ANGLE CONDITIONS IN FINITE ELEMENT ANALYSIS

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Angle conditions play an important role in the analysis of the finite element method (FEM). They enable us e.g. to derive the optimal interpolation order and prove convergence of FEM, to derive various a posteriori error estimates, to perform regular mesh refinements, etc.

In 1968, Miloš Zlámal introduced the minimum angle condition for triangular elements. From that time onward many other useful geometric angle conditions on the shape of finite elements appeared.

In this talk, we shall present and discuss some of these conditions used for simplicial partitions (meshes).

As the main result, we will demonstrate that even the famous *maximum angle condition* (the weakest one !) in the finite element analysis is not necessary to achieve the optimal convergence rate when simplicial finite elements are used to solve elliptic problems. This condition is only sufficient. In fact, finite element approximations may converge even though some dihedral angles of simplicial elements (triangles, tetrahedra, etc) tend to π.

A. Hannukainen, S. Korotov, M. Křížek. The maximum angle condition is not necessary for convergence of the finite element method. Numer. Math. 120 (2012), 79–88.

In order to remind the situation with the convergence of FEM, we first consider a family $\mathcal{F} = \{\mathcal{T}_h\}_{h\to 0}$ of conforming (face-to-face) triangulations of a polygonal domain.

In 1968 M. Zlámal introduced the following *minimum angle condition* which states that there should exist a constant α_0 such that for any triangulation $\mathcal{T}_h \in \mathcal{F}$ and any triangle $K \in \mathcal{T}_h$ we have

$$0 < \alpha_0 \leq \alpha_K, \tag{1}$$

where α_K is the minimal angle of K.

Under this (sufficient) condition he derived the optimal order bounds of the interpolation error in the Sobolev H^1-norm (and H^2-norm) and therefore also of the discretization error for FEM applied to second (and fourth) order elliptic equation with some boundary conditions.

M. Zlámal. On the finite element method. Numer. Math. 12 (1968), 394–409.

The same condition was also introduced by Alexander Ženíšek for FEM applied to a system of linear elasticity equations of second order, published in 1969. This paper was submitted already on April 3, 1968, whereas Zlámal's paper on April 17, 1968.

Nevertheless, condition (1) is known as *Zlámal's minimum angle condition*, since paper by Ženíšek was published in Czech and one year later than that of Zlámal.

A. Ženíšek. The convergence of the finite element method for boundary value problems of a system of elliptic equations (in Czech). Apl. Mat. 14 (1969), 355–377.

In 1976, three research groups independently found that a weaker condition than the minimum angle condition can used in proofs of the optimal rate of the interpolation error which by the Céa's lemma yields also some rate of the discretization error.

They proposed the so-called *maximum angle condition:* There exists a constant γ_0 such that for any triangulation $\mathcal{T}_h \in \mathcal{F}$ and any triangle $K \in \mathcal{T}_h$ we have

$$\gamma_K \leq \gamma_0 < \pi, \tag{2}$$

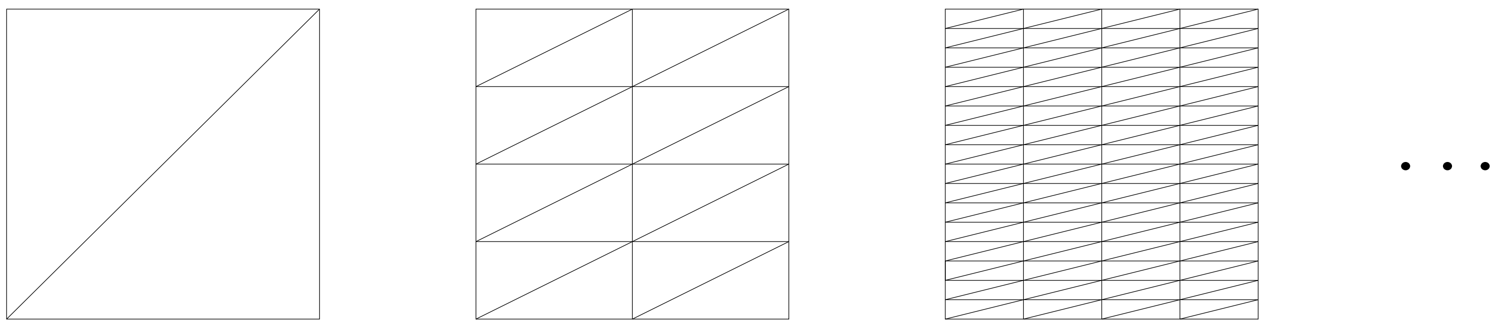
where γ_K is the maximum angle of K.

