Global exact simultaneous controllability of an arbitrary number of 1D bilinear Schrödinger equations

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Partial differential equations, optimal design and numerics
Benasque
Joint work with Vahagn Nersesyan (UVSQ).
Model studied : $N$ identical and independent $1D$ particles in a potential

\[
\begin{cases}
  i \partial_t \psi^j = (-\partial_{xx}^2 + V(x)) \psi^j - u(t)\mu(x)\psi^j, & x \in (0, 1), \\
  \psi^j(t, 0) = \psi^j(t, 1) = 0, & j \in \{1, \ldots, N\},
\end{cases}
\]

where

- **State** : $(\psi^1, \ldots, \psi^N) \in S^N$,
- **control** : $u : (0, T) \to \mathbb{R}$,
- $V : (0, 1) \to \mathbb{R}$ potential,
- $\mu : (0, 1) \to \mathbb{R}$ dipole moment.

**Goal** : Simultaneous control of $(\psi^1, \ldots, \psi^N)$ with a single control $u$. 

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Global exact simultaneous controllability of bilinear Schrödinger equations
1. Introduction
   - Notations
   - Main result
   - Previous results

2. Approximate controllability towards finite sums of eigenvectors

3. Local exact controllability around finite sums of eigenvectors.
   - Results
   - Rotation and compactness
   - Local controllability around eigenvectors: the return method

4. Global exact controllability
   - Global exact controllability under favourable hypotheses
   - Global exact controllability for an arbitrary potential
Introduction

- Notations
- Main result
- Previous results

Approximate controllability towards finite sums of eigenvectors

Local exact controllability around finite sums of eigenvectors.

Global exact controllability
- $S : L^2((0, 1), \mathbb{C})$ unit sphere.
- $\lambda_{k,V} \in \mathbb{R}$ and $\varphi_{k,V} \in S$ eigenvalues and eigenvectors of
  \[ A_V \psi := (-\partial_{xx}^2 + V) \psi, \quad D(A_V) := H^2 \cap H^1_0((0, 1), \mathbb{C}) \]
- Functional framework
  \[ H_s(V) := D(A_V^{s/2}), \quad \| \cdot \|_{H_s(V)}^2 := \sum_{k=1}^{\infty} |k^s \langle \cdot, \varphi_{k,V} \rangle|^2, \quad \forall s > 0. \]
Let $\Phi_k, v(t, x) := e^{-i\lambda_k, v t} \varphi_k, v(x)$. $(\Phi_1, v, \ldots, \Phi_N, v)$ solution with $u \equiv 0$.

Bold notations: $\psi := (\psi^1, \ldots, \psi^N)$, $H := H^N$.

Unique weak solution $C^0([0, T], H^3_{(V)})$ for $u \in L^2((0, T), \mathbb{R})$, $\psi_0 \in H^3_{(V)}$, $\psi(\cdot, \psi_0, u)$.

Unitary equivalent vectors $\psi_0, \psi_f$: there exists $U : L^2 \to L^2$ unitary map such that $\psi_f = U \psi_0$ i.e.

$$\psi_f^j = U \psi_0^j, \quad \forall j \in \{1, \ldots, N\}.$$
Main result and strategy

Main Theorem

Let $N \in \mathbb{N}^*$. For every $V \in H^4((0, 1), \mathbb{R})$, system $(S_N)$ is globally exactly controllable in $H^4_{(V)}$, generically with respect to $\mu \in H^4((0, 1), \mathbb{R})$. More precisely, there exists a set $Q_V$ residual in $H^4((0, 1), \mathbb{R})$ such that for every $\mu \in Q_V$

$$\forall \psi_0, \psi_f \text{ unitarily equivalent, } \exists T > 0, \exists u \in L^2((0, T), \mathbb{R});$$

$$\psi(T, \psi_0, u) = \psi_f.$$
Main result and strategy

Main Theorem

Let $N \in \mathbb{N}^*$. For every $V \in H^4((0, 1), \mathbb{R})$, system $(S_N)$ is globally exactly controllable in $H^4_V$, generically with respect to $\mu \in H^4((0, 1), \mathbb{R})$. More precisely, there exists a set $Q_V$ residual in $H^4((0, 1), \mathbb{R})$ such that for every $\mu \in Q_V$

\[ \forall \psi_0, \psi_f \text{ unitarily equivalent}, \exists T > 0, \exists u \in L^2((0, T), \mathbb{R}); \]
\[ \psi(T, \psi_0, u) = \psi_f. \]

Overall strategy:

- Global approximate controllability towards finite sums of eigenvectors
- Use of a suitable Lyapunov function
Main result and strategy

Main Theorem

Let $N \in \mathbb{N}^*$. For every $V \in H^4((0, 1), \mathbb{R})$, system $(S_N)$ is globally exactly controllable in $H^4_{(V)}$, generically with respect to $\mu \in H^4((0, 1), \mathbb{R})$. More precisely, there exists a set $Q_V$ residual in $H^4((0, 1), \mathbb{R})$ such that for every $\mu \in Q_V$

$$\forall \psi_0, \psi_f \text{ unitarily equivalent}, \exists T > 0, \exists u \in L^2((0, T), \mathbb{R});$$

$$\psi(T, \psi_0, u) = \psi_f.$$ 

Overall strategy:

- Global approximate controllability towards finite sums of eigenvectors
- Exact controllability around finite sums of eigenvectors
  - Coron’s return method: local exact controllability around finite sums of eigenvectors
Main result and strategy

