Compactness estimates for Hamilton-Jacobi equations

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Outline

1. Hamilton-Jacobi equations
2. Compactness estimates for hyperbolic conservation laws
3. Compactness estimates for Hamilton-Jacobi equations
Outline

1. Hamilton-Jacobi equations
2. Compactness estimates for hyperbolic conservation laws
3. Compactness estimates for Hamilton-Jacobi equations
Hamilton-Jacobi equations

\[
\begin{aligned}
&\left\{ \begin{array}{l}
    u_t(t, x) + H(t, x, \nabla u(t, x)) = 0 \\
    u(0, x) = u_0(x)
\end{array} \right. \\
&\ (t, x) \in [0, T] \times \mathbb{R}^n \\
&\ x \in \mathbb{R}^n
\end{aligned}
\]

where

- \( H : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \) is a \( C^2 \) smooth function such that

\[
(a) \quad \lim_{|p| \to \infty} \inf_{(t,x) \in [0, T] \times \mathbb{R}^n} \frac{H(t, x, p)}{|p|} = +\infty
\]

\[
(b) \quad D_p^2 H(t, x, p) \geq \alpha \cdot \mathbb{I}_n, \quad \forall (t, x, p) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n
\]

with \( \alpha > 0 \)

- \( u_0 : \mathbb{R}^n \to \mathbb{R} \) is a Lipschitz function

play an important role in Dynamic Optimization
Hamilton-Jacobi equations

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&u_t(t, x) + H(t, x, \nabla u(t, x)) = 0 \quad (t, x) \in [0, T] \times \mathbb{R}^n \\
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  \( (a) \lim_{|p| \to \infty} \inf_{(t,x) \in [0, T] \times \mathbb{R}^n} \frac{H(t, x, p)}{|p|} = +\infty \)
  
  \( (b) \quad D_p^2 H(t, x, p) \geq \alpha \cdot I_n, \quad \forall (t, x, p) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \)

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\end{cases}
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  \((a)\) \quad \lim_{|p| \rightarrow \infty} \inf_{(t, x) \in [0, T] \times \mathbb{R}^n} \frac{H(t, x, p)}{|p|} = +\infty

  \((b)\) \quad D^2_p H(t, x, p) \succeq \alpha \cdot \mathbb{I}_n, \quad \forall (t, x, p) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n

  with \( \alpha > 0 \)

- \( u_0 : \mathbb{R}^n \rightarrow \mathbb{R} \) is a Lipschitz function

play an important role in Dynamic Optimization
Given a $C^2$ smooth function $L : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ such that

(a) $\lim_{|q| \to \infty} \inf_{(t,x) \in [0,T] \times \mathbb{R}^n} \frac{L(t, x, q)}{|q|} = +\infty$

(b) $D_q^2 L(t, x, q) \geq \lambda \cdot \mathbb{I}_n, \quad \forall (t, x, q) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \quad (\lambda > 0)$

consider the problem of minimizing the functional

$$J(\xi) = \int_0^T L(t, \xi(t), \xi'(t)) \, dt$$

over all absolutely continuous arcs $\xi : [0, T] \to \mathbb{R}^n$ satisfying

$$\xi(0) = x_0 \quad \text{and} \quad \xi(T) = x_T$$

with $x_0, x_T \in \mathbb{R}^n$.
The simplest problem in the calculus of variations

Given a $C^2$ smooth function $L : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ such that

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$$J(\xi) = \int_0^T L(t, \xi(t), \xi'(t)) \, dt$$

over all absolutely continuous arcs $\xi : [0, T] \to \mathbb{R}^n$ satisfying

$\xi(0) = x_0$ and $\xi(T) = x_T$

with $x_0, x_T \in \mathbb{R}^n$
A particle moving from time 0 to time $T$ between two points $x_0, x_T \in \mathbb{R}^3$ subject to a conservative force

$$F(x) = -\nabla V(x)$$

among all the (admissible) trajectories $\xi(t)$, follows the one that minimizes the action, i.e. the functional

$$J(\xi) = \int_0^T \left[ \frac{1}{2} m |\xi'(t)|^2 - V(\xi(t)) \right] dt,$$

where $m$ is the mass of the particle and $\frac{1}{2} m |x'(t)|^2$ is its kinetic energy.
The action functional

A particle moving from time 0 to time \( T \) between two points \( x_0, x_T \in \mathbb{R}^3 \) subject to a conservative force

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where $m$ is the mass of the particle and $\frac{1}{2} m |x'(t)|^2$ is its kinetic energy.
Given $a, b \in \mathbb{R}$ with $a < b$, consider in the space $\mathbb{R}^3$ the circles

\[
\begin{align*}
&\left\{ \begin{array}{l}
y^2 + z^2 = A^2 \\
x = a
\end{array} \right. \\
&\left\{ \begin{array}{l}
y^2 + z^2 = B^2 \\
x = b
\end{array} \right.
\]