I. Babuška, A. K. Aziz. On the angle condition in the finite element method. *SIAM J. Numer. Anal.* 13 (1976), 214–226.

R. E. Barnhill, J. A. Gregory. Sard kernel theorems on triangular domains with applications to finite element error bounds. *Numer. Math.* 25 (1976), 215–229.

P. Jamet. Estimation de l'erreur pour des éléments finis droits presque dégénérés. *RAIRO Anal. Numér.* 10 (1976), 43–60.

Clearly, the minimum angle condition implies the maximum angle condition, since $\gamma_K \le \pi - 2\alpha_K \le \pi - 2\alpha_0 \equiv \gamma_0$. But the converse implication does not hold, see the figure in below.

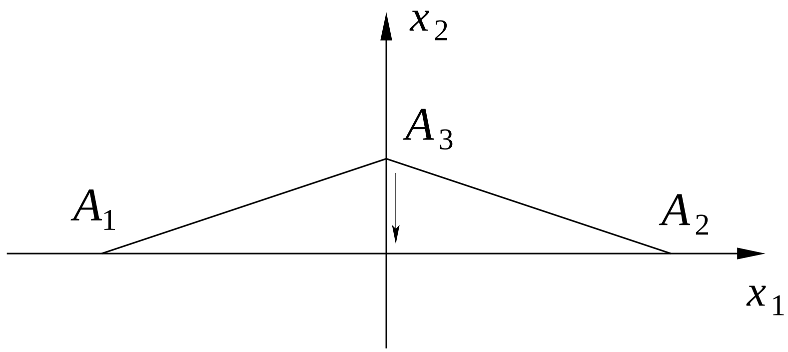


Note that John L. Synge already in 1957 proved the optimal order of nodal linear interpolation under condition (2), but without any application to FEMs.

J. L. Synge. The Hypercircle in Mathematical Physics. Cambridge Univ. Press, Cambridge, 1957.

In FEM literature there are examples showing that if the maximum angle condition does not hold then the linear triangular finite elements loose their optimal interpolation order.

The main idea: take $\varepsilon > 0$ and triangle K with $A_1 = (-1, 0)$, $A_2 = (1, 0)$, and $A_3 = (0, \varepsilon)$, hence, $\gamma_K \to \pi$ when $\varepsilon \to 0$.



I. Babuška, A. K. Aziz. On the angle condition in the finite element method. SIAM J. Numer. Anal. 13 (1976), 214–226.

G. Strang, G. Fix. An Analysis of the Finite Element Method. Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1973.

A. Ženíšek. The convergence of the finite element method for boundary value problems of a system of elliptic equations. Apl. Mat. 14 (1969), 355–377.

Consider the function $v(x_1, x_2) = x_1^2$ and its linear interpolant

$$(L_\varepsilon v)(x_1, x_2) = -\frac{x_2}{\varepsilon} + 1 \quad \text{on } K, \tag{3}$$

i.e.,

$$(L_\varepsilon v)(A_i) = v(A_i), \quad i = 1, 2, 3.$$

Using the standard Sobolev space notation, (3), and the fact $\frac{\partial v}{\partial x_2} = 0$, we find that

$$\|v - L_\varepsilon v\|_{1,K}^2 \geq \left|\frac{\partial L_\varepsilon v}{\partial x_2}\right|_{0,K}^2 = \frac{1}{\varepsilon^2}\operatorname{meas} K = \frac{1}{\varepsilon} \to \infty \quad \text{as} \quad \varepsilon \to 0. \tag{4}$$

We conclude that one badly shaped triangle in every triangulation $\mathcal{T}_h \in \mathcal{F}$ can yield an arbitrary large interpolation error in the Sobolev H^1-norm.