Main Theorem

Let \( N \in \mathbb{N}^* \). For every \( V \in H^4((0, 1), \mathbb{R}) \), system \((S_N)\) is globally exactly controllable in \( H^4_{(V)} \), generically with respect to \( \mu \in H^4((0, 1), \mathbb{R}) \). More precisely, there exists a set \( Q_V \) residual in \( H^4((0, 1), \mathbb{R}) \) such that for every \( \mu \in Q_V \)

\[
\forall \psi_0, \psi_f \text{ unitarily equivalent, } \exists T > 0, \exists u \in L^2((0, T), \mathbb{R}); \quad \psi(T, \psi_0, u) = \psi_f.
\]

Overall strategy:

- Global approximate controllability towards finite sums of eigenvectors
- Exact controllability around finite sums of eigenvectors
  - Coron’s return method: local exact controllability around finite sums of eigenvectors
  - Connectedness and compactness: exact controllability around \( z_0 \) (initial conditions) and \( z_f \) (targets) with \( z_0^J, z_f^J \) finite sums of eigenvectors

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Main result and strategy

Main Theorem

Let \( N \in \mathbb{N}^* \). For every \( V \in H^4((0, 1), \mathbb{R}) \), system \( (S_N) \) is globally exactly controllable in \( H^4(V) \), generically with respect to \( \mu \in H^4((0, 1), \mathbb{R}) \). More precisely, there exists a set \( Q_V \) residual in \( H^4((0, 1), \mathbb{R}) \) such that for every \( \mu \in Q_V \)

\[
\forall \psi_0, \psi_f \text{ unitarily equivalent, } \exists T > 0, \exists u \in L^2((0, T), \mathbb{R}); \\
\psi(T, \psi_0, u) = \psi_f.
\]

Overall strategy:

- Global approximate controllability towards finite sums of eigenvectors
- Exact controllability around finite sums of eigenvectors
- Time reversibility

\[
\psi(T, \overline{\psi_f}, u) = \overline{\psi_0} \implies \psi(T, \psi_0, u(T - \cdot)) = \psi_f.
\]
Perturbation and favourable hypotheses

Dealing with an arbitrary potential $V$. Consider the control $u(t) := \tilde{u}(t) - 1$.

\[
\begin{aligned}
    i\partial_t\tilde{\psi}^j &= (-\partial_{xx}^2 + V(x)) \tilde{\psi}^j - (\tilde{u}(t) - 1)\mu(x)\tilde{\psi}^j, \\
    &= (-\partial_{xx}^2 + V(x) + \mu(x)) \tilde{\psi}^j - \tilde{u}(t)\mu(x)\tilde{\psi}^j, \\
    \tilde{\psi}^j(t, 0) = \tilde{\psi}^j(t, 1) &= 0, \\
    j &\in \{1, \ldots, N\},
\end{aligned}
\]

'New potential' : $V + \mu$

- Study of global approximate and local exact controllability of $(S_N)$ under favourable hypothesis on the potential for arbitrary $V$. 
Previous results: finite dimension and approximate controllability

- **Finite dimension**
  - Turinici, Rabitz (2004)
    Control of the orientation of an ensemble of molecules (finite dimension)
  - Silveira, Pereira da Silva, Rouchon (2009)
    Stabilization of density matrices (finite dimension)

- **Approximate controllability in infinite dimension**
    Simultaneous approximate controllability in $L^2$.
    Approximate control of density matrices (through control of Galerkin approximations)
  - Boussaïd, Caponigro, Chambrion (2013)
    Higher Sobolev norms for 'weakly coupled' systems.
Previous results: a single particle ($N = 1$)

$V = 0$. $\mu \in H^3(0,1)$ satisfies $\exists c > 0$ such that

$$\langle \mu \phi_1, \phi_k \rangle \geq \frac{c}{k^3}, \quad \forall k \in \mathbb{N}^*.$$ 

- **Beauchard Laurent** (2010), local exact controllability: $\forall T > 0$, $\exists \delta > 0$ such that

$$\forall \psi_f \in S \cap H^3_0 \text{ with } \|\psi_f - \Phi_1(T)\|_{H^3_0} < \delta,$$

there exists $u \in L^2((0, T), \mathbb{R})$ such that

\[
\begin{cases}
  i\partial_t \psi = -\partial_{xx}^2 \psi - u(t)\mu(x)\psi, \\
  \psi(t, 0) = \psi(t, 1) = 0, \quad \Rightarrow \quad \psi(T) = \psi_f.
\end{cases}
\]

$C^1$ regularity of the map $\psi_f \mapsto u$.

- **Nersesyan** (2010), global exact controllability in $S \cap H^3_{0+\varepsilon}$ for generic $\mu$. 

