Some inverse boundary value problems in two dimensions: theory and applications

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Summary

- Introduction

- Uniqueness, reconstruction and stability for the Calderón and Gel’fand-Calderón problem

- Reconstruction at high energies

- Virtual Hybrid Edge Detection

« If D represents an in-homogeneous conducting body with electrical conductivity $\sigma$, determine $\sigma$ by means of direct current steady state electrical measurements carried out on the surface of D, that is, without penetrating D.»
The Calderón problem

<table>
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<th>$D \subset \mathbb{R}^d, d \geq 2$</th>
<th>open bounded domain with smooth boundary.</th>
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<td>$\sigma$</td>
<td>electrical conductivity on $D$.</td>
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- $\sigma(x)$ is a symmetric positive definite matrix, s.t.
  
  $$c\|\xi\|^2 \leq \langle \xi, \sigma(x)\xi \rangle \leq C\|\xi\|^2,$$

  for $x \in D$, $\xi \in \mathbb{R}^d$, where $c, C > 0$.

| $\Lambda_\sigma f = \sigma \nabla u \cdot \nu|_{\partial D}$ | Dirichlet-to-Neumann operator. |
|---|---|
| (voltage-to-current map) |

- $f \in H^{1/2}(\partial D)$, $\nu$ exterior normal at $\partial D$, $u$ the unique $H^1$ solution of

  $$\begin{cases} 
  \nabla \cdot (\sigma \nabla u) = 0 & \text{in } D, \\
  u|_{\partial D} = f. 
  \end{cases}$$

- $\Lambda_\sigma : H^{1/2}(\partial D) \to H^{-1/2}(\partial D)$

Calderón problem

Given $\Lambda_\sigma$, determine $\sigma$ on $D$. 
Physical interpretation

- Conductivity equation (Ohm’s law + conservation of electric charge):

\[ \nabla \cdot (\sigma \nabla u) = 0 \quad \text{in } D. \quad (1) \]

- Action of \( \Lambda_\sigma \):
  (voltage at \( \partial D \)) \( f = u|_{\partial D} \mapsto \Lambda_\sigma f \) (associated current)

- Boundary data, Cauchy space (graph of \( \Lambda_\sigma \)):

\[ \{(u|_{\partial D}, (\sigma \nabla u \cdot \nu)|_{\partial D}) | u \text{ solution of (1)}\}. \]

The set of all possible pairs of voltage and current measurements at the boundary of \( D \).
Conductivity imaging: motivations

Fields of application:
- medical imaging
- geophysical prospecting
- non-destructive testing

Pros: high contrast, cheap, safe. Cons: low resolution.

Electrical Impedance Tomography (EIT)

Monitoring lung ventilation distribution

credits: Zhao et al. Crit Care. 2010
**Definition**

σ is called *isotropic* if \( \sigma = \sigma(x)I \) for some function \( \sigma(x) \), where \( I \) is the identity matrix. Otherwise \( \sigma \) is called *anisotropic*.

If \( \sigma \) is isotropic and \( C^2 \) we can substitute \( \tilde{u} = u \sqrt{\sigma} \) in \( \nabla \cdot (\sigma \nabla u) = 0 \). It satisfies the following Schrödinger equation:

\[
(-\Delta + \nu) \tilde{u} = 0 \quad \text{in } D, \quad \text{where } \nu = \frac{\Delta \sqrt{\sigma}}{\sqrt{\sigma}},
\]

If \( \Phi_{\nu} \) is the Dirichlet-to-Neumann operator associated to \( \nu \), we have

\[
\Phi_{\nu} = \sigma^{-1/2} \left( \Lambda_{\sigma} \sigma^{-1/2} + \frac{\partial \sigma^{1/2}}{\partial \nu} \right).
\]
Gel’fand–Calderón Problem

Schrödinger equation at fixed energy $E \in \mathbb{R}$,

$$(-\Delta + \nu)\psi = E\psi \quad \text{in } D,$$

where $\nu \in L^\infty(D)$ is a potential (energy), $D \subset \mathbb{R}^d$ open bounded domain.

