = \{ B' \in \tilde{\mathcal{B}}_k : f(B) \cap B' \neq \emptyset \text{ for some } B \in \tilde{\mathcal{B}}_k \} \]

How to define the image of a box?

Box image \[ \mathcal{F}_\mathcal{B}(B) = \{ B' \in \mathcal{B} : f(B) \cap B' \neq \emptyset \} \]

How to compute \( f(B) \) and then \( \mathcal{F}_\mathcal{B}(B) \) ?

The method is to choose a finite set \( T \subset B \) of test points and define

\[ \tilde{\mathcal{F}}_\mathcal{B}(B) = \{ B' \in \mathcal{B} : f(T) \cap B' \neq \emptyset \} \]

Clearly

\[ \tilde{\mathcal{F}}_\mathcal{B}(B) \subset \mathcal{F}_\mathcal{B}(B) \]

but possibly

\[ \tilde{\mathcal{F}}_\mathcal{B}(B) \neq \mathcal{F}_\mathcal{B}(B) \]
Box Map and Box Intersection

The Subdivision and the Continuation algorithm are based on the theoretical selection step

\[ B_k = \{ B' \in \tilde{B}_k : f(B) \cap B' \neq \emptyset \text{ for some } B \in \tilde{B}_k \} \]

How to define the image of a box?

Box image \[ \mathcal{F}_B(B) = \{ B' \in B : f(B) \cap B' \neq \emptyset \} \]

How to compute \( f(B) \) and then \( \mathcal{F}_B(B) \)?

The method is to choose a finite set \( T \subset B \) of test points and define

\[ \tilde{\mathcal{F}}_B(B) = \{ B' \in B : f(T) \cap B' \neq \emptyset \} \]

Clearly

\[ \tilde{\mathcal{F}}_B(B) \subset \mathcal{F}_B(B) \]

but possibly

\[ \tilde{\mathcal{F}}_B(B) \neq \mathcal{F}_B(B) \]
Box Map and Box Intersection

The Subdivision and the Continuation algorithm are based on the theoretical selection step

\[ \mathcal{B}_k = \{ B' \in \tilde{\mathcal{B}}_k : f(B) \cap B' \neq \emptyset \text{ for some } B \in \tilde{\mathcal{B}}_k \} \]

How to define the image of a box?

Box image \( \mathcal{F}_B(B) = \{ B' \in \mathcal{B} : f(B) \cap B' \neq \emptyset \} \)

How to compute \( f(B) \) and then \( \mathcal{F}_B(B) \)?

The method is to choose a finite set \( T \subset B \) of test points and define

\[ \tilde{\mathcal{F}}_B(B) = \{ B' \in \mathcal{B} : f(T) \cap B' \neq \emptyset \} \]

Clearly

\[ \tilde{\mathcal{F}}_B(B) \subset \mathcal{F}_B(B) \]

but possibly

\[ \tilde{\mathcal{F}}_B(B) \neq \mathcal{F}_B(B) \]
Choice of test-points

In low dimensional phase space \((d \leq 3)\)
- N points on the edges of the boxes + the center
- on uniform grid within the box

In higher dimension
- randomly distributed

Remark: Rigorous choice of test point in such a way that no boxes are lost due to the discretization could be done if the Lipschitz constant of the map \(f\) is known.
Choice of test-points

In low dimensional phase space \((d \leq 3)\)

- \(N\) points on the edges of the boxes + the center
- on uniform grid within the box

In higher dimension

- randomly distributed

Remark: Rigorous choice of test point in such a way that no boxes are lost due to the discretization could be done if the Lipschitz constant of the map \(f\) is known.
OUTLINE

Invariant Sets
  Relative Global Attractor
  Invariant manifold

GAIO Implementation

Application: detecting connecting orbit

Bibliography
The **Hat Algorithm**

**Goal:** Detect (homo-)heteroclinic orbit connecting two hyperbolic fixed points.

Consider

\[ \dot{x} = f(x, \lambda), \quad \lambda \in \Lambda \]

and \( x_\lambda, y_\lambda \) two one-parameter families of hyperbolic fixed points. If for \( \lambda = \bar{\lambda} \) there exist a heteroclinic connection \( \gamma \), then

\[ \gamma \in W^u(x_{\bar{\lambda}}) \cap W^s(y_{\bar{\lambda}}) \]

**Discretization:** Analysis of

\[ I_k(\lambda) = U_k^{(j)}(x_\lambda) \cap S_k^{(j)}(y_\lambda) \]

k subdivision steps

j iteration steps

when \( \lambda \in \tilde{\Lambda} \) and \( k \) vary.
The Hat Algorithm

- Fix $\lambda \in \tilde{\Lambda}$ and compute $U_k^{(j)}(x_\lambda)$ and $S_k^{(j)}(y_\lambda)$ for $k$ small. (rough covering)
- **Refine** the covering up to the maximum $k = m(\lambda)$ s.t $I_k(\lambda) \neq \emptyset$
- Change $\lambda$ and repeat the procedure
- Look at the maximum of $m(\lambda)$. 

1st February 2011

Introduction to set Oriented Numerics

Roberto Castelli
Heteroclinic connection between two Lyapunov orbits in the CRTBP. $x_\lambda, y_\lambda$ Lyapunov orbits, $\lambda$ Jacobi constant

(a) Graph of the function $m(\lambda)$  
(b) Heteroclinic connection for $J = 3.16988$

**Figure:** The *Hat Algorithm* applied to detect heteroclinic connections between two Lyapunov orbits in the Earth-Moon CRTBP
Invariant Sets
  Relative Global Attractor
  Invariant manifold

GAIO Implementation

Application: detecting connecting orbit

Bibliography


REFERENCES

APPLICATIONS


Schütze, O., Mostaghim, S., Dellnitz, M., Teich, J.: Covering Pareto Sets by Multilevel Evolutionary Subdivision Techniques Proceedings of EMO 2003, Faro, Portugal


Thank you

⇒ Relative Global attractor
⇒ Invariant Manifolds
⇒ GAIO Implementation
⇒ Connecting Orbits