The maximum angle condition corresponding to 3D setting requires that all dihedral angles between faces and all angles between edges (within triangular faces) are bounded from above by a constant less than π. The optimal interpolation rate in the H^1-norm for linear elements is preserved under this condition.

M. Křížek. On the maximum angle condition for linear tetrahedral elements. SIAM J. Numer. Anal. 29 (1992), 513–520.

For tetrahedral elements similar examples as in (3) can also be constructed. Namely, if the maximal angle between two faces or the maximal angle between edges (within triangular faces) tends to π, then the interpolation error may tend to ∞ like in (4).

• Examples, analogous to the above mentioned, caused numerical analysts to believe that large angles of triangular elements (i.e., when the maximum angle condition is not satisfied) produce also large discretization error when solving second order elliptic problems by FEM. For instance, Babuška and Aziz state that the maximum angle condition is essential for convergence of FEM, whereas D'Azevedo and Simpson assert that it is necessary and sufficient for convergence.

I. Babuška, A. K. Aziz. On the angle condition in the finite element method. *SIAM J. Numer. Anal.* 13 (1976), 214–226.

E. F. D'Azevedo, R. B. Simpson. On optimal interpolation triangle incidences. *SIAM J. Sci. Statist. Comput.* 10 (1989), 1063–1075.

To the contrary, we show here that the finite element method may converge even when the maximum angle condition is violated for a quite large number of elements in the used partitions.

Let us emphasize that the Céa's lemma

$$\|u - u_h\|_V \leq C \inf_{v_h \in V_h} \|u - v_h\|_V$$

gives only an upper bound of the discretization error by means of the interpolation error.

Note that the discretization error can be, in some cases, of the same order as the interpolation error.

But in principle, the discretization error can also be much smaller than the interpolation error, as we will see later !

WHY IS THE MAXIMUM ANGLE CONDITION NOT NECESSARY ?