Previous results: a first step ($N = 2$ and $N = 3$)

\[ V = 0, \mu \in H^3(0, 1) \text{ satisfies } \exists c > 0 \text{ such that} \]

\[ |\langle \mu \varphi_j, \varphi_k \rangle| \geq \frac{c}{k^3}, \quad \forall j \in \{1, \ldots, N\}, \forall k \in \mathbb{N}^*. \]

\[ (\psi_0^1, \ldots, \psi_0^N) = (\varphi_1, \ldots, \varphi_N). \]

- Unreachable targets with small controls in small time for $N \geq 2$.
- $N = 2$: local controllability in arbitrary time up to a global phase i.e. $\forall T > 0, \exists \theta \in \mathbb{R}, \exists \delta > 0$;

\[ \forall (\psi^1_f, \psi^2_f) \in \left( S \cap H^3_{(0)} \right)^2 \text{ with } \langle \psi^1_f, \psi^2_f \rangle = 0 \text{ and} \]

\[ \|\psi^1_f - e^{i\theta} \Phi_1(T)\|_{H^3_{(0)}} + \|\psi^2_f - e^{i\theta} \Phi_2(T)\|_{H^3_{(0)}} < \delta, \]

\[ \exists u \in L^2((0, T), \mathbb{R}) \text{ such that } (\psi^1, \psi^2)(T) = (\psi^1_f, \psi^2_f). \]
Previous results: a first step ($N = 2$ and $N = 3$) II

- $N = 2$: local exact controllability up to a global delay i.e. $\exists T^* > 0; \forall T \geq 0$, $\exists \delta > 0$;

$$\forall (\psi^1_f, \psi^2_f) \in \left( S \cap H^3_{(0)} \right)^2 \text{ with } \langle \psi^1_f, \psi^2_f \rangle = 0 \text{ and } \|\psi^1_f - \Phi_1(T)\|_{H^3_{(0)}} + \|\psi^2_f - \Phi_2(T)\|_{H^3_{(0)}} < \delta,$$

$\exists u \in L^2((0, T^* + T), \mathbb{R})$ such that $(\psi^1, \psi^2)(T^* + T) = (\psi^1_f, \psi^2_f)$.

- $N = 3$: local controllability up to a global phase and a global delay i.e. $\exists T^* > 0, \exists \theta \in \mathbb{R}; \forall T \geq 0, \exists \delta > 0$;

$$\forall (\psi^1_f, \psi^2_f, \psi^3_f) \in \left( S \cap H^3_{(0)} \right)^3 \text{ with } \langle \psi^j_f, \psi^k_f \rangle = \delta_{j=k} \text{ and } \|\psi^1_f - e^{i\theta} \Phi_1(T)\|_{H^3_{(0)}} + \|\psi^2_f - e^{i\theta} \Phi_2(T)\|_{H^3_{(0)}} + \|\psi^3_f - e^{i\theta} \Phi_3(T)\|_{H^3_{(0)}} < \delta,$$

$\exists u \in L^2((0, T^* + T), \mathbb{R})$ such that $(\psi^1, \psi^2, \psi^3)(T^* + T) = (\psi^1_f, \psi^2_f, \psi^3_f)$. 
1 Introduction

2 Approximate controllability towards finite sums of eigenvectors

3 Local exact controllability around finite sums of eigenvectors.

4 Global exact controllability
Approximate controllability towards finite sums of eigenvectors

$N \in \mathbb{N}^*$. $V, \mu \in H^4((0, 1), \mathbb{R})$ such that

(C$_1$) $\langle \mu \varphi_j, V, \varphi_k, V \rangle \neq 0$ for all $j \in \{1, \ldots, N\}$, $k \in \mathbb{N}^*$.

(C$_2$) $\lambda_j, V - \lambda_k, V \neq \lambda_p, V - \lambda_q, V$ for all $j \in \{1, \ldots, N\}$, $k, p, q \in \mathbb{N}^*$ such that $\{j, k\} \neq \{p, q\}$ and $k \neq j$.

Theorem

Let $C_M := \text{Span}\{\varphi_1, V, \ldots, \varphi_M, V\}$. Under Conditions (C$_1$) and (C$_2$), for any $\psi_0 \in S \cap H^4_0(V)$ with $\langle \psi_0^j, \varphi_j, V \rangle \neq 0$, for all $j \in \{1, \ldots, N\}$, there are $M \in \mathbb{N}^*$, $\psi_f \in C_M$, sequences $T_n > 0$ and $u_n \in C_0^\infty((0, T_n), \mathbb{R})$ such that

$$\psi(T_n, \psi_0, u_n) \xrightarrow{n \to \infty} \psi_f \quad \text{in} \quad H^3.$$ 

Sketch of proof \textit{I}

- Lyapunov strategy.

\[
\mathcal{L}(z) := \alpha \sum_{j=1}^{N} \| (-\partial_{xx}^2 + V)^2 P_N z_j \|_{L^2}^2 + 1 - \prod_{j=1}^{N} |\langle z_j, \varphi_j, V \rangle|^2,
\]

with $P_N$ orthogonal projection in $L^2$ onto $\text{Span}\{\varphi_k, V ; k \geq N + 1\}$.

- Decrease: $z \in S \cap H^4_{(V)}$ with $\langle z_j, \varphi_j, V \rangle \neq 0$, for all $j \in \{1, \ldots, N\}$.