If 0 is not a Dirichlet eigenvalue of $-\Delta + \nu - E$ in $D$, we can define:

$$\Phi_v(E)f = \frac{\partial u}{\partial \nu}|_{\partial D}$$

Dirichlet-to-Neumann map

- $f \in H^{1/2}(\partial D)$, $\nu$ exterior normal to $\partial D$, $u$ the unique solution $H^1$ of

$$\begin{cases}
(-\Delta + \nu)u = Eu & \text{in } D, \\
u|_{\partial D} = f.
\end{cases}$$

- $\Phi_v(E) : H^{1/2}(\partial D) \to H^{-1/2}(\partial D)$

Gel’fand–Calderón inverse problem

Given $\Phi_v(E)$ for a fixed $E \in \mathbb{R}$, determine $\nu$ on $D$. 
Motivations: acoustic tomography

- Reduced acoustic equation for time-harmonic waves with frequency \( \omega \in \mathbb{R} \),

\[
\nabla \cdot \left( \frac{1}{\rho} \nabla p \right) + \omega^2 \kappa p = 0 \quad \text{in} \ \Omega,
\]

- \( p \) pressure
- \( \rho(x) \geq c > 0 \) density
- \( \kappa(x) \) compressibility
- \( c = (\kappa \rho)^{-1/2} \) speed of sound
- \( \Omega \subset \mathbb{R}^2 \) open bounded domain

- Dirichlet-to-Neumann map

\[
\Lambda_{\omega,\kappa,\rho} : f \rightarrow \frac{1}{\rho} \frac{\partial p}{\partial \nu} \bigg|_{\partial \Omega}.
\]

- Inverse Problem: recover \( \rho \) and \( \kappa \) from \( \Lambda_{\omega,\kappa,\rho} \).
Questions

▶ Uniqueness:
  ▶ injectivity of $\sigma \mapsto \Lambda_\sigma$.

▶ Reconstruction

▶ Stability:
  ▶ there exists $f$ such that

$$\|\sigma_2 - \sigma_1\|_{L^\infty(D)} \leq f(\|\Lambda_2 - \Lambda_1\|),$$

$$f(t) \to 0 \text{ as } t \to 0^+. $$
Historical remarks for Calderón problem

First precise formulation: Calderón (1980).
Similar formulations : Slichter (1933), Tikhonov (1949), Druskin (1982).

First global results (isotropic case, full boundary data) :

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<th>$d = 2$</th>
<th>$d \geq 3$</th>
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<td><strong>Uniqueness:</strong></td>
<td>Nachman (1996)</td>
<td>Sylvester-Uhlmann (1987)</td>
</tr>
<tr>
<td><strong>Stability:</strong></td>
<td>Liu (1997)</td>
<td>Alessandrini (1988)</td>
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Uniqueness for $\sigma \in L^\infty(D), D \subset \mathbb{R}^2$: Astala-Paivarinta (2006).
Uniqueness for $\sigma \in Lip(D), D \subset \mathbb{R}^d, d \geq 3$: Caro-Rogers (2014).
Uniqueness for $\sigma \in L^\infty(D), D \subset \mathbb{R}^d, d \geq 3$: open problem.

Several uniqueness results in the case of partial data on the boundary.
Historical remarks for Gel’fand-Calderón

First formulation: Gel’fand (1954).