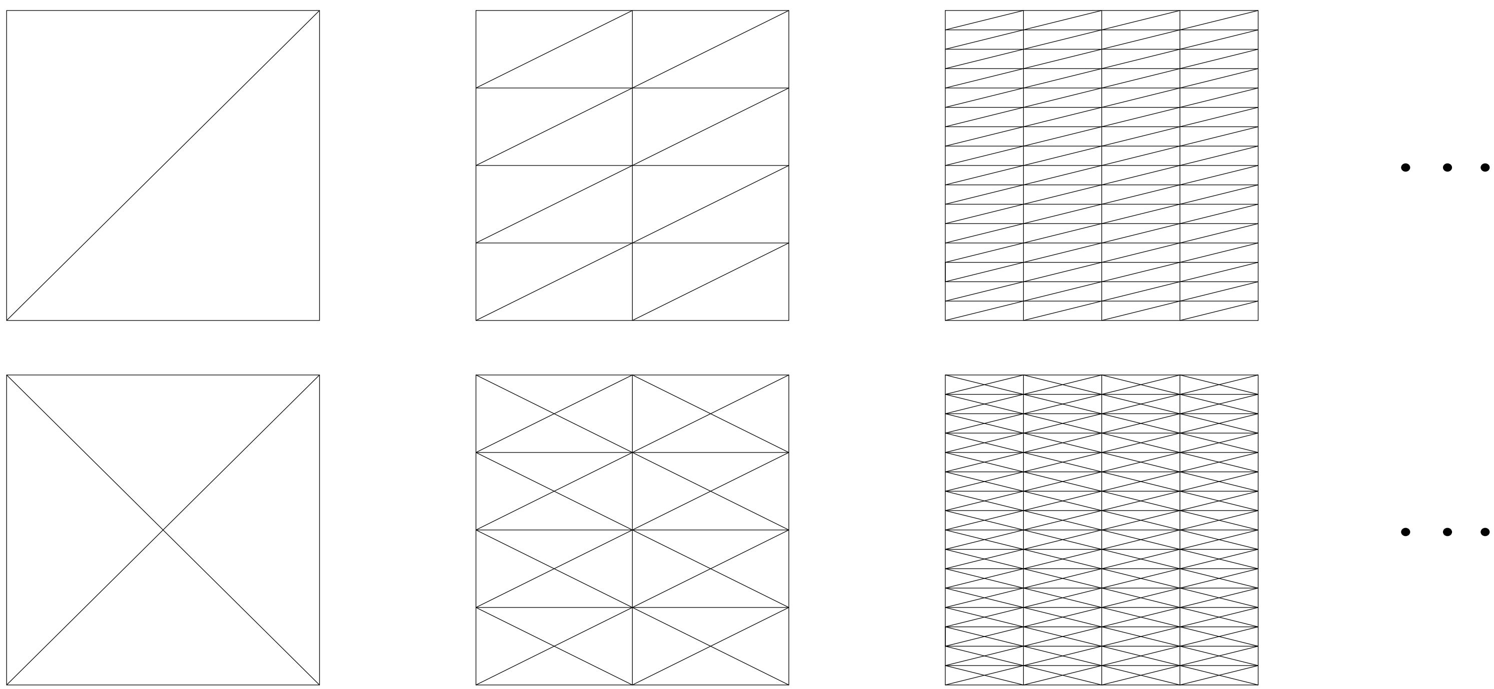
We now show that the discretization error can be very small, whereas the interpolation error is large.

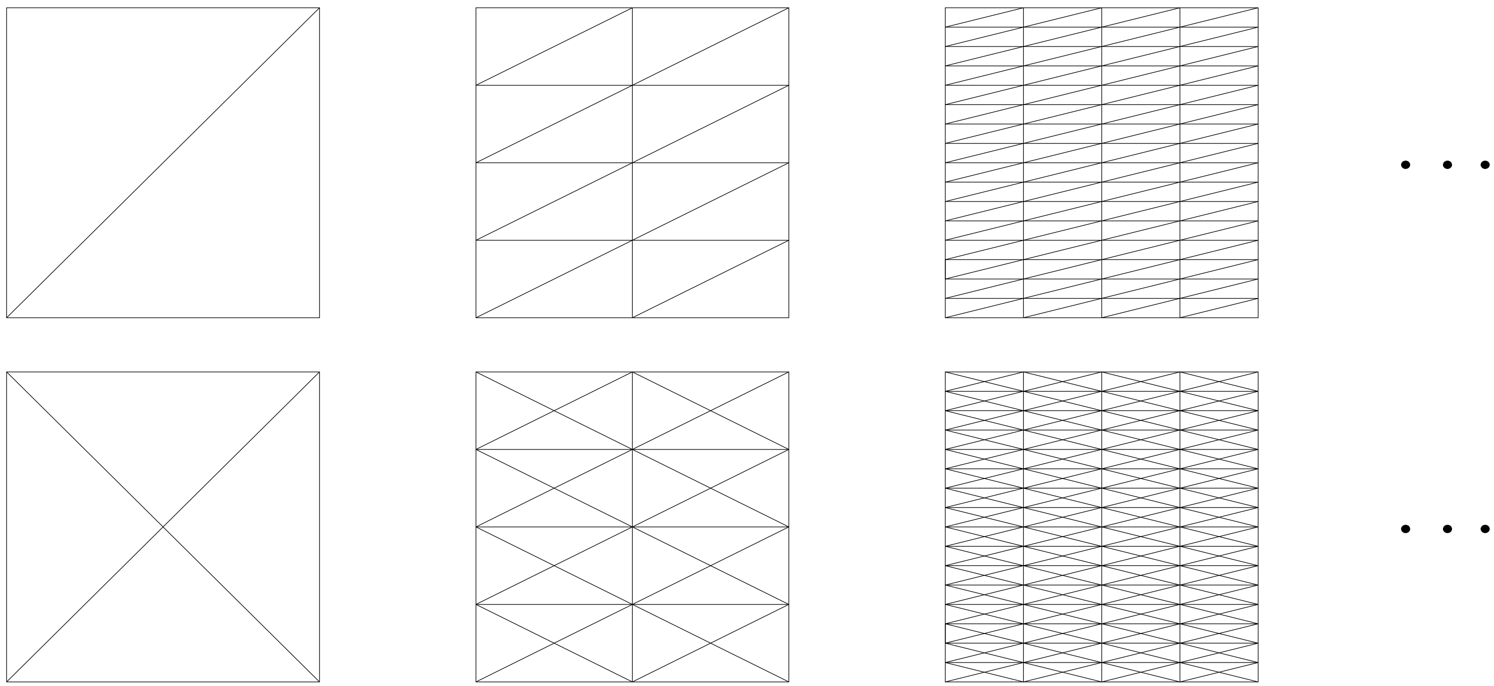
For simplicity, consider the Poisson equation with the homogeneous Dirichlet boundary conditions in the unit square $\Omega = (0,1) \times (0,1)$,