Either

\[
z \in \bigcup_{M \in \mathbb{N}^*} C_M,
\]

or $\exists T > 0, \exists u \in C_{0}^\infty((0, T), \mathbb{R})$ such that

\[
\mathcal{L}(\psi(T, z, u)) < \mathcal{L}(z).
\]
Sketch of proof II

idea: existence of $T$ and $w \in C_0^\infty((0, T), \mathbb{R})$ such that

$$\frac{d}{d\sigma} \mathcal{L}(\psi(T, \psi_0, \sigma w))\bigg|_{\sigma=0} \neq 0.$$ 

- We define

$$\mathcal{K} := \left\{ \psi \in H^4_V; \psi(T_n, \psi_0, u_n) \xrightarrow{n \to \infty} \psi \text{ in } H^3, \text{ for } T_n \geq 0, u_n \in C_0^\infty((0, T_n), \mathbb{R}) \right\}.$$ 

- $e \in \mathcal{K}$ such that $\mathcal{L}(e) = \inf_{\psi \in \mathcal{K}} \mathcal{L}(\psi)$. Then

$$e \in \bigcup_{M \in \mathbb{N}^*} \mathcal{C}_M.$$
Introduction

2 Approximate controllability towards finite sums of eigenvectors

3 Local exact controllability around finite sums of eigenvectors.
   - Results
   - Rotation and compactness
   - Local controllability around eigenvectors: the return method

4 Global exact controllability
Local exact controllability around finite sums of eigenvectors

$N \in \mathbb{N}^*$. $V, \mu \in H^3((0, 1), \mathbb{R})$ such that

- \textbf{(C$_3$)} there exists $c > 0$ such that
  \[ |\langle \mu \varphi_j, V \varphi_k, V \rangle| \geq \frac{c}{k^3}, \quad \forall j \in \{1, \ldots, N\}, \forall k \in \mathbb{N}^*, \]

- \textbf{(C$_4$)} $\lambda_k, V - \lambda_j, V \neq \lambda_p, V - \lambda_n, V$ for all $j, n \in \{1, \ldots, N\}$, $k \geq j + 1$, $p \geq n + 1$ with $\{j, k\} \neq \{p, n\}$,

- \textbf{(C$_5$)} $1, \lambda_1, V, \ldots, \lambda_N, V$ are rationally independent.

\vspace{1cm}

\textbf{Theorem}

Let $C_0, C_f \in U_N$ and $z_0 := C_0 \varphi_V$, $z_f := C_f \varphi_V$. Under Conditions \textbf{(C$_3$)}-\textbf{(C$_5$)}, there exists $T > 0$, $\delta > 0$ such that, if

\[ O_{\delta, C} := \left\{ \phi \in H^3_V ; \langle \phi^j, \phi^k \rangle = \delta_{j=k} \text{ and } \sum_{j=1}^N \|\phi^j - (C \varphi_V)^j\|_{H^3_V} < \delta \right\}, \]

for every $\psi_0 \in O_{\delta, C_0}$, $\psi_f \in O_{\delta, C_f}$, there exists $u \in L^2((0, T), \mathbb{R})$ such that the associated solution satisfies $\psi(T) = \psi_f$. 
Proposition

Assume Conditions \((C_3)-(C_4)\).

- \(T > 0\), there are \(\theta_1, \ldots, \theta_N \in \mathbb{R}\), \(\delta > 0\);
  \[
  \forall \psi_0 \in H^3(V); \quad \langle \psi_0^j, \psi_0^k \rangle = \delta_{j=k} \quad \text{and} \quad \sum_{j=1}^N \| \psi_0^j - \varphi_j, V \|_{H^3(V)} < \delta,
  \]

- \(\forall \psi_f \in H^3(V); \quad \langle \psi_f^j, \psi_f^k \rangle = \delta_{j=k} \quad \text{and} \quad \sum_{j=1}^N \| \psi_f^j - e^{i\theta_j} \varphi_j, V \|_{H^3(V)} < \delta,
  \]

there exists \(u \in L^2((0, T), \mathbb{R})\) such that \(\psi(T, \psi_0, u) = \psi_f\).

- \(C^1\) regularity of the map \((\psi_0, \psi_f) \mapsto u\).

Similar to \(\textbf{M} \) (2013) for \(N = 2, 3\). No condition on the phase terms \(\theta_j\).
Proof: rotation

1. Proof in the case $C_0 = C_f = I_N$. $\psi_0, \psi_f \approx \varphi_V$.
   - Use of the proposition.
     \[
     \psi_0 \approx \varphi_V \quad \overset{T^*,u}{\sim} \quad \left(e^{i\theta_1}\varphi_1, V, \ldots, e^{i\theta_N}\varphi_N, V\right).
     \]
   - Rotation and rational independence of eigenvalues: Condition $(C_5)$.
     \[
     \left(e^{i\theta_1}\varphi_1, V, \ldots, e^{i\theta_N}\varphi_N, V\right) \quad \overset{T_r,u=0}{\sim} \quad \zeta := \left(e^{i(\theta_1-\lambda_1)V T_r}\varphi_1, V, \ldots, e^{i(\theta_N-\lambda_N)V T_r}\varphi_N, V\right)
     \]
     \[
     \approx \left(e^{-i\theta_1}\varphi_1, V, \ldots, e^{-i\theta_N}\varphi_N, V\right)
     \]
   - Use of the proposition.
     \[
     \bar{\psi}_f \approx \varphi_V \quad \overset{T^*,v}{\sim} \quad \bar{\zeta}.
     \]
   - Conclusion: time-reversibility
     \[
     \zeta \quad \overset{T^*,v(T^*\cdot)}{\sim} \quad \psi_f.
     \]
2. Proof in the case \( C_0 = C_f = C \in U_N \). Let \( z := C \varphi_V \). \( \psi_0, \psi_f \approx z \).

