A globally convergent algorithm to solve an inverse problem for waves with potential.

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Workshop: PDE, Optimal design and Numerics
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Outline

1. Introduction
   - The wave equation with potential
   - The inverse problem
   - Classical uniqueness and stability result
   - Classical resolution method

2. A Carleman estimate

3. Our algorithm

4. Numerical issues

5. Conclusion
Let $\Omega$ be a smooth bounded domain of $\mathbb{R}^n$, $n \geq 1$, and $T > 0$. We consider the wave equation with potential

$$\begin{array}{ll}
\frac{\partial^2 t}{\partial t^2} w - \Delta w + pw = g, & \text{in } \Omega \times (0, T), \\
w = 0, & \text{on } \partial \Omega \times (0, T), \\
w(0) = w_0, \quad \partial_t w(0) = w_1, & \text{in } \Omega.
\end{array}$$  \hspace{1cm} (1)

Here, $w$ denotes the amplitude of the waves, $p$ is a potential supposed to be in $L^\infty(\Omega)$, $g$ is a source term for instance in $L^2(\Omega \times (0, T))$ and $(w_0, w_1)$ are the initial data lying in $H^1_0(\Omega) \times L^2(\Omega)$.

D'Alembertian operator:

$$\square = \partial_t^2 - \Delta.$$
Given the source term $g$ and the initial data $(w_0, w_1)$, can we determine the unknown potential $p(x), \forall x \in \Omega$, from the additional knowledge of the flux

$$\mu = \partial_{\nu} w, \quad \text{on } \Gamma_0 \times (0, T),$$

where $\Gamma_0$ is a part of $\partial \Omega$?

Uniqueness? Stability? Numerical resolution?
Theorem (Baudouin-Puel)

Geometric condition:

\[
\exists x_0 \notin \overline{\Omega} \text{ such that } \Gamma_0 \supset \{x \in \partial \Omega, \ (x - x_0) \cdot \nu(x) \geq 0\},
\]
Introduction
Classical uniqueness and stability result

Theorem (Baudouin-Puel)

- **Geometric condition:**
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  \exists x_0 \not\in \bar{\Omega} \text{ such that } \Gamma_0 \supset \{ x \in \partial \Omega, \ (x - x_0) \cdot \nu(x) \geq 0 \},
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- **Time condition:** \( T > \sup_{x \in \Omega} |x - x_0| , \)
Introduction
Classical uniqueness and stability result

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- **Regularity assumption:** \( w \in H^1((0, T); L^\infty(\Omega)), \)
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- **Time condition:** \( T > \sup_{x \in \Omega} |x - x_0|, \)

- **Regularity assumption:** \( w \in H^1((0, T); L^\infty(\Omega)), \)

- **Positivity condition:** \( \exists \alpha > 0 \text{ such that } |w_0| > \alpha \text{ in } \Omega. \)

Then for \( m > 0, \) there exists a positive constant \( M = M(\Omega, T, x_0, m) \) such that for all \( p \) and \( q \) in \( L^\infty_m(\Omega) = \{ p \in L^\infty(\Omega), \| p \|_{L^\infty(\Omega)} \leq m \} : \)

\[ \| p - q \|_{L^2(\Omega)} \leq M \| \partial_t (\partial_\nu w[p] - \partial_\nu w[q]) \|_{L^2(\Gamma_0 \times (0, T))}, \]

where \( w[p] \) and \( w[q] \) denote the corresponding solutions of (1).
A classical method for solving this inverse problem consists in minimizing

\[ J(q) = \| \partial_t (\partial_\nu w[q] - \mu) \|_{L^2(\Gamma_0 \times (0, T))}^2, \]

where \( \mu = \partial_\nu w[p] \) is the observation.
A classical method for solving this inverse problem consists in minimizing

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where \( \mu = \partial_\nu w[p] \) is the observation. Unfortunately, \( J \) is not convex and may have several local minima. Classical minimization algorithms are not guaranteed to converge toward the global minimum of \( J \).
A classical method for solving this inverse problem consists in minimizing

\[ J(q) = \| \partial_t (\partial_N w[q] - \mu) \|_{L^2(\Gamma_0 \times (0, T))}^2, \]

where \( \mu = \partial_N w[p] \) is the observation. Unfortunately, \( J \) is not convex and may have several local minima. Classical minimization algorithms are not guaranteed to converge toward the global minimum of \( J \).