First global results (scalar case, full boundary data):

\[ D \subset \mathbb{R}^d \]

\[
\begin{array}{ccc}
  d = 2 & d \geq 3 \\
& & Novikov (1988) \\
\end{array}
\]

In two dimension, uniqueness for potentials in \( L^p, p > 2 \):
Several uniqueness results in the case of partial data on the boundary.
Stability estimate in 2D

Theorem 1 (Novikov-S. 2010, S. 2011)

Let $D \subset \mathbb{R}^2$ be an open bounded domain with a $C^2$ boundary, $v_1, v_2 \in C^2(\bar{D}, M_n(\mathbb{C}))$ two matrix-valued potentials with $\|v_j\|_{C^2(\bar{D})} \leq N$ for $j = 1, 2$, and $\Phi_1, \Phi_2$ the corresponding Dirichlet-to-Neumann operators. For simplicity we also assume that $v_1\vert_{\partial D} = v_2\vert_{\partial D}$ and $\frac{\partial}{\partial \nu} v_1\vert_{\partial D} = \frac{\partial}{\partial \nu} v_2\vert_{\partial D}$. Then there exists a constant $C = C(D, N, n)$ such that

$$\|v_2 - v_1\|_{L^\infty(D)} \leq C(\log(3 + \|\Phi_2 - \Phi_1\|_*^{-1}))^{-\frac{3}{4}} \log(3 \log(3 + \|\Phi_2 - \Phi_1\|_*^{-1}))^2,$$

where $\|\cdot\|_*$ is the induced operator norm on $L^\infty(\partial D, M_n(\mathbb{C}))$ and $\|v\|_{L^\infty(D)} = \max_{1 \leq i, j \leq n} \|v_{i,j}\|_{L^\infty(D)}$ (likewise for $\|v\|_{C^2(\bar{D})}$) for a matrix-valued potential $v$. 

Remarks

- First stability estimate in dimension two for general potentials.

- Similar results had been obtained only for potentials with special forms, namely conductivity-type potentials, i.e. $\nu = \frac{\Delta \sqrt{\sigma}}{\sqrt{\sigma}}$ for an isotropic conductivity $\sigma$.

- The logarithm is sharp (Mandache 2001).
General strategy (uniqueness at $E = 0$)

Aim: $\Phi_2 = \Phi_1 \Rightarrow v_2 = v_1$.

Starting point: Alessandrini’s identity.

$$
\int_D (v_2 - v_1)u_2u_1 = \int_{\partial D} u_2(\Phi_2 - \Phi_1)u_1 = 0,
$$

for every $u_j$ solution of $(-\Delta + v_j)u_j = 0$ in $\bar{D}$, $j = 1, 2$.

Idea: prove that the products of solutions of the Schrödinger equation are dense in $L^2(D)$.

Exponentially growing solutions or complex geometrical optics (CGO) solutions (Faddeev, Beals-Coifman, Sylvester-Uhlmann):

$$
u_j(x, k) \sim e^{ixk}, \quad \text{for } |k| \to +\infty,
$$

$$
k \in \mathbb{C}^d, k^2 = k_1^2 + \ldots + k_d^2 = 0.
$$
Uniqueness in dimension $d \geq 3$

Fix $p \in \mathbb{R}^d, k, l \in \mathbb{C}^d$ such that $k^2 = l^2 = 0$ and $k - l = p$ (possible only in dimension $d \geq 3$, since $|k|, |l| \to +\infty$),

$$u_1 \sim e^{ixk}, \quad u_2 \sim e^{-ixl}.$$ 

We obtain:

$$0 = \int_D (v_2 - v_1)u_2u_1 \sim \int_D (v_2 - v_1)e^{ixp} = \widehat{v_2 - v_1}(p).$$

Taking the limit $|k|, |l| \to +\infty$ we have $\widehat{v_2 - v_1} = 0$, thus $v_2 = v_1$.