- Let \( \delta_z > 0 \) such that

\[
C^* \left( B_{H^3_{(V)}} (z, \delta_z) \right) \subset B_{H^3_{(V)}} (\varphi_V, \delta),
\]

and

\[
\tilde{\psi}_0 := C^* \psi_0, \quad \tilde{\psi}_f := C^* \psi_f.
\]

- Step 1. \( \tilde{T} := 2 T^* + T_r, \exists u \in L^2((0, \tilde{T}), \mathbb{R}) \) such that

\[
\tilde{\psi}_0 \xrightarrow{\tilde{T}, u} \tilde{\psi}_f.
\]

- Linearity of \((S_N)\) with respect to the state

\[
\psi(\tilde{T}, \psi_0, u) = \psi(\tilde{T}, C \tilde{\psi}_0, u) = C \psi(\tilde{T}, \tilde{\psi}_0, u) = C \tilde{\psi}_f = \psi_f.
\]
Proof: connectedness and compactness

3. Conclusion: \( C_0, C_f \in U_N \).

- Connectedness in the set of unitary matrices and compactness.
3. Conclusion: $C_0, C_f \in U_N$.
- Connectedness in the set of unitary matrices and compactness.

\[ C : t \in [0, 1] \mapsto C(t) \in U_N \]
with $C(0) = C_0$ and $C(1) = C_f$
3. **Conclusion**: $C_0, C_f \in U_N$.
   - Connectedness in the set of unitary matrices and compactness.
3. **Conclusion**: $C_0, C_f \in U_N$.

- Connectedness in the set of unitary matrices and compactness.
3. **Conclusion**: $C_0, C_f \in U_N$.

- Connectedness in the set of unitary matrices and compactness.
Proposition

$N \in \mathbb{N}^*$. $V, \mu \in H^3((0, 1), \mathbb{R})$ such that

(C3) there exists $c > 0$ such that

$$|\langle \mu \varphi_j, V, \varphi_k, V \rangle| \geq \frac{c}{k^3}, \quad \forall j \in \{1, \ldots, N\}, \forall k \in \mathbb{N}^*,$$

(C4) $\lambda_{k, V} - \lambda_{j, V} \neq \lambda_{p, V} - \lambda_{n, V}$ for all $j, n \in \{1, \ldots, N\}, k \geq j + 1, p \geq n + 1$ with $\{j, k\} \neq \{p, n\}$.

$T > 0$, there are $\theta_1, \ldots, \theta_N \in \mathbb{R}$, $\delta > 0$;

$$\forall \psi_0 \in H^3_{(V)}; \langle \psi_0^j, \psi_0^k \rangle = \delta_{j=k} \text{ and } \sum_{j=1}^{N} \| \psi_0^j - \varphi_j, V \|_{H^3_{(V)}} < \delta,$$

$$\forall \psi_f \in H^3_{(V)}; \langle \psi_f^j, \psi_f^k \rangle = \delta_{j=k} \text{ and } \sum_{j=1}^{N} \| \psi_f^j - e^{i\theta_j} \varphi_j, V \|_{H^3_{(V)}} < \delta,$$

there exists $u \in L^2((0, T), \mathbb{R})$ such that $\psi(T, \psi_0, u) = \psi_f$. 
Natural strategy : linear test

- Linearized system around \((\Phi_1, \nu, \ldots, \Phi_N, \nu, u \equiv 0)\)

\[
\begin{aligned}
\left\{ \begin{array}{l}
    i\partial_t \psi_j = -\partial_{xx}^2 \psi_j - \nu(t) \mu(x) \Phi_j, \nu, \quad j \in \{1, \ldots, N\} \\
    \psi_j(t, 0) = \psi_j(t, 1) = 0, \\
    \psi_j(0, x) = 0.
\end{array} \right.
\end{aligned}
\]

\[
\psi^j(T) = i \sum_{k=1}^{\infty} \langle \mu \varphi_j, \nu, \varphi_k, \nu \rangle \int_0^T \nu(t) e^{i(\lambda_k, \nu - \lambda_j, \nu) t} dt \Phi_k, \nu(T).
\]
Natural strategy : linear test

- Linearized system around \((\Phi_1, \nu, \ldots, \Phi_N, \nu, u \equiv 0)\)

\[
\begin{aligned}
  i \partial_t \psi^j &= -\partial_{xx}^2 \psi^j - \nu(t) \mu(x) \Phi_j, \nu, \quad j \in \{1, \ldots, N\} \\
  \psi^j(t, 0) &= \psi^j(t, 1) = 0, \\
  \psi^j(0, x) &= 0.
\end{aligned}
\]

\[
\psi^j(T) = i \sum_{k=1}^{\infty} \langle \mu \varphi_j, \nu, \varphi_k, \nu \rangle \int_0^T \nu(t) e^{i(\lambda_k, \nu - \lambda_j, \nu)t} dt \Phi_k, \nu(T).
\]

- Gap condition + null upper density (Conditions \((C_3)-(C_4)) \leadsto Solution of moment problem for non redundant frequencies

\[
\left\{ \lambda_k, \nu - \lambda_j, \nu ; j \in \{1, \ldots, N\}, k \geq j + 1 \text{ and } k = j = N \right\}.
\]
Natural strategy : linear test