$$-\Delta u = f \quad \text{in} \quad \Omega, \qquad u = 0 \quad \text{on} \quad \partial\Omega, \tag{5}$$

where $f \in L^2(\Omega)$. Since Ω is convex, its weak solution is from the Sobolev space $H^2(\Omega)$ and thus continuous by the Sobolev imbedding theorem.

Example 1: Define two families $\mathcal{F}_1$ and $\mathcal{F}_2$ of triangulations of $\overline{\Omega}$. To this end we first introduce uniform rectangular meshes of the given unit square consisting of congruent rectangles. Its horizontal sides are divided into 2^k equal parts and the vertical parts are divided into 4^k equal parts for $k = 0, 1, 2, \ldots$ To construct $\mathcal{F}_1$ we divide each rectangle by its diagonal with a positive slope, whereas for $\mathcal{F}_2$ we take both diagonals.





We observe that $\mathcal{F}_1$ satisfies the maximum angle condition with $\gamma_0 = \pi/2$ for all k, whereas for $\mathcal{F}_2$ we observe that $\gamma_K \to \pi$ for every second triangle from any $\mathcal{T}_h \in \mathcal{F}_2$. Let V_h and W_h be finite element spaces of continuous and piecewise linear functions over triangulations from $\mathcal{F}_1$ and $\mathcal{F}_2$, respectively. Obviously,

$$V_h \subset W_h. \tag{6}$$

Denote by $u_h \in W_h$ the standard Galerkin approximation of the weak solution $u \in H^2(\Omega)$ of our PDE problem. Let $L_h u$ stands for the linear interpolant of u in V_h. Then by Céa's lemma and due to $V_h \subset W_h$ there exists a constant $C > 0$ such that

$$\|u - u_h\|_1 \le C \inf_{w_h \in W_h} \|u - w_h\|_1 \le C \inf_{v_h \in V_h} \|u - v_h\|_1 \le$$

$$\le C\|u - L_h u\|_1 \le C' h |u|_2 \text{ as } h \to 0, \qquad (*)$$

where the last inequality can be proved under the maximum angle condition for a constant $C' > 0$ independent of h.

• This example shows that the discretization error tends to 0 at least linearly in the H^1-norm even though the maximal angle of every second triangle from any $\mathcal{T}_h \in \mathcal{F}_2$ tends to π.

In Figure 1 we observe the practical rates of convergence on $\mathcal{F}_1$ and $\mathcal{F}_2$ for our test problem with RHS $f(x_1, x_2) = \pi^2 \sin \pi x_1 \sin \pi x_2$.