For any smooth $\xi : [a, b] \to \mathbb{R}$, with $\xi(x) > 0$, $\xi(a) = A$ and $\xi(b) = B$, consider

- the regular curve $\vec{X}(x) = (x, 0, \xi(x))$ in the $xz$-plane
- the surface of revolution $\Sigma(\xi)$ generated by the rotation of $\vec{X}$ around the $x$-axis

Finding the surface of revolution of minimal area amounts to minimizing

\[
A(\Sigma(\xi)) = 2\pi \int_a^b \xi(x) \sqrt{1 + (\xi'(x))^2} \, dx
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Given \( a, b \in \mathbb{R} \) with \( a < b \), consider in the space \( \mathbb{R}^3 \) the circles

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Dynamic programming

Replacing the initial point constraint with the initial cost $u_0$, consider

$$
\inf_{\xi(t)=x} \left\{ \int_0^t L(s, \xi(s), \xi'(s)) \, dt + u_0(\xi(0)) \right\} = V(t, x)
$$

\[
egin{align*}
V_t(t, x) + \sup_{q \in \mathbb{R}^n} \left\{ \langle q, \nabla V(t, x) \rangle - L(t, x, q) \right\} &= 0 & (t, x) \in [0, T] \times \mathbb{R}^n \text{ a.e.} \\
V(0, x) &= u_0(x) \\
H(t, x, \nabla V(t, x))
\end{align*}
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Dynamic programming

Replacing the initial point constraint with the initial cost $u_0$, consider

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\end{cases}
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Weak solutions to Hamilton-Jacobi equations

\[
\begin{cases}
  u_t(t, x) + H(t, x, \nabla u(t, x)) = 0 & (t, x) \in [0, T] \times \mathbb{R}^n \\
  u(0, x) = u_0(x) & x \in \mathbb{R}^n
\end{cases}
\]

(HJ)

- has no global smooth solution due to crossing of characteristics
- may have infinitely many Lipschitz solutions satisfying (HJ) a.e.
  - Dacorogna and Marcellini (1999)
- has a unique viscosity solution
- the viscosity solution is the unique semiconcave \( u \) satisfying (HJ) a.e.
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\end{aligned} \]  

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Hamilton-Jacobi equations

Weak solutions to Hamilton-Jacobi equations

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(HJ)

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  - Dacorogna and Marcellini (1999)
- has a unique viscosity solution
- the viscosity solution is the unique semiconcave \( u \) satisfying (HJ) a.e.
A function \( u \in C([0, T] \times \mathbb{R}^n) \) is a viscosity solution of

\[
    u_t + H(t, x, \nabla u) = 0 \quad \text{in } [0, T] \times \mathbb{R}^n
\]

if for every \((t, x) \in (0, T) \times \mathbb{R}^n\) and every \( \phi \in C^1((0, T) \times \mathbb{R}^n) \)

- \( u - \phi \) has a local maximum at \((t, x)\) \( \Rightarrow \) \( \phi_t(t, x) + H(t, x, \nabla \phi(t, x)) \leq 0 \)
- \( u - \phi \) has a local minimum at \((t, x)\) \( \Rightarrow \) \( \phi_t(t, x) + H(t, x, \nabla \phi(t, x)) \geq 0 \)
A function \( u \in C([0, T] \times \mathbb{R}^n) \) is a viscosity solution if for every \((t, x) \in (0, T) \times \mathbb{R}^n\) and every \( \phi \in C^1((0, T) \times \mathbb{R}^n) \):

- \( u - \phi \) has a local maximum at \((t, x)\) \(\Rightarrow\) \( \phi_t(t, x) + H(t, x, \nabla \phi(t, x)) \leq 0 \)
- \( u - \phi \) has a local minimum at \((t, x)\) \(\Rightarrow\) \( \phi_t(t, x) + H(t, x, \nabla \phi(t, x)) \geq 0 \)
Viscosity solutions

A function $u \in C([0, T] \times \mathbb{R}^n)$ is a viscosity solution of

$$u_t + H(t, x, \nabla u) = 0 \quad \text{in } ]0, T[ \times \mathbb{R}^n$$

if for every $(t, x) \in (0, T) \times \mathbb{R}^n$ and every $\phi \in C^1((0, T) \times \mathbb{R}^n)$

- $u - \phi$ has a local maximum at $(t, x)$ $\Rightarrow$ $\phi_t(t, x) + H(t, x, \nabla \phi(t, x)) \leq 0$
- $u - \phi$ has a local minimum at $(t, x)$ $\Rightarrow$ $\phi_t(t, x) + H(t, x, \nabla \phi(t, x)) \geq 0$
Semiconcave functions