We propose a new algorithm to solve the inverse problem and prove its global convergence. It is based on Carleman estimates.
Outline

1. Introduction

2. A Carleman estimate
   - Carleman weight function for waves
   - An estimate with pointwise term in time 0

3. Our algorithm

4. Numerical issues

5. Conclusion
A Carleman estimate
Carleman weight function for waves

We define, for \((x, t) \in \Omega \times (0, T)\),

\[
\psi(x, t) = |x - x_0|^2 - \beta t^2 + C_0,
\]

and

\[
\varphi(x, t) = e^{\lambda \psi(x, t)},
\]

where \(\beta > 0, \lambda > 0\) and \(C_0 > 0\) is chosen such that \(\psi \geq 1\) in \(\Omega \times (0, T)\).

Function \(\psi\) for \(x_0 = 0, \beta = 1\) and \(C_0 = 0\)

\[
\psi(t) \leq \psi(0), \quad \forall t \in (0, T).
\]
A Carleman estimate
An estimate with pointwise term in time 0

Theorem

Assume the geometric and time conditions. Suppose $\beta \in (0, 1)$ and

$$\beta T > \sup_{x \in \Omega} |x - x_0|.$$
A Carleman estimate
An estimate with pointwise term in time 0

Theorem

Assume the geometric and time conditions. Suppose $\beta \in (0, 1)$ and

$$\beta T > \sup_{x \in \Omega} |x - x_0|.$$  

Then with $m > 0$, there exists a constant $M > 0$ such that for all $s$ and $\lambda$ large enough, for all $q \in L^\infty_m(\Omega)$ and for all $z \in L^2(0, T; H^1_0(\Omega))$ satisfying

$${\Box}z + qz \in L^2(\Omega \times (0, T)), \quad \partial_\nu z \in L^2(\Gamma_0 \times (0, T))$$

and $z(0) = 0$ in $\Omega$:

$$s^{1/2} \int_\Omega e^{2s\varphi(0)} |\partial_t z(0)|^2 \, dx$$

\text{initial energy}

$$\leq M \int_0^T \int_\Omega e^{2s\varphi} |{\Box}z + qz|^2 \, dx \, dt + M s \int_0^T \int_{\Gamma_0} e^{2s\varphi} |\partial_\nu z|^2 \, d\gamma \, dt.$$  

\text{source} \quad \text{observations}

Proof
Outline

1 Introduction

2 A Carleman estimate

3 Our algorithm
   - Iterative loop
   - Convergence result
   - Proof of the convergence result

4 Numerical issues

5 Conclusion
Our algorithm
Iterative loop

Initialization: \( q^0 = 0 \).

Iteration: Given \( q^k \),

\[
\begin{align*}
\text{Initialization: } & \quad q^0 = 0. \\
\text{Iteration: } & \quad \text{Given } q^k, \\
\end{align*}
\]
Our algorithm

Iterative loop

**Initialization:** \( q^0 = 0 \).

**Iteration:** Given \( q^k \),

1. Compute \( w[q^k] \) the solution of

\[
\begin{cases}
\partial_t^2 w - \Delta w + q^k w = g, & \text{in } \Omega \times (0, T), \\
w = 0, & \text{on } \partial\Omega \times (0, T), \\
w(0) = w_0, \quad \partial_t w(0) = w_1, & \text{in } \Omega,
\end{cases}
\]

and set \( \mu^k = \partial_t (\partial_\nu w[q^k] - \partial_\nu w[p]) \) on \( \Gamma_0 \times (0, T) \).
Our algorithm
Iterative loop

Initialization: \( q^0 = 0 \).

Iteration: Given \( q^k \),

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\partial_t^2 w - \Delta w + q^k w = g, & \text{in } \Omega \times (0, T), \\
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\end{cases}
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and set \( \mu^k = \partial_t (\partial_\nu w[q^k] - \partial_\nu w[p]) \) on \( \Gamma_0 \times (0, T) \).