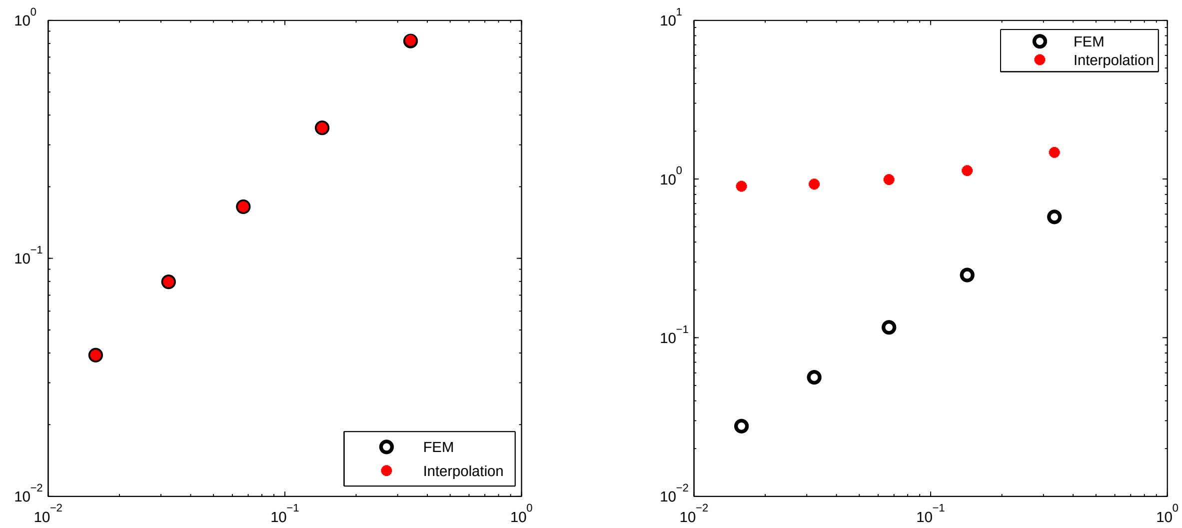
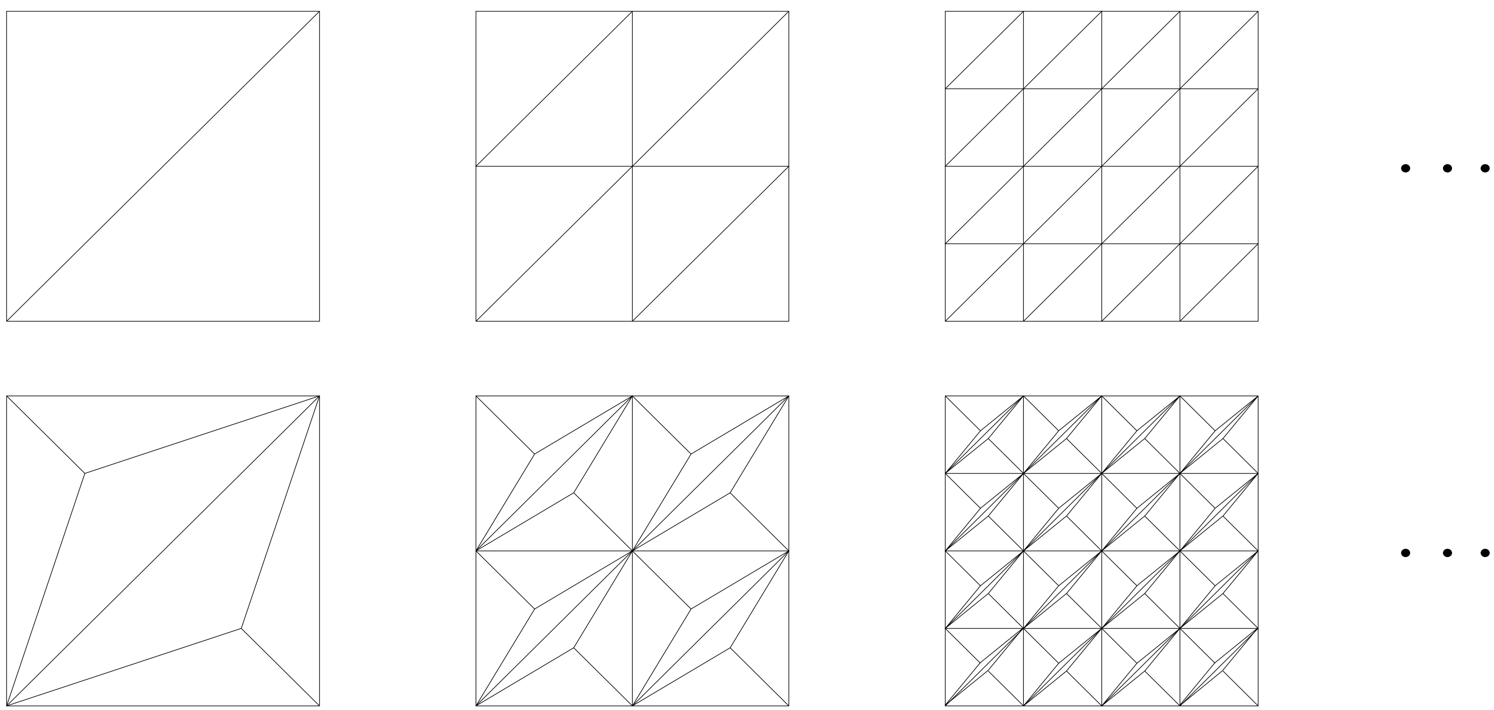