Definition

We say that \( u : \mathbb{R}^N \to \mathbb{R} \) is (linearly) semiconcave if there exists a constant \( K > 0 \) (a semiconcavity constant for \( u \)) such that

\[
\lambda u(x) + (1 - \lambda)u(y) - u(\lambda x + (1 - \lambda)y) \leq K\lambda(1 - \lambda)\frac{|y - x|^2}{2}
\]

for all \( x, y \in \mathbb{R}^N \) and all \( \lambda \in [0, 1] \)

- \( u \) is semiconcave with semiconcavity \( K \) if any only if the function
  \[
  \tilde{u}(x) = u(x) - \frac{K}{2}|x|^2
  \]
  is concave

- \( v \) is semiconvex with semiconvexity constant \( K \) if \( -v \) is semiconcave with semiconcavity constant \( K \)
Semiconcave functions

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$$\lambda u(x) + (1 - \lambda)u(y) - u(\lambda x + (1 - \lambda)y) \leq K\lambda(1 - \lambda)\frac{|y - x|^2}{2}$$

for all $x, y \in \mathbb{R}^N$ and all $\lambda \in [0, 1]$

- $u$ is semiconcave with semiconcavity $K$ if any only if the function
  $$\tilde{u}(x) = u(x) - \frac{K}{2}|x|^2$$
  is concave

- $v$ is semiconvex with semiconvexity constant $K$ if $-v$ is semiconcave with semiconcavity constant $K$
Semiconcave functions

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\[
\lambda u(x) + (1 - \lambda)u(y) - u(\lambda x + (1 - \lambda)y) \leq K\lambda(1 - \lambda)|y - x|^2
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for all \( x, y \in \mathbb{R}^N \) and all \( \lambda \in [0, 1] \)

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- \( v \) is semiconvex with semiconvexity constant \( K \) if \( -v \) is semiconcave with semiconcavity constant \( K \)
For more on semiconcave functions see

- **Control theory**

- **Nonsmooth and variational analysis**
  Rockafellar (1982)


- **Monographs**
  C – Sinestrari (Birkhäuser, 2004)
  Villani (Springer, 2009)
When $n = 1$ the Hamilton-Jacobi equation

\[
\begin{cases}
  u_t(t, x) + H(t, x, u_x(t, x)) = 0 & (t, x) \in [0, T] \times \mathbb{R} \\
  u(0, x) = u_0(x) & x \in \mathbb{R}
\end{cases}
\]

can be reduced to the conservation law

\[
\begin{cases}
  v_t(t, x) + H(t, x, v(t, x)) = 0 & (t, x) \in [0, T] \times \mathbb{R} \\
  v(0, x) = u'_0(x) & x \in \mathbb{R}
\end{cases}
\]

taking $v(t, x) = u_x(t, z)$
Scalar conservation laws

\( u \) is an entropy solution of

\[
\frac{\partial u}{\partial t} + f(u) \frac{\partial u}{\partial x} = 0 \quad \text{in} \quad [0, +\infty) \times \mathbb{R}
\]

where \( f : \mathbb{R} \to \mathbb{R} \) is is (uniformly) strictly convex

\[
f''(u) \geq c > 0 \quad \forall u \in \mathbb{R}
\]

if

- \( u \) distributional solution

\[
\int \int \left[ u \varphi_t + f(u) \varphi_x \right] \, dx \, dt = 0 \quad \forall \varphi \in C^1_c([0, +\infty) \times \mathbb{R}) \quad (D)
\]

- Lax stability condition

\[
u(t, x-) \geq u(t, x+) \quad \text{for a.e } t > 0, \quad \forall x \in \mathbb{R}
\]
Scalar conservation laws

$u$ is an entropy solution of

$$u_t + f(u)_x = 0 \quad \text{in} \quad [0, +\infty) \times \mathbb{R}$$

where $f : \mathbb{R} \to \mathbb{R}$ is is (uniformly) strictly convex

$$f''(u) \geq c > 0 \quad \forall u \in \mathbb{R}$$

if

- $u$ distributional solution

$$\int \int [u\varphi_t + f(u)\varphi_x] \, dx \, dt = 0 \quad \forall \varphi \in C^1_c([0, +\infty) \times \mathbb{R}) \quad (D)$$

- Lax stability condition

$$u(t, x-) \geq u(t, x+) \quad \text{for a.e} \ t > 0, \quad \forall x \in \mathbb{R}$$
Compactness for the semigroup \((S_t)_{t \geq 0}\)

For every initial data \(u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})\), the map \(S_t : L^1(\mathbb{R}) \rightarrow L^1(\mathbb{R})\) associates to every initial data \(u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})\) the unique entropy solution \(u(t) = S_t(u_0)\) of

\[
\begin{cases}
  u_t + f(u)_x = 0 & [0, +\infty) \times \mathbb{R} \\
  u(0, x) = u_0(x) & x \in \mathbb{R}
\end{cases}
\]