2 - Introduce the functional

\[
J^k_0(z) = \int_0^T \int_\Omega e^{2s\varphi} |\Box z + q^k z|^2 \, dxdt + s \int_0^T \int_{\Gamma_0} e^{2s\varphi} |\partial_\nu z - \mu^k|^2 \, d\gamma dt,
\]

on the space \( T^k = \{ z \in L^2(0, T; H^1_0(\Omega)), z(0) = 0, \Box z + q^k z \in L^2(\Omega \times (0, T)), \partial_\nu z \in L^2(\Gamma_0 \times (0, T)) \} \).
Our algorithm
Iterative loop

Theorem

Assume the geometric and time conditions. Then, for all \( s > 0 \) and \( k \in \mathbb{N} \), the functional \( J^k_0 \) is continuous, strictly convex and coercive on \( T^k \) endowed with a suitable weighted norm.
Our algorithm
Iterative loop

**Theorem**

Assume the **geometric and time conditions**. Then, for all \( s > 0 \) and \( k \in \mathbb{N} \), the functional \( J^k_0 \) is continuous, strictly convex and coercive on \( T^k \) endowed with a suitable weighted norm.

3 - Let \( Z^k \) be the unique minimizer of the functional \( J^k_0 \), and then set

\[
\tilde{q}^{k+1} = q^k + \frac{\partial_t Z^k(0)}{w_0},
\]

where \( w_0 \) is the initial condition of (1).
Our algorithm
Iterative loop

Theorem

Assume the geometric and time conditions. Then, for all \( s > 0 \) and \( k \in \mathbb{N} \), the functional \( J_0^k \) is continuous, strictly convex and coercive on \( T^k \) endowed with a suitable weighted norm.

3 - Let \( Z^k \) be the unique minimizer of the functional \( J_0^k \), and then set

\[
\tilde{q}^{k+1} = q^k + \frac{\partial_t Z^k(0)}{w_0},
\]

where \( w_0 \) is the initial condition of (1).

4 - Finally, set

\[
q^{k+1} = T_m(\tilde{q}^{k+1}), \quad \text{where } T_m(q) = \begin{cases} 
q, & \text{if } |q| \leq m, \\
\text{sign}(q)m, & \text{if } |q| \geq m.
\end{cases}
\]
Theorem

Assume the geometric and time conditions, the regularity assumption and the positivity condition. Let $p \in L^\infty_m(\Omega)$. There exists a constant $M > 0$ such that for all $s$ large enough and for all $k \in \mathbb{N}$,

$$\int_\Omega e^{2s\varphi(0)}(q^k - p)^2 \, dx \leq \left( \frac{M}{\sqrt{s}} \right)^k \int_\Omega e^{2s\varphi(0)} p^2 \, dx.$$

In particular, if $s$ is large enough, $q^k$ converges toward $p$ when $k$ goes to infinity.
Our algorithm

Proof of the convergence result

The algorithm is based on the fact that \( z^k = \partial_t (w[q^k] - w[p]) \) solves

\[
\begin{aligned}
\partial_t^2 z^k - \Delta z^k + q^k z^k &= g^k, & \text{in } \Omega \times (0, T), \\
z^k &= 0, & \text{on } \partial\Omega \times (0, T), \\
z^k(0) = 0, & \quad \partial_t z^k(0) = z_1^k, & \text{in } \Omega,
\end{aligned}
\]

where

\[
g^k = (p - q^k) \partial_t w[p], \quad z_1^k = (p - q^k) w_0.
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\end{align*}
$$

where

$$g^k = (p - q^k) \partial_t w[p], \quad z_1^k = (p - q^k) w_0.$$

Moreover, by definition,

$$\mu^k = \partial_\nu z^k \text{ on } \Gamma_0 \times (0, T),$$

and we notice that $z^k$ is the unique minimizer of the functional:

$$J_{g^k}(z) = \int_0^T \int_{\Omega} e^{2s\varphi} |\Box z + q^k z - g^k|^2 \, dx dt + s \int_0^T \int_{\Gamma_0} e^{2s\varphi} |\partial_\nu z - \mu^k|^2 \, d\gamma dt,$$
Let us write the Euler Lagrange equations satisfied by:

- \( Z^k \) minimizer of \( J_0^k \):

\[
\nabla J_0^k(Z^k, z) = \int_0^T \int_\Omega e^{2s\varphi}(\Box Z^k + q^k Z^k)(\Box z + q^k z) \, dx \, dt \\
+ s \int_0^T \int_{\Gamma_0} e^{2s\varphi}(\partial_\nu Z^k - \mu^k)\partial_\nu z \, d\gamma \, dt = 0,
\]
Let us write the Euler Lagrange equations satisfied by:

- $Z^k$ minimizer of $J^k_0$:

$$\nabla J^k_0(Z^k, z) = \int_0^T \int_\Omega e^{2s\varphi}(\Box Z^k + q^k Z^k)(\Box z + q^k z) \, dx \, dt$$

$$+ s \int_0^T \int_{\Gamma_0} e^{2s\varphi}(\partial_\nu Z^k - \mu^k)\partial_\nu z \, d\gamma \, dt = 0,$$

- and $z^k$ minimizer of $J^k_{g_k}$:

$$\nabla J^k_{g_k}(z^k, z) = \int_0^T \int_\Omega e^{2s\varphi}(\Box z^k + q^k z^k - g^k)(\Box z + q^k z) \, dx \, dt$$

$$+ s \int_0^T \int_{\Gamma_0} e^{2s\varphi}(\partial_\nu z^k - \mu^k)\partial_\nu z \, d\gamma \, dt = 0,$$

for all $z \in \mathcal{T}^k$. 

Our algorithm

Proof of the convergence result
Applying these equations to $z = Z^k - z^k$ and subtracting the two identities, we obtain:

$$
\int_0^T \int_\Omega e^{2s\varphi} |\Box z + q^k z|^2 \, dx \, dt + s \int_0^T \int_{\Gamma_0} e^{2s\varphi} |\partial_\nu z|^2 \, d\gamma \, dt
$$

$$
= \int_0^T \int_\Omega e^{2s\varphi} g^k (\Box z + q^k z) \, dx \, dt.
$$
Our algorithm

Proof of the convergence result

Applying these equations to $z = Z^k - z^k$ and subtracting the two identities, we obtain:

$$
\int_0^T \int_\Omega e^{2s\varphi} |\Box z + q^k z|^2 \, dx dt + s \int_0^T \int_{\Gamma_0} e^{2s\varphi} |\partial_\nu z|^2 \, d\gamma dt
= \int_0^T \int_\Omega e^{2s\varphi} g^k (\Box z + q^k z) \, dx dt.
$$

This implies $(2ab \leq a^2 + b^2)$ that

$$
\frac{1}{2} \int_0^T \int_\Omega e^{2s\varphi} |\Box z + q^k z|^2 \, dx dt + s \int_0^T \int_{\Gamma_0} e^{2s\varphi} |\partial_\nu z|^2 \, d\gamma dt \\
\leq \frac{1}{2} \int_0^T \int_\Omega e^{2s\varphi} |g^k|^2 \, dx dt.
$$
The left hand side precisely is the right hand side of the Carleman estimate.
Hence, we deduce:

\[
s^{1/2} \int_{\Omega} e^{2s\varphi(0)} |\partial_t z(0)|^2 \, dx \leq M \int_0^T \int_{\Omega} e^{2s\varphi} |g^k|^2 \, dx \, dt,
\]

where

\[
\partial_t z(0) = \partial_t Z^k(0) - \partial_t z^k(0).
\]
The left hand side precisely is the right hand side of the Carleman estimate. Hence, we deduce:

\[ s^{1/2} \int_{\Omega} e^{2s\varphi(0)} |\partial_t z(0)|^2 \, dx \leq M \int_{0}^{T} \int_{\Omega} e^{2s\varphi} |g^k|^2 \, dxdt, \]

where

\[ \partial_t z(0) = \partial_t Z^k(0) - \partial_t z^k(0). \]

Moreover

\[ \partial_t Z^k(0) = (\tilde{q}^{k+1} - q^k)w_0, \text{ by definition of } \tilde{q}^{k+1}, \]

\[ \partial_t z^k(0) = z_1^k = (p - q^k)w_0, \]

\[ g^k = (p - q^k)\partial_t w[p]. \]
Therefore, since $\varphi(t) \leq \varphi(0)$ for all $t \in (0, T)$ we have:

$$s^{1/2} \int_{\Omega} e^{2s\varphi(0)} |w_0|^2 (\tilde{q}^{k+1} - p)^2 \, dx$$

$$\leq M \|\partial_t w[p]\|_{L^2(0, T; L^\infty(\Omega))}^2 \int_{\Omega} e^{2s\varphi(0)} (q^k - p)^2 \, dx.$$
Therefore, since $\varphi(t) \leq \varphi(0)$ for all $t \in (0, T)$ we have:

\[
\begin{align*}
  s^{1/2} \int_{\Omega} e^{2s\varphi(0)} |w_0|^2 (\tilde{q}^{k+1} - p)^2 \, dx \\
  &\leq M \| \partial_t w[p] \|_{L^2(0, T; L^\infty(\Omega))}^2 \int_{\Omega} e^{2s\varphi(0)} (q^k - p)^2 \, dx.
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\]