Figure 1: The practical convergence rates for the families $\mathcal{F}_1$ (left) and $\mathcal{F}_2$ (right). The horizontal axis corresponds to the discretization parameter and the vertical axis corresponds to the H^1-norm of the discretization and interpolation errors. The difference between interpolation and discretization errors on the left figure is extremely small, which cannot be seen from the graph.

Example 2: Another supportive example is illustrated in below. In this case, the family $\mathcal{F}_3$ satisfies even the minimum angle condition and the maximal angle of every third triangle from any $\mathcal{T}_h \in \mathcal{F}_4$ tends to π. We can define V_h and W_h over triangulations from $\mathcal{F}_3$ and $\mathcal{F}_4$ as in the previous example so that $V_h \subset W_h$, and derive $(*)$ again.



• 3D examples in the above manner can also be constructed.

• In fact, a more universal statement, applicable also for nonsimplicial Lagrange and Hermite elements (possibly degenerating in various ways), can be formulated as follows.

Consider a general elliptic problem in a weak form: Find $u \in V$ such that

$$a(u, v) = F(v) \qquad \forall v \in V, \tag{7}$$

where V is a Hilbert space with the induced norm $\| \cdot \|_V$, $a(\cdot, \cdot)$ is a continuous V-elliptic bilinear form, and $F(\cdot)$ is a linear continuous functional over V. Then we have:

Theorem: *Let* $\{V_h\}_{h\to 0}$ *and* $\{W_h\}_{h\to 0}$ *be two families of finite element spaces such that* $V_h \subset W_h \subset V$*. Assume that for each* $v \in V$

$$\lim_{h\to 0} \inf_{v_h \in V_h} \|v - v_h\|_V = 0, \tag{8}$$

i.e., the union $\bigcup_{h>0} V_h$ *is dense in* V*. Then*

$$\|u - u_h\|_V \to 0,$$

where $u_h \in W_h$ *is the standard finite element approximation of the weak solution* $u \in V$ *of elliptic boundary value problem (7).*

P r o o f : From Cea's lemma and (8), we obtain

$$\|u-u_h\|_V \le C \inf_{w_h \in W_h} \|u-w_h\|_V \le C \inf_{v_h \in V_h} \|u-v_h\|_V \to 0 \text{ as } h \to 0. \qquad \square$$

THANK YOU FOR YOUR ATTENTION !