- Strategy of the first global uniqueness result [Sylvester-Uhlmann 1987].
- Direct reconstruction method using similar ideas.
Ill-posedness of the problem

Mandache (2001) and Isaev (2011): for fixed $E$, it is not possible to have a stability estimate like

$$
\|v_2 - v_1\|_{L^\infty(D)} \leq c(\log(3 + \|\Phi_2(E) - \Phi_1(E)\|_{-1}^-))^{-\alpha},
$$

(2)

for $\alpha > m$, if $v_1, v_2 \in C^m(D, \mathbb{C})$ and $\Phi_1(E), \Phi_2(E)$ the corresponding Dirichlet-to-Neumann operators ($\|\cdot\|_* = \|\cdot\|_{H^{1/2}(\partial D) \rightarrow H^{-1/2}(\partial D)}$).

Question:

- What is the behaviour of estimate (2) with respect to $E$?
Rapidly converging algorithm for a stable reconstruction

Main idea: to consider the Schrödinger equation

\[ (-\Delta + v)\psi = E\psi \quad \text{on } D \subset \mathbb{R}^2, \]

for \( E > 0 \) sufficiently large.

We showed that

\[ \Phi(E) \xrightarrow{\text{Lipschitz}} v_{\text{appr}}(\cdot, E), \]

\[ \|v_{\text{appr}}(\cdot, E) - v(\cdot)\|_{L^\infty(D)} \leq O(E^{-\frac{(m-2)}{2}}), \]

for \( E \) sufficiently large, where \( m \) is related to the regularity of \( v \).
Convergence results

Theorem 2 (Novikov-S. 2012)

Let \( v \in W^{m,1}(\mathbb{R}^2, M_n(\mathbb{C})) \), \( m \geq 3 \), \( \text{supp } v \subset D \) and let \( \Phi(E) \) be the corresponding Dirichlet-to-Neumann operator at fixed energy \( E \), where \( E \geq E_2 \) and \( E \) is not a Dirichlet eigenvalue of \(-\Delta + V\) and \(-\Delta\) in \( D \).

Then \( v \) is reconstructed from \( \Phi(E) \) with Lipschitz stability via Algorithm 1 up to \( O(E^{-(m-2)/2}) \) in the uniform norm as \( E \to +\infty \).

- Theoretical tools: \( \tilde{\partial} \), non-local Riemann-Hilbert problem and inverse scattering theories.
Numerical Results (with S. Siltanen and M. Lassas)

- $D$ the unit disk

- Truncated Fourier basis to approximate the Dirichlet-to-Neumann map (using FEM):
  \[ \phi^{(n)}(\theta) = \frac{1}{\sqrt{2\pi}} e^{in\theta}, \quad n = -N, ..., N. \]
  Generally $N \leq 16$.

- Integral equations are solved representing operators as matrix in truncated Fourier basis.
Resolution test

Reconstruction from boundary data at $E = 10, 20, 30, 100, 200, 300, 500$. 
Resolution test 2

Reconstruction from boundary data at
\( E = 30, 40, 50, 75, 100, 150, 200, 300 \).
Virtual Hybrid Edge Detection

New method to detect singularities in conductivities from voltage and current data.

Collaborators:

- Allan Greenleaf  
  (University of Rochester)
- Matti Lassas  
  (University of Helsinki)
- Samuli Siltanen  
  (University of Helsinki)
- Gunther Uhlmann  
  (University of Washington)
Main features

- Only know method able to detect inclusions inside inclusions in EIT.
- It combines idea from Calderón problem (CGO solutions), X-ray tomography and compressed sensing (super-resolution).
- At present it works in 2D (using Astala-Paivarinta results).

Basic idea: compute 1D Fourier transforms of CGO solutions along every line passing through the origin in the parameter space (at every fixed boundary point). These data contains high resolution information about singularities of the conductivity. We use super-resolution techniques (Candès,Fernandez-Granda 2014) and X-ray tomography inversion to reconstruct the singularities.
Figure: Phantom conductivity
Figure: Sinogram and smooth reconstruction
Figure: Super-resolution sinogram and sharp reconstruction
Figure: Phantom, super-resolution reconstruction, difference image
Future work: test VHED with real data and try 3D reconstruction.

Thank you!