- Linearized system around \((\Phi_1, \nu, \ldots, \Phi_N, \nu, u \equiv 0)\)

\[
\begin{cases}
    i\partial_t \psi^j = -\partial_{xx}^2 \psi^j - \nu(t)\mu(x)\Phi_j, \nu, & j \in \{1, \ldots, N\} \\
    \psi^j(t, 0) = \psi^j(t, 1) = 0, \\
    \psi^j(0, x) = 0.
\end{cases}
\]

\[
\psi^j(T) = i \sum_{k=1}^{\infty} \langle \mu \varphi_j, \nu, \varphi_k, \nu \rangle \int_0^T \nu(t) e^{i(\lambda_k, \nu - \lambda_j, \nu)t} dt \Phi_k, \nu(T).
\]

- Gap condition + null upper density (Conditions \((C_3)-(C_4)) \leadsto Solution of moment problem for non redundant frequencies

\[
\left\{ \lambda_k, \nu - \lambda_j, \nu ; j \in \{1, \ldots, N\}, k \geq j + 1 \text{ and } k = j = N \right\}.
\]

- Lost directions.

\[
\frac{\langle \psi^j(T), \Phi_j, \nu(T) \rangle}{\langle \mu \varphi_j, \nu, \varphi_j, \nu \rangle} = \frac{\langle \psi^k(T), \Phi_k, \nu(T) \rangle}{\langle \mu \varphi_k, \nu, \varphi_k, \nu \rangle}, \quad \forall j, k \in \{1, \ldots, N\}.
\]
The return method

The return method


Local controllability around $z_0$ and $z_f$
linearized system not controllable
The return method


Construction of a reference trajectory with controllable linearized system
The reference trajectory

Let $T > 0$ and $0 < \varepsilon_1 < \cdots < \varepsilon_{N-1} < T$.

Under Conditions $(C_3)$ and $(C_4)$, there exist $\bar{\eta} > 0$, $C > 0$ such that $\forall \eta \in (0, \bar{\eta})$, $\exists \theta_1^\eta, \ldots, \theta_N^\eta \in \mathbb{R}$, $\exists u_{ref}^\eta \in L^2((0, T), \mathbb{R})$ with

$$\|u_{ref}^\eta\|_{L^2} \leq C\eta,$$

such that $\forall j \in \{1, \ldots, N\}$, $\forall k \in \{1, \ldots, N - 1\}$,

$$\langle \mu \psi_{ref}^{j,\eta}(\varepsilon_k), \psi_{ref}^{j,\eta}(\varepsilon_k) \rangle = \langle \mu \varphi_j, \nu_k, \varphi_j, \nu_k \rangle + \eta \delta_{j=k},$$

and

$$\psi_{ref}^\eta(T) = \left( e^{i\theta_1^\eta} \varphi_1, \nu, \ldots, e^{i\theta_N^\eta} \varphi_N, \nu \right).$$

**Main ideas**: Small perturbations + partial control results (moment problem and invariants)

$$\psi_{ref}^\eta(T) = \left( e^{i\theta_1^\eta} \varphi_1, \nu, \ldots, e^{i\theta_N^\eta} \varphi_N, \nu \right) \iff \langle \psi_{ref}^{j,\eta}(T), \Phi_k, \nu(T) \rangle = 0, \forall k \geq j + 1.$$
Proof of the construction of the reference trajectory

- $[0, \varepsilon_{N-1}]$ : Small perturbation (partial control result) such that
  \[
  \langle \mu \psi^j,\eta \psi^j(\varepsilon_k) \rangle = \langle \mu \varphi_j,\nu, \varphi_j,\nu \rangle + \eta \delta_{j=k}, \quad \forall j \in \{1, \ldots, N\}, \forall k \in \{1, \ldots, N-1\}.
  \]

- $[\varepsilon_{N-1}, T]$ : Reaching the target.
  \[
  \psi^\eta_{\text{ref}}(T) = \left( e^{i\theta_1^\eta} \varphi_1,\nu, \ldots, e^{i\theta_N^\eta} \varphi_N,\nu \right) \iff P_j(\psi^j,\eta_{\text{ref}}(T)) = 0, \quad \forall j \in \{1, \ldots, N\},
  \]

where
  \[
  P_j(\psi) = \sum_{k \geq j+1} \langle \psi, \varphi_k,\nu \rangle \varphi_k,\nu.
  \]

Inverse mapping theorem at $(0, \Phi_1,\nu(\varepsilon_{N-1}), \ldots, \Phi_N,\nu(\varepsilon))$ to
  \[
  \Theta(u, \psi_0) := \left( \psi_0, P_1(\psi^1(\tau)), \ldots, P_N(\psi^N(\tau)) \right).
  \]

Continuous right inverse of $d\Theta(0, \Phi_1,\nu(\varepsilon_{N-1}), \ldots, \Phi_N,\nu(\varepsilon))$ : solve a trigonometric moment problem with frequencies
  \[
  \{ \lambda_k,\nu - \lambda_j,\nu \mid j \in \{1, \ldots, N\}, k \geq j + 1 \}.
  \]
Controllability of the linearized system around the reference trajectory

\[
\begin{cases}
    i \partial_t \psi^{j,\eta} = (-\partial_{xx}^2 + V(x)) \psi^{j,\eta} - u_{\text{ref}}^{\eta}(t) \mu(x) \psi^{j,\eta} - \nu(t) \mu(x) \psi_{\text{ref}}^{j,\eta}, \\
    \psi^{j,\eta}(t, 0) = \psi^{j,\eta}(t, 1) = 0, \\
    \psi^{j,\eta}(0, x) = \psi_0^{j,\eta}(x).
\end{cases}
\]