**Theorem (Lax, 1954)**

The map \(S_t : L^1(\mathbb{R}) \rightarrow L^1_{\text{loc}}(\mathbb{R})\) is compact for every \(t > 0\)

A question (by P. Lax):

Is it possible to give a quantitative estimate of the compactness of \(S_t\)?
Compactness of the semigroup \((S_t)_{t \geq 0}\)

\[ S_t : L^1(\mathbb{R}) \to L^1(\mathbb{R}) \] associates to every initial data \(u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})\) the unique entropy solution \(u(t) = S_t(u_0)\) of

\[
\begin{align*}
  u_t + f(u)x &= 0 & [0, +\infty) \times \mathbb{R} \\
  u(0, x) &= u_0(x) & x \in \mathbb{R}
\end{align*}
\]

**Theorem (Lax, 1954)**

*The map \(S_t : L^1(\mathbb{R}) \longrightarrow L^1_{\text{loc}}(\mathbb{R})\) is compact for every \(t > 0\)*

A question (by P. Lax) :

is it possible to give a quantitative estimate of the compactness of \(S_t\) ?
Compactness for conservation laws

Kolmogorov $\varepsilon$-entropy

Let $(X, d)$ be a metric space and $K$ a totally bounded subset of $X$

For any $\varepsilon > 0$, let $N_{\varepsilon}(K)$ be the minimal number of sets in a cover of $K$ by subsets of $X$ having diameter no larger than $2\varepsilon$

Definition

The $\varepsilon$-entropy of $K$ is defined as

$$\mathcal{H}_\varepsilon(K \mid X) = \log_2 N_{\varepsilon}(K)$$
Kolmogorov $\varepsilon$-entropy

Let $(X, d)$ be a metric space and $K$ a totally bounded subset of $X$.

For any $\varepsilon > 0$, let $N_\varepsilon(K)$ be the minimal number of sets in a cover of $K$ by subsets of $X$ having diameter no larger than $2\varepsilon$.

**Definition**

The $\varepsilon$-entropy of $K$ is defined as

$$H_\varepsilon(K | X) = \log_2 N_\varepsilon(K)$$
Applications

one relies on Kolmogorov’s $\varepsilon$-entropy to:

- provide estimates on the accuracy and resolution of numerical methods
- analyze computational complexity of conservation laws (derive number of needed operations to compute solutions with an error $< \varepsilon$)
Upper estimate

Given $L, m, M > 0$, define

$$C_{[L,m,M]} = \left\{ u_0 \in L^1(\mathbb{R}) : \text{spt}(u_0) \subset [-L, L], \|u_0\|_{L^1} \leq m, \|u_0\|_{L^\infty} \leq M \right\}$$

Our goal: to give an upper bound for

$$\mathcal{H}_\varepsilon \left( S_T(C_{[L,m,M]}) \| L^1(\mathbb{R}) \right)$$

Theorem (De Lellis and Golse, 2005)

For any $\varepsilon > 0$ and $T > 0$, one has

$$\mathcal{H}_\varepsilon \left( S_T(C_{[L,m,M]}) \| L^1(\mathbb{R}) \right) \leq \frac{C_T}{\varepsilon}$$

for some constant $C_T > 0$
Upper estimate

Given $L, m, M > 0$, define

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Lower estimate

Given any $L, m, M > 0$, recall

$$
\mathcal{C}_{[L,m,M]} = \left\{ u_0 \in L^1(\mathbb{R}) : \text{spt}(u_0) \subset [-L, L], \|u_0\|_{L^1} \leq m, \|u_0\|_{L^\infty} \leq M \right\}
$$

Theorem (Ancona, Glass and Khai T. Nguyen, 2012)

For any $T > 0$ and for $\varepsilon > 0$ sufficiently small, one has

$$
\mathcal{H}_\varepsilon \left( S_T(\mathcal{C}_{[L,m,M]} \mid L^1(\mathbb{R})) \right) \geq \frac{c_T}{\varepsilon}
$$

for some constant $c_T > 0$

By the upper and lower bounds, we conclude

$$
\mathcal{H}_\varepsilon \left( S_T(\mathcal{C}_{[L,m,M]} \mid L^1(\mathbb{R})) \right) \approx \varepsilon^{-1}
$$
Lower estimate

Given any $L, m, M > 0$, recall

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**Theorem (Ancona, Glass and Khai T. Nguyen, 2012)**

