Convergence of monotone operators with respect to measures

Mohamed Mamchaoui

Department of Mathematics, Faculty of Sciences
University Abou Bakr Belkaïd Tlemcen, Algeria
m_mamchaoui@mail.univ-tlemcen.dz
Table of Contents

1 References

2 Homogenization of Monotone Operators
   • Setting of the Homogenization Problem
   • The Homogenized Problem

3 Convergence in the variable $L^p$

4 Our Goal

5 Monotonicity and convergence
   • First Case :
   • Second Case :

6 Diagram

7 Conclusion


Let $\Omega$ be a bounded open set of $\mathbb{R}^n$ and $Y = ]0, 1[^n$ the unit cube. For a fixed $\epsilon > 0$, let us consider the nonlinear monotone operator:

$$A^\epsilon(u) := -\text{div}(a(\frac{x}{\epsilon}, Du)) , \forall u \in H^1_0(\Omega)$$

where $a(x, \cdot)$ is $Y$-periodic and satisfies the following properties:

1. $a : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that: $\forall \xi \in \mathbb{R}^n$, $a(\cdot, \xi)$ is Lebesgue measurable and $Y$-periodic.

2. $(a(x, \xi_1) - a(x, \xi_2), \xi_1 - \xi_2) \geq \alpha |\xi_1 - \xi_2|^2$ a.e $x \in \mathbb{R}^n$ and $\forall \xi_1, \xi_2 \in \mathbb{R}^n$.

3. $|a(x, \xi_1) - a(x, \xi_2)| \leq \beta |\xi_1 - \xi_2|$ a.e $x \in \mathbb{R}^n$ and $\forall \xi_1, \xi_2 \in \mathbb{R}^n$.

4. $a(x, 0) = 0$ a.e $x \in \mathbb{R}^n$.

If $a \in N^\#_4$, then for all real positive $\epsilon$ and $f_\epsilon \in L^2(\Omega)$, we consider the following problem:

$$\begin{cases}
-\text{div}(a(\frac{x}{\epsilon}, Du_\epsilon)) = f_\epsilon & \text{in } \Omega \\
u_\epsilon \in H^1_0(\Omega)
\end{cases} \quad (1)$$
The Homogenized Problem

**Theorem:**

Let \( a \in N_\# \) and let \((\epsilon_h)_h\) be a sequence of positive real numbers converging to 0. Assume that \( f_\epsilon \to f \) strongly in \( H^{-1}(\Omega) \).

Let \((u_\epsilon)_\epsilon\) be the solution to (1). Then,

\[
\begin{align*}
  u_\epsilon & \rightharpoonup u_* \text{ in } H^1_0(\Omega) \\
  a(\frac{x}{\epsilon}, Du_\epsilon) & \rightharpoonup b(Du_*) \text{ in } L^2(\Omega; \mathbb{R}^n)
\end{align*}
\]

where \( u_* \) is the unique solution to the homogenized problem:

\[
\begin{align*}
  \left\{ \begin{array}{l}
    -\text{div}(b(Du_*)) = f \quad \text{in } \Omega \\
    u_* \in H^1_0(\Omega)
  \end{array} \right.
\]

(2)

The operator \( b \) is defined by:

\[
b : \mathbb{R}^n \to \mathbb{R}^n
\]

\[
\xi \mapsto b(\xi) = \int_Y a(y, \xi + Dw^\xi(y)) dy
\]

(3)

where \( w^\xi \) is the unique solution to the local problem:

\[
\begin{align*}
  \left\{ \begin{array}{l}
    \int_Y (a(y, \xi + Dw^\xi(y)), Dv(y)) dy = 0 \quad \forall v \in H^1_\#(Y) \\
    w^\xi \in H^1_\#(Y)
  \end{array} \right.
\]

(4)
Proposition:

The homogenized operator $b$ has the following properties:

1. $b$ is strictly monotone

2. There exists a constant $\gamma = \frac{\beta^2}{\alpha} > 0$ such that:
\[ |b(\xi_1) - b(\xi_2)| \leq \gamma |\xi_1 - \xi_2| \quad \forall \xi_1, \xi_2 \in \mathbb{R}^n \]

3. $b(0) = 0$. 
Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^N$. It will always assumed that $p$ and $q$ are conjugate indexes, i.e. $1/p + 1/q = 1$ and that $1 < p < \infty$. Let $\mu_\delta$ and $\mu$ be Randon measures in $\Omega$ and $\mu_\delta \overset{\delta \rightarrow 0}{\rightarrow} \mu$. We say that a sequence $u_\delta$ is bounded in $L^p(\Omega, d\mu_\delta)$ if

$$\limsup_{\delta \rightarrow 0} \int_{\Omega} |u_\delta|^p \, d\mu_\delta < \infty.$$ 

**Definition**

A bounded sequence $u_\delta \in L^p(\Omega, d\mu_\delta)$ is weakly convergent to $u \in L^p(\Omega, d\mu)$, $u_\delta \rightharpoonup u$ if

$$\lim_{\delta \rightarrow 0} \int_{\Omega} u_\delta \varphi \, d\mu_\delta = \int_{\Omega} u \varphi \, d\mu.$$ 

(5)

for any test function $\varphi \in C_0^\infty(\Omega)$.

We have the following results:

- each bounded sequence has a weakly convergent subsequence.
- if the sequence $u_\delta$ weakly converges to $u$, then

$$\liminf_{\delta \rightarrow 0} \int_{\Omega} |u_\delta|^p \, d\mu_\delta \geq \int_{\Omega} |u|^p \, d\mu.$$
Definition

A bounded sequence $u_\delta \in L^p(\Omega, d\mu_\delta)$ is called strongly convergent to $u \in L^p(\Omega, d\mu)$, $u_\delta \rightharpoonup u$ if

$$\lim_{\delta \to 0} \int_\Omega u_\delta v_\delta d\mu_\delta = \int_\Omega uv d\mu, \quad \text{if } v_\delta \rightharpoonup v \text{ in } L^q(\Omega, d\mu_\delta).$$

We have

- strong convergence implies weak convergence.
- the weak convergence $u_\delta \rightharpoonup u$, together with the relation

$$\lim_{\delta \to 0} \int_\Omega |u_\delta|^p d\mu_\delta = \int_\Omega |u|^p d\mu,$$

is equivalent to the strong convergence $u_\delta \to u$. 
$a\left(\frac{x}{\varepsilon}, Du_{\varepsilon, \delta}\right) \xrightarrow{\delta \to 0} a\left(\frac{x}{\varepsilon}, z_{\varepsilon}\right)$
Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^N$, where the Lebesgue measure of the boundary is zero, $Y = [0, 1)^N$ the semi-open cube in $\mathbb{R}^N$. It will always assumed that $p$ and $q$ are conjugate indexes, i.e. $1/p + 1/q = 1$ and that $1 < p < \infty$. Let $\mu_\delta$ and $\mu$ be Randon measures in $\Omega$ and $\mu_\delta \to_\delta 0 \mu$.

Consider a family of sequence $(u_\delta)$ such that

$$\int_\Omega (|u_\delta|^p + |Du_\delta|^p) \, d\mu_\delta \leq c.$$ 

Let $a : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ be a function such that $a(., \xi)$ is $Y$-periodic and $\mu_\delta$-measurable for every $\xi \in \mathbb{R}^N$. Moreover, assume that there exists constants $c_1, c_2 > 0$ and two more constants $\alpha$ and $\beta$, with $0 \leq \alpha \leq \min \{1, p - 1\}$ and $\max \{p, 2\} \leq \beta < \infty$ such that $a$ satisfies the following continuity and monotonicity assumptions:

$$a(y, 0) = 0 \quad (6)$$

$$|a(y, \xi_1) - a(y, \xi_2)| \leq c_1(1 + |\xi_