Linearization of the invariants:

\[
\text{Re}(\langle \psi^{j,\eta}, \psi^{j,\eta}_{\text{ref}}(t) \rangle) = 0, \quad \forall 1 \leq j \leq N,
\]

\[
\langle \psi^{j,\eta}, \psi^{k,\eta}_{\text{ref}}(t) \rangle + \langle \psi^{k,\eta}, \psi^{j,\eta}_{\text{ref}}(t) \rangle = 0, \quad \forall 1 \leq k < j \leq N.
\]

**Controllability**: There exists \( \hat{\eta} \in (0, \bar{\eta}) \) such that for any \( \eta \in (0, \hat{\eta}) \), for any suitable \( (\Psi_0, \Psi_f) \in H^3(V) \), there exists \( \nu \in L^2((0, T), \mathbb{R}) \) such that the solution initiated from \( \Psi_0 \) satisfies

\[
\Psi^{\eta}(T) = \Psi_f.
\]
Controllability of $\langle \psi^j, \eta(T), \Phi_k, V(T) \rangle$ for $j \in \{1, \ldots, N\}$ and $k \in \mathbb{N}^*$.

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Choice of $\eta$ small enough + moment problem

Minimal family for diagonal directions.

Invariants
### Sketch of proof

Controllability of $\langle \Psi^j, \eta(T), \Phi_k, V(T) \rangle$ for $j \in \{1, \ldots, N\}$ and $k \in \mathbb{N}^*$.

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- Choice of $\eta$ small enough + moment problem
  - For $\eta = 0$: $j \in \{1, \ldots, N\}$, $k \geq j + 1$ and $k = j = N$

$$
\langle \Psi^j, 0(T), \Phi_k, V(T) \rangle = i \langle \mu \varphi_j, V, \varphi_k, V \rangle \int_0^T v(t)e^{i(\lambda_k, V - \lambda_j, V)t}dt,
$$

solve a trigonometric moment problem (Conditions $(C_3)$ and $(C_4)$).
## Sketch of proof

Controllability of \( \langle \Psi^j, \eta(T), \Phi_k, V(T) \rangle \) for \( j \in \{1, \ldots, N\} \) and \( k \in \mathbb{N}^* \).

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- **Choice of \( \eta \) small enough + moment problem**
  - For \( \eta = 0 \): \( j \in \{1, \ldots, N\} \), \( k \geq j + 1 \) and \( k = j = N \)
    \[
    \langle \Psi^{j,0}, (T), \Phi_k, V(T) \rangle = i \langle \mu \varphi_j, V, \varphi_k, V \rangle \int_0^T v(t) e^{i(\lambda_k, V - \lambda_j, V)t} dt,
    \]
    solve a trigonometric moment problem (Conditions \( (C_3) \) and \( (C_4) \)).
  - **Choice of \( \eta \) sufficiently small \( \Longrightarrow \) controllability of \( \langle \Psi^j, \eta(T), \Phi_k, V(T) \rangle \).
Controllability of $\langle \Psi^j, \eta(T), \Phi_k, V(T) \rangle$ for $j \in \{1, \ldots, N\}$ and $k \in \mathbb{N}^*$. 

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- Choice of $\eta$ small enough + moment problem
- Minimal family for diagonal directions.

- For $\eta = 0$:
  \[
  \langle \Psi^j, 0(T), \Phi_j, V(T) \rangle \sim \langle \mu \phi_j, \nu, \phi_j, \nu \rangle \int_0^T \nu(t) dt, \quad \forall j \in \{1, \ldots, N\}.
  \]
Sketch of proof

Controllability of $\langle \Psi^j, \eta(T), \Phi_k, \nu(T) \rangle$ for $j \in \{1, \ldots, N\}$ and $k \in \mathbb{N}^*$.

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- Choice of $\eta$ small enough + moment problem
- Minimal family for diagonal directions.

For $\eta = 0$:

$\langle \Psi^{j, 0}(T), \Phi_j, \nu(T) \rangle \sim \langle \mu \varphi_j, \nu, \varphi_j, \nu \rangle \int_0^T \nu(t) dt$, $\forall j \in \{1, \ldots, N\}$.

For $\eta > 0$:

$\langle \Psi^{j, \eta}(T), \Phi_j, \nu(T) \rangle \sim \int_0^T \nu(t) \langle \mu \psi_{ref}^{j, \eta}(t), \psi_{ref}^{j, \eta}(t) \rangle dt$, $\forall j \in \{1, \ldots, N\}$.

Independence condition on $\langle \mu \psi_{ref}^{j, \eta}(t), \psi_{ref}^{j, \eta}(t) \rangle$ in the construction of $\psi_{ref}^{\eta}$.
Sketch of proof

Controllability of \( \langle \Psi^j, \eta(T), \Phi_k, \nu(T) \rangle \) for \( j \in \{1, \ldots, N\} \) and \( k \in \mathbb{N}^* \).