*For any $T > 0$ and for $\varepsilon > 0$ sufficiently small, one has*

$$
\mathcal{H}_{\varepsilon} \left( S_T(C_{[L,m,M]}) \mid L^1(\mathbb{R}) \right) \geq \frac{c_T}{\varepsilon}
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$$
Consider the Hamilton-Jacobi equation \((n \geq 1)\)

\[
    u_t(t, x) + H(\nabla u(t, x)) = 0 \quad (t, x) \in [0, +\infty) \times \mathbb{R}^n
\]

with \(H \in C^2(\mathbb{R}^n)\) satisfying

(H1) superlinearity: \(\lim_{|p| \to \infty} \frac{H(p)}{|p|} = +\infty\)

(H2) uniform convexity: \(D^2 H(p) \geq \alpha \cdot \mathbb{I}_n, \quad \forall p \in \mathbb{R}^n\)

where \(\alpha > 0\) and \(\mathbb{I}_n\) is the identity \(n \times n\) matrix

Legendre transform of \(H\)

\[
    H^*(q) = \max_{p \in \mathbb{R}^n} \{ \langle p, q \rangle - H(p) \} \quad (q \in \mathbb{R}^n)
\]

is in turn superlinear and satisfies

\[
    H^* \in C^2(\mathbb{R}^n) \quad \text{and} \quad D^2 H^* \leq \frac{1}{\alpha} \mathbb{I}_n
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Compactness for Hamilton-Jacobi

Hopf-Lax semigroup

For any \( u_0 \in \text{Lip}(\mathbb{R}^n) \) the Cauchy problem

\[
\begin{cases}
  u_t(t, x) + H(\nabla u(t, x)) = 0 & (t, x) \in [0, +\infty) \times \mathbb{R}^n \\
  u(0, x) = u_0(x) & x \in \mathbb{R}^N
\end{cases}
\]

admits a unique viscosity solution given by

\[
u(t, x) = \min_{y \in \mathbb{R}^n} \left\{ t \cdot H^*(\frac{x - y}{t}) + u_0(y) \right\}, \quad \forall (t, x) \in ]0, +\infty[ \times \mathbb{R}^n
\]

Our goal: to obtain upper and lower compactness estimates for

\[ S_t : \text{Lip}(\mathbb{R}^n) \to \text{Lip}(\mathbb{R}^n) \]

\[
S_t(u_0)(x) = \min_{y \in \mathbb{R}^n} \left\{ t \cdot H^*(\frac{x - y}{t}) + u_0(y) \right\}, \quad x \in \mathbb{R}^n
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$$u(t, x) = \min_{y \in \mathbb{R}^n} \left\{ t \cdot H^* \left( \frac{x-y}{t} \right) + u_0(y) \right\}, \quad \forall (t, x) \in [0, +\infty) \times \mathbb{R}^n$$

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Semiconcavity of the Hopf-Lax semigroup

Given $K, L, M > 0$, define

$$C_{[L,M]} = \{ u \in \text{Lip}(\mathbb{R}^n) : \text{spt}(u) \subset [-L, L]^n, \| \nabla u \|_{L^\infty(\mathbb{R}^n)} \leq M \}$$

$$SC_{[K,L,M]} = \{ u \in C_{[L,M]} : u \text{ semiconcave with constant } K \}$$

**Proposition**

For any $L, M, T > 0$ and every $u \in C_{[L,M]}$

1. $S_T(u)$ is semiconcave with constant $\frac{1}{\alpha T}$
2. $\| \nabla S_T(u) \|_{L^\infty(\mathbb{R}^n)} \leq M$
3. $\text{spt}(S_T(u) + T \cdot H(0)) \subset [-L_T, L_T]^n$ where $L_T = L + T \cdot \sup_{|p| \leq M} |DH(p)|$

**Hopf-Lax semigroup**

$$\begin{cases}
S_t : \text{Lip}(\mathbb{R}^n) \to \text{Lip}(\mathbb{R}^n) \\
S_t(u)(x) = \min_{y \in \mathbb{R}^n} \{ t \cdot H^* \left( \frac{x-y}{t} \right) + u(y) \} \quad x \in \mathbb{R}^n
\end{cases} \quad t \geq 0$$
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Compactness for Hamilton-Jacobi flow associated with H-L semigroup

\[ S_T(C_{[L,M]}) + T \cdot H(0) \subset SC_{\frac{1}{\alpha T},L_T,M} \]

\[ SC_{[K,L,M]} = \{ u \in C_{[L,M]} : u \text{ semiconcave with constant } K \} \]

\[ C_{[L,M]} = \{ u \in \text{Lip}(\mathbb{R}^n) : \text{spt}(u) \subset [-L,L]^n, \| \nabla u \|_{L^\infty(\mathbb{R}^n)} \leq M \} \]
Upper estimate

\[ C_{[L,M]} = \{ u \in \text{Lip}(\mathbb{R}^n) : \text{spt}(u) \subset [-L, L]^n, \| \nabla u \|_{L^\infty(\mathbb{R}^N)} \leq M \} \]