Using the positivity condition on $w_0$ and the fact that

\[|q^{k+1} - p| = |T_m(\tilde{q}^{k+1}) - T_m(p)| \leq |\tilde{q}^{k+1} - p|\]

because $T_m$ is Lipschitz and $T_m(p) = p$, we immediately deduce

\[
\int_{\Omega} e^{2s\varphi(0)} (q^{k+1} - p)^2 \, dx \leq \left( \frac{M}{\sqrt{s}} \right)^{k+1} \int_{\Omega} e^{2s\varphi(0)} (q^0 - p)^2 \, dx.
\]
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2 A Carleman estimate

3 Our algorithm

4 Numerical issues
   - Discretization of the problem
   - Discrete Carleman estimate
   - Examples in 1D
   - Example in 2D

5 Conclusion
Numerical issues
Discretization of the problem

- $\Omega = [0, 1]$, $x_0 = -0.1$, $\Gamma_0 = \{x = 1\}$, $\beta = 0.99$, $T = 1.5$, $\lambda = 0.1$, $s = 1$
Numerical issues
Discretization of the problem

- $\Omega = [0, 1]$, $x_0 = -0.1$, $\Gamma_0 = \{x = 1\}$, $\beta = 0.99$, $T = 1.5$, $\lambda = 0.1$, $s = 1$

- finite differences in space $h = 0.02$, explicit Euler scheme in time $\tau = 0.01$
- $g = 0$, $w_1 = 0$, $w_0(x) = \sin(x\pi)$
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**Numerical issues**

**Discretization of the problem**

- $\Omega = [0, 1]$, $x_0 = -0.1$, $\Gamma_0 = \{x = 1\}$, $\beta = 0.99$, $T = 1.5$, $\lambda = 0.1$, $s = 1$

- finite differences in space $h = 0.02$, explicit Euler scheme in time $\tau = 0.01$
- $g = 0$, $w_1 = 0$, $w_0(x) = \sin(x\pi)$
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  $x_0 \quad 0 \quad 1 \quad \Gamma_0$

- finite differences in space $h = 0.02$, explicit Euler scheme in time $\tau = 0.01$
- $g = 0$, $w_1 = 0$, $w_0(x) = \sin(x\pi)$

- additional noise on the observation data:

  $\mu = (1 + \alpha \ \text{Normal}(0, 0.5)) \ \mu, \quad \alpha \geq 0.$
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- minimization of $J$ by a conjugate gradient
Baudouin-Ervedoza
A regularization term must be added to make the Carleman estimates uniform with respect to the discretization parameter $h$.

$$J_{0,h}^k(z_h) = \int_0^T \int_0^1 e^{2s\varphi} |\Box_h z_h + q^k z_h|^2 dt + s \int_0^T e^{2s\varphi(t,1)} |\partial_h^- z_h(t,1) - \mu^k|^2 dt$$

$$+ s \int_0^T \int_0^1 e^{2s\varphi} |h\partial_h^+ \partial_t z_h|^2 dt.$$  

This term is needed due to spurious waves created by the discretization process (Ervedoza-Zuazua).
Numerical issues
Examples in 1D

\[ p(x) = \sin(2\pi x) \]

without regularization with regularization

oise \alpha = 0\%
Numerical issues
Examples in 1D

2% noise
Numerical issues
Examples in 1D

2% noise

10% noise
Numerical issues
Examples in 1D

\[ p(x) = 1 \]

\[ p(x) = 0 \text{ or } 1 \]

\[ p(x) = \sin(1 - 1/x) \]
Numerical issues
Example in 2D

$p$ in 2D-view

$p$ in 3D-view
Numerical issues

Example in 2D

$p$ in 2D-view

$p$ in 3D-view

$p_h$

$q_h$
Outline

1. Introduction
2. A Carleman estimate
3. Our algorithm
4. Numerical issues
5. Conclusion
   - Drawbacks of the method
   - Prospects
Conclusion

Drawbacks of the method

- We have to derive in time the observation flux: $\partial_t(\partial_\nu w[p])$

  observation at $x = 1$ \hspace{1cm} time derivative

$\Rightarrow$ we regularize the signal by convolutions with a gaussian.
Conclusion

Drawbacks of the method

- We have to derive in time the observation flux: $\partial_t(\partial_\nu w[p])$

observation at $x = 1$  

time derivative

$\Rightarrow$ we regularize the signal by convolutions with a gaussian.