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- Choice of \( \eta \) small enough + moment problem
- Minimal family for diagonal directions.
- Invariants

\[
\langle \Psi^j, \eta, \psi^k, \eta_{\text{ref}}(t) \rangle + \langle \Psi^k, \eta, \psi^j, \eta_{\text{ref}}(t) \rangle = 0, \quad \forall 1 \leq k < j \leq N.
\]
1 Introduction

2 Approximate controllability towards finite sums of eigenvectors

3 Local exact controllability around finite sums of eigenvectors.

4 Global exact controllability
   - Global exact controllability under favourable hypotheses
   - Global exact controllability for an arbitrary potential
$V, \mu \in H^4((0,1), \mathbb{R})$ such that

(C$_6$) for any $j \in \mathbb{N}^*$, $\exists c_j > 0$;

$$|\langle \mu \varphi_j, \nu, \varphi_k, \nu \rangle| \geq \frac{c_j}{k^3}, \quad \forall k \in \mathbb{N}^*,$$

(C$_7$) \{$1, (\lambda_j, \nu)_{j \in \mathbb{N}^*}\} are rationally independent: $\forall M \in \mathbb{N}^*$,

$$\forall \mathbf{r} \in \mathbb{Q}^{M+1}\setminus \{\mathbf{0}\},$$

$$r_0 + \sum_{j=1}^{M} r_j \lambda_j, \nu \neq 0.$$

Conditions (C$_6$)-(C$_7$) $\implies$ Conditions (C$_1$)-(C$_5$), for any $N \in \mathbb{N}^*$.

**Theorem**

Let $N \in \mathbb{N}^*$. Under Conditions (C$_6$)-(C$_7$), for any unitarily equivalent vectors $\varphi_0, \varphi_f \in S \cap H^4_{(V)}$, there are $T > 0, u \in L^2((0, T), \mathbb{R})$ such that

$$\psi(T, \varphi_0, u) = \varphi_f.$$
Sketch of proof

\[ \psi_0 \quad \psi_f \]
Global approximate controllability

Existence of $M \in \mathbb{N}^*$

$z_0, z_f \in \mathcal{C}_M$

\[ z_0, z_f \in \mathcal{C}_M \]
Sketch of proof

Exact controllability of \( (S_M) \) around \( z_0, z_f \)
Sketch of proof

Time reversibility

$\psi_0$ $\rightarrow$ $\psi$ $\rightarrow$ $\psi_f$ $\rightarrow$ $\overline{\psi}_f$
Dealing with an arbitrary potential $V$

$V \in H^4((0,1), \mathbb{R})$ arbitrary

\[
\begin{cases}
  i\partial_t \psi^j = - (\partial_{xx}^2 + V(x) + \mu(x)) \psi^j - u(t)\mu(x)\psi, & (t, x) \in (0, T) \times (0,1), \\
  \psi^j(t, 0) = \psi^j(t, 1) = 0, & j \in \{1, \ldots, N\}.
\end{cases}
\]

Link between propagators of $(S_N)$ and $(\tilde{S}_N)$:

$$\tilde{\psi}(T, \psi_0, u) = \psi(T, \psi_0, u - 1).$$

$Q_V$ : set of $\mu \in H^4((0,1), \mathbb{R})$ such that Conditions $(C_6)$ and $(C_7)$ are satisfied for $V$ replaced by $V + \mu$ i.e.

$$\forall j \in \mathbb{N}^*, \exists c_j > 0; \ |\langle \mu \varphi_j, V + \mu, \varphi_k, V + \mu \rangle| \geq \frac{c_j}{k^3}, \quad \forall k \in \mathbb{N}^*,$$

$$\{1, (\lambda_j, V + \mu)_{j \in \mathbb{N}^*}\}$$ are rationally independent.

$\mu \in Q_V$ : global exact controllability of $(\tilde{S}_N)$ in $S \cap H^4(V + \mu)$. 
Assume $\mu \in Q_V$. Let $\psi_0, \psi_f \in S \cap H^4(V)$. Let $u_1 \in H^1((0, 1), \mathbb{R})$ with $u_1(0) = 0$, $u_1(1) = -1$. Then,
\[
\tilde{\psi}_0 := \psi(1, \psi_0, u_1), \quad \tilde{\psi}_f := \psi(1, \psi_f, u_1) \in S \cap H^4(V + \mu).
\]

Reaching the 'right space' : $\psi_0 \xrightarrow{1,u_1} \tilde{\psi}_0$, for $(S_N)$,

Global exact controllability of $(\tilde{S}_N)$ : $\exists \tilde{T} > 0$, $\exists \tilde{u} \in L^2((0, \tilde{T}), \mathbb{R})$ such that
\[
\tilde{\psi}_0 \xrightarrow{\tilde{T}, \tilde{u}} \tilde{\psi}_f, \quad \text{for} \ (\tilde{S}_N),
\]
i.e.
\[
\tilde{\psi}_0 \xrightarrow{\tilde{T}, \tilde{u}^{-1}} \tilde{\psi}_f, \quad \text{for} \ (S_N).
\]

Time reversibility : $\tilde{\psi}_f \xrightarrow{1,u_1(1-\cdot)} \psi_f$, for $(S_N)$.

$Q_V$ is residual in $H^4((0, 1), \mathbb{R})$. 

Morgan MORANCEY  Global exact simultaneous controllability of bilinear Schrödinger equations 34
Open problems and perspectives

Conclusion
- Global exact controllability
- Arbitrary number of equations
- No restriction on the potential

Open problems
- Large time: Lyapunov strategy, rotation (Kronecker diophantine approximation), compactness argument.
- Optimal spaces: $H^4(V)$, $H^3(V)$ (Lyapunov strategy in infinite dimension)
Open problems and perspectives

Conclusion

- Global exact controllability
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Open problems

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Thank you for your attention.

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