Theorem (Ancona, C and Khai T. Nguyen)

For any \( L, M, T > 0 \) there exist constant \( \varepsilon_0 = \varepsilon_0(L, M, T) > 0 \) and \( C = C(L, M, T) > 0 \) such that, for all \( \varepsilon \in (0, \varepsilon_0) \),

\[ \mathcal{H}_\varepsilon \left( S_T(C_{[L,M]}) + T \cdot H(0) \right| W^{1,1}(\mathbb{R}^n)) \leq \frac{C}{\varepsilon^n} \]

Hopf-Lax semigroup

\[
\left\{ \begin{array}{ll}
S_t : \text{Lip}(\mathbb{R}^n) \to \text{Lip}(\mathbb{R}^n) \\
S_t(u)(x) = \min_{y \in \mathbb{R}^n} \left\{ t \cdot H^* \left( \frac{x-y}{t} \right) + u(y) \right\} & x \in \mathbb{R}^n \\
\end{array} \right. \]
Compactness for Hamilton-Jacobi

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\[
\begin{aligned}
  \{ S_t : \operatorname{Lip}(\mathbb{R}^n) &\to \operatorname{Lip}(\mathbb{R}^n) \\
  S_t(u)(x) = \min_{y \in \mathbb{R}^n} \left\{ t \cdot H^*(\frac{x-y}{t}) + u(y) \right\} & \quad x \in \mathbb{R}^n
\end{aligned}
\]
Main steps of the proof

\[ C_{[L,M]} = \{ u \in \text{Lip}(\mathbb{R}^n) : \text{sp}(u) \subset [-L, L]^n, \| \nabla u \|_{L^\infty(\mathbb{R}^N)} \leq M \} \]

\[ SC_{[K,L,M]} = \{ u \in C_{[L,M]} : u \text{ semiconcave with constant } K \} \]

- Semiconcavity of the Hopf-Lax semigroup

\[ S_T(C_{[L,M]}) + T \cdot H(0) \subset SC_{[\frac{1}{\alpha T}, L_T, M]} \]

where \( L_T = L + T \cdot \sup_{|p| \leq M} |DH(p)| \)

- Upper bound for the \( \varepsilon \)-entropy of semiconcave functions

\[ \mathcal{H}_\varepsilon \left( SC_{[K,L,M]} \mid W^{1,1}(\mathbb{R}^n) \right) \leq \frac{C(K, L, M)}{\varepsilon^n} \]

for \( \varepsilon > 0 \) sufficiently small
Main steps of the proof

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\[ H_\varepsilon\left( SC_{[K,L,M]} \mid W^{1,1}(\mathbb{R}^n) \right) \leq \frac{C(K, L, M)}{\varepsilon^n} \]

for \( \varepsilon > 0 \) sufficiently small
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for \( \varepsilon > 0 \) sufficiently small
Compactness for Hamilton-Jacobi

Lower estimate

remind

\[ C_{[L,M]} = \{ u \in \text{Lip}(\mathbb{R}^n) : \text{spt}(u) \subset [-L, L]^n, \| \nabla u \|_{L^\infty(\mathbb{R}^N)} \leq M \} \]

\[
\begin{aligned}
&\quad \{ S_t : \text{Lip}(\mathbb{R}^n) \to \text{Lip}(\mathbb{R}^n) \} \\
&\quad S_t(u)(x) = \min_{y \in \mathbb{R}^n} \left\{ t \cdot H^* \left( \frac{x-y}{t} \right) + u(y) \right\} \quad x \in \mathbb{R}^n
\end{aligned}
\]

Theorem (Ancona, C and Khai T. Nguyen)

Let \( M > 0 \) be fixed
Then, for all \( T > 0 \) there exist constants \( \Gamma_T > 0 \) and \( \Lambda_T \geq 0 \) such that

\[
\mathcal{H}_\varepsilon \left( S_T(C_{[L,M]}) + T \cdot H(0) \mid W^{1,1}(\mathbb{R}^n) \right) \geq \frac{\Gamma_T}{\varepsilon^n}
\]

for all \( L > \Lambda_T \) and all \( \varepsilon > 0 \)
Compactness for Hamilton-Jacobi

Lower estimate

reminder

\[ C_{[L,M]} = \{ u \in \text{Lip}(\mathbb{R}^n) : \text{spt}(u) \subset [-L, L]^n, \| \nabla u \|_{L^\infty(\mathbb{R}^N)} \leq M \} \]

\[
\begin{align*}
S_t : \text{Lip}(\mathbb{R}^n) &\rightarrow \text{Lip}(\mathbb{R}^n) \\
S_t(u)(x) &= \min_{y \in \mathbb{R}^n} \left\{ t \cdot H^* \left( \frac{x-y}{t} \right) + u(y) \right\} \quad x \in \mathbb{R}^n
\end{align*}
\]