- For $\lambda = 1$ and $s = 3$, $\max(\exp(2s\varphi))/\min(\exp(2s\varphi)) = 10^{110}$ !

$\Rightarrow$ we tried to work with the conjugate variable $\tilde{z} = e^{s\varphi} z$,

$\Rightarrow$ we are trying to change the weights (coming soon...hopefully).
Conclusion
Prospects

- Recovery of the wave propagation speed $c(x)$

\[
\begin{aligned}
\partial_t^2 w - \nabla \cdot (c^2 \nabla w) &= g, & \text{in } \Omega \times (0, T), \\
w &= 0, & \text{on } \partial \Omega \times (0, T), \\
w(0) = w_0, \quad \partial_t w(0) = w_1, & \text{in } \Omega.
\end{aligned}
\]

Application to medical imaging or radar.

⋆⋆⋆
Assume the geometric and time conditions. Define the weight functions $\varphi$ with $\beta \in (0, 1)$ being such that

$$\beta T > \sup_{x \in \Omega} |x - x_0|.$$
Proofs
A global Carleman estimate for the wave equation

Theorem

Assume the geometric and time conditions. Define the weight functions $\varphi$ with $\beta \in (0, 1)$ being such that

$$\beta T > \sup_{x \in \Omega} |x - x_0|.$$ 

Then there exist a constant $M > 0$ such that for all $s$ and $\lambda$ large enough:

$$s \int_{-T}^{T} \int_{\Omega} e^{2s\varphi} \left( |\partial_t z|^2 + |\nabla z|^2 \right) \, dx \, dt + s^3 \int_{-T}^{T} \int_{\Omega} e^{2s\varphi} |z|^2 \, dx \, dt$$

$$\leq M \int_{-T}^{T} \int_{\Omega} e^{2s\varphi} |\Box z|^2 \, dx \, dt + Ms \int_{-T}^{T} \int_{\Gamma_0} e^{2s\varphi} |\partial_\nu z|^2 \, d\gamma \, dt,$$

for all $z \in L^2((-T, T); H^1_0(\Omega))$ satisfying

$\Box z = \partial_t^2 z - \Delta z \in L^2(\Omega \times (-T, T))$ and $\partial_\nu z \in L^2(\partial \Omega \times (-T, T))$. 

Proofs

Sketch of the proof of the global Carleman estimate

Define, for $s > 0$, the conjugate variable $w = e^{s\varphi} \chi z$, where $\chi$ is an cut-off function in time.
Proofs
Sketch of the proof of the global Carleman estimate

- Define, for $s > 0$, the conjugate variable $w = e^{s\varphi} \chi z$, where $\chi$ is an cut-off function in time.

- Introduce the conjugate operator:

  \[ Pw = e^{s\varphi} \Box (e^{-s\varphi} w) = \partial_t^2 w - \Delta w + s^2 ((\partial_t \varphi)^2 - |\nabla \varphi|^2) w \]
  \[- s(\partial_t^2 \varphi - \Delta \varphi) w + s^2 ((\partial_t \varphi)^2 - |\nabla \varphi|^2) w - 2s \partial_t w \partial_t \varphi + 2s \nabla w \cdot \nabla \varphi. \]
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- Using integrations by part, develop the term $\int_{-T}^{T} \int_{\Omega} |Pw|^2 \, dxdt$. 

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- Using integrations by part, develop the term $\int_{-T}^T \int_{\Omega} |Pw|^2 \, dx \, dt$.

- Show that the terms in $|w|^2$, $|\nabla w|^2$ and $|\partial_t w|^2$ can be bounded by below when $s$ is large enough.