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\]

for all \( L > \Lambda_T \) and all \( \varepsilon > 0 \)
Main ideas of the proof of the lower estimate

1. **Controllability type result**: introduce a parameterized class $U$ of smooth functions and show that any element of such a class can be attained, at any given time $T > 0$, by the Hopf-Lax flow $S_T(u)$ for a suitable $u \in C_{[L,M]}$

2. **Combinatorial computation**: provide an optimal (w.r.t. parameters) estimate of the maximum number of functions in $U$ that can be contained in a ball of radius $2\varepsilon$ (with respect to the norm of $W^{1,1}(\mathbb{R}^n)$)
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Theorem

Given $K, L, M > 0$, let $T > 0$ be such that

$$K T \leq \frac{1}{2\alpha_M} \quad \text{where} \quad \alpha_M = \sup_{|p| \leq M} \|D^2H(p)\|$$

Then

$$SC_{[K,L,M]} - T \cdot H(0) \subset ST(C_{[LT,M]})$$

with $LT = L + T \cdot \sup_{|p| \leq M} |DH(p)|$

Our goal: for any $u_T \in SC_{[K,L,M]} - T \cdot H(0)$ to find $u_0 \in C_{[LT,M]}$ such that $ST(u_0) = u_T$
Reachability of semiconcave functions

**Theorem**

Given $K, L, M > 0$, let $T > 0$ be such that

$$K T \leq \frac{1}{2 \alpha_M}$$

where $\alpha_M = \sup_{|\rho| \leq M} \| D^2 H(p) \|$

Then

$$SC_{[K,L,M]} - T \cdot H(0) \subset S_T(C_{[L_T,M]})$$

with $L_T = L + T \cdot \sup_{|\rho| \leq M} |DH(p)|$

Our goal: for any $u_T \in SC_{[K,L,M]} - T \cdot H(0)$ to find $u_0 \in C_{[L_T,M]}$ such that $S_T(u_0) = u_T$
Reachability of semiconcave functions

Theorem

Given $K, L, M > 0$, let $T > 0$ be such that

$$KT \leq \frac{1}{2\alpha_M}$$

where

$$\alpha_M = \sup_{|p| \leq M} \|D^2H(p)\|$$

Then

$$SC_{[K,L,M]} - T \cdot H(0) \subset ST(C_{[L_T,M]})$$

with $L_T = L + T \cdot \sup_{|p| \leq M} |DH(p)|$

Our goal: for any $u_T \in SC_{[K,L,M]} - T \cdot H(0)$ to find $u_0 \in C_{[L_T,M]}$ such that $S_T(u_0) = u_T$
Backward construction

Solve the equation backwards: set \( v(t, x) = S_t(v_0)(x) \) with

\[
v_0(x) = -u_T(-x)
\]

and define

\[
u(t, x) = -v(T - t, -x) \quad (t, x) \in [0, T] \times \mathbb{R}^n
\]

Then

- \( u(T, \cdot) = u_T \)
- \( u_0 = u(0, \cdot) \in C_{[L, M]} \) by the properties of \( S_T \)
- \( u_t(t, x) + H(\nabla u(t, x)) = 0 \) for a.e. \( (t, x) \in [0, T] \times \mathbb{R}^n \)

Therefore,

\[
u \text{ viscosity solution} \implies u_T = S_T(u_0)
\]

The viscosity property follows from the semiconvexity of \( v(t, \cdot) \).
Backward construction

Solve the equation backwards: set

\[ \nu(t, x) = S_t(\nu_0)(x) \]

with

\[ \nu_0(x) = -u_T(-x) \]

and define

\[ u(t, x) = -\nu(T - t, -x) \quad (t, x) \in [0, T] \times \mathbb{R}^n \]

Then

1. \( u(T, \cdot) = u_T \)
2. \( u_0 = u(0, \cdot) \in C_{[L_T, M]} \) by the properties of \( S_T \)
3. \( u_t(t, x) + H(\nabla u(t, x)) = 0 \) for a.e. \( (t, x) \in [0, T] \times \mathbb{R}^n \)

Therefore,

\[ u \text{ viscosity solution} \implies u_T = S_T(u_0) \]

The viscosity property follows from the semiconvexity of \( \nu(t, \cdot) \)
Backward construction

Solve the equation backwards: set \( v(t, x) = S_t(v_0)(x) \) with

\[
v_0(x) = -u_T(-x)
\]

and define

\[
u(t, x) = -v(T - t, -x) \quad (t, x) \in [0, T] \times \mathbb{R}^n
\]

Then

\[
\begin{align*}
& u(T, \cdot) = u_T \\
& u_0 = u(0, \cdot) \in C_{[L_T, M]} \quad \text{by the properties of } S_T \\
& u_t(t, x) + H(\nabla u(t, x)) = 0 \quad \text{for a.e. } (t, x) \in [0, T] \times \mathbb{R}^n
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The viscosity property follows from the semiconvexity of \( v(t, \cdot) \)
Lower bound for $\mathcal{H}_\varepsilon \left( SC_{[K,L,M]} \mid W^{1,1}(\mathbb{R}^n) \right)$

**Proposition**

Given $K, L, M > 0$, for any $\varepsilon > 0$

$$\mathcal{H}_\varepsilon \left( SC_{[K,L,M]} \mid W^{1,1}(\mathbb{R}^n) \right) \geq \frac{\Gamma(K, L, M)}{\varepsilon^n}$$

Given $N \geq 1$ integer, divide $[-L, L]^2$ into $N^2$ squares of side $\frac{2L}{N}$

$$[-L, L]^2 = \bigcup_{i,j=1,...,N} \Box_{ij}$$

Construct bump functions $b_{ij} : \Box_{ij} \rightarrow \mathbb{R}$ such that

- $\|\nabla b_{ij}\|_\infty \leq \frac{KL}{12N}$, $\|b_{ij}\|_{W^{1,1}} \leq \frac{C}{N^3}$
- $\nabla b_{ij}$ Lipschitz with constant $K$
The class $\mathcal{U}_N$ of smooth functions

Let

$$\Delta_N = \left\{ \delta = (\delta_{ij})_{i,j=1}^N : \delta_{ij} \in \{-1, 1\} \right\}$$

Consider the class of smooth functions

$$\mathcal{U}_N = \left\{ u_\delta = \sum_{i,j=1}^N \delta_{ij} \cdot b_{ij} : \delta \in \Delta_N \right\}$$

Then $\#(\mathcal{U}_N) = 2^{N^2}$. Also, one can show that

- $\mathcal{U}_N \subset SC_{[K,L,M]}$
- $\|u_\delta' - u_\delta\|_{W^{1,1}(\mathbb{R}^2)} \leq \epsilon$ if $\#\{(i,j) : \delta_{ij}' \neq \delta_{ij}\} \leq C_{K,L} N^{n+1} \epsilon$

Choosing $N \approx \frac{1}{\epsilon}$, by a combinatorial argument one can show that

$$\#\{\delta' \in \Delta_N : \|u_\delta' - u_\delta\|_{W^{1,1}(\mathbb{R}^2)} \leq \epsilon\} \leq 2^{N^2} e^{-N^2/8} = e^{-N^2/8} \#(\mathcal{U}_N)$$

which yields

$$\mathcal{H}_\epsilon \left( \mathcal{U}_N \mid W^{1,1}(\mathbb{R}^2) \right) \geq \frac{\Gamma}{\epsilon^2}$$

with $\Gamma = \Gamma(K,L,M) > 0$. Therefore,

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End of the proof of the lower estimate

want to show

Let $M > 0$ be fixed. Then, $\forall T > 0$ there exist constants $\Gamma_T > 0$ and $\Lambda_T \geq 0$ such that

$$H_\epsilon\left(S_T(C_{[L,M]}) + T \cdot H(0) \mid W^{1,1}(\mathbb{R}^n)\right) \geq \frac{\Gamma_T}{\varepsilon^n} \quad \forall L > \Lambda_T, \forall \varepsilon > 0$$

Choose $0 < h \leq M$ such that $\sup_{\|p\| \leq h} \|DH^2(p)\| \leq 2 \cdot \|DH^2(0)\|$ and define

$$\Lambda_T = 2T \cdot \sup_{\|p\| \leq h} |DH(p)| \quad \text{and} \quad K_T = \frac{1}{4T|D^2H(0)|}$$

By the reachability of semiconcave functions we have that, $\forall L \geq \Lambda_T$,

$$SC_{[K_T, \frac{L}{2}, h]} \subset S_T(C_{[L,h]}) + T \cdot H(0) \subset S_T(C_{[L,M]}) + T \cdot H(0)$$

Recalling the lower bound for the $\varepsilon$-entropy of semiconcave functions

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Concluding remarks

- combining the upper and lower estimates (near $\varepsilon = 0$)

$$ \mathcal{H}_\varepsilon \left( S_T(C_{[L,M]}) + T \cdot H(0) \mid W^{1,1}(\mathbb{R}^n) \right) \approx \varepsilon^{-n} $$

- compactness estimates can be extended to

$$ u_t(t, x) + H(t, x, \nabla u(t, x)) = 0 \quad (t, x) \in [0, T] \times \mathbb{R}^n $$

(no Hopf-Lax formula available)

- reachability example of a controllability result for Hamilton-Jacobi equations
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Thank you for your attention and thanks to for the hospitality