- Finally, come back to the initial variable $z$ and absorb the residual terms thanks to the weights $s$. 
Proofs

Proof of the estimate with pointwise term in time 0

Since \( z(0) = 0 \) in \( \Omega \), we can extend the function \( z \) by \( z(t) = z(-t) \) for \( t \in (-T, 0) \) and apply the Carleman estimate to this extended function \( z \). Of course, since each term is odd or even, the integrals on \(( -T, T )\) simply are twice the integrals on \(( 0, T )\).
Since $z(0) = 0$ in $\Omega$, we can extend the function $z$ by $z(t) = z(-t)$ for $t \in (-T, 0)$ and apply the Carleman estimate to this extended function $z$. Of course, since each term is odd or even, the integrals on $(-T, T)$ simply are twice the integrals on $(0, T)$.

The Carleman estimate for the operator $\Box + p$ with $p \in L^\infty(\Omega)$ is a direct consequence noticing that in $\Omega \times (0, T)$,

$$|\Box z|^2 \leq 2|\Box z + pz|^2 + 2\|p\|_{L^\infty(\Omega)}^2 |z|^2 \leq 2|\Box z + pz|^2 + 2m^2 |z|^2.$$
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\]

Then choosing \( s \) large enough, one can absorb the term

\[
2Mm^2 \int_0^T \int_\Omega e^{2s\varphi}|z|^2 \, dx \, dt,
\]

by the left hand side.
Proofs

Proof of the estimate with pointwise term in time 0

- We set

\[ w = e^{s\varphi} \chi z \quad \text{and} \quad P_1w = \partial_t^2 w - \Delta w + s^2 w(\|\partial_t \varphi\|^2 - \|\nabla \varphi\|^2). \]
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Under the condition \( z(0) = 0 \) in \( \Omega \), we get \( w(0) = 0 \) in \( \Omega \). This allows us to do the following computations

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\int_{-T}^0 \int_\Omega P_1 w \partial_t w \, dx \, dt = \int_{-T}^0 \int_\Omega (\partial_t^2 w - \Delta w + s^2 w(\|\partial_t \varphi\|^2 - \|\nabla \varphi\|^2) \partial_t w \, dx \, dt \\
\geq \frac{1}{2} \int_\Omega |\partial_t w(0)|^2 \, dx - M s^2 \int_{-T}^0 \int_\Omega |w|^2 \, dx \, dt,
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\geq \frac{1}{2} \int_{\Omega} |\partial_t w(0)|^2 \, dx - Ms^2 \int_{-T}^{0} \int_{\Omega} |w|^2 \, dx \, dt,
\]

implying in particular, by Cauchy-Schwarz, that

\[
s^{1/2} \int_{\Omega} |\partial_t w(0)|^2 \, dx \leq \int_{-T}^{T} \int_{\Omega} |P_1 w|^2 \, dx \, dt + s \int_{-T}^{T} \int_{\Omega} |\partial_t w|^2 \, dx \, dt \\
+ Ms^{5/2} \int_{-T}^{T} \int_{\Omega} |w|^2 \, dx \, dt.
\]
Proofs

Proof of the estimate with pointwise term in time 0

We can use the Carleman estimate on $w$ and, bounding each term from above and from below, we get:

$$s^{1/2} \int_{\Omega} |\partial_t w(0)|^2 \, dx + s \int_{-T}^{T} \int_{\Omega} (|\partial_t w|^2 + |\nabla w|^2) \, dx \, dt + s^3 \int_{-T}^{T} \int_{\Omega} |w|^2 \, dx \, dt$$

$$+ \int_{-T}^{T} \int_{\Omega} |P_1 w|^2 \, dx \, dt \leq M \int_{-T}^{T} \int_{\Omega} |Pw|^2 \, dx \, dt + Ms \int_{-T}^{T} \int_{\Gamma_0} |\partial_\nu w|^2 \, d\gamma \, dt.$$
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Coming back to the initial variable $z$, we obtain

$$s^{1/2} \int_{\Omega} e^{2s\varphi(0)} |\partial_t z(0)|^2 \, dx$$

$$\leq M \int_{-T}^{T} \int_{\Omega} e^{2s\varphi} |\Box z|^2 \, dxdt + Ms \int_{-T}^{T} \int_{\Gamma_0} e^{2s\varphi} |\partial_\nu z|^2 \, d\gamma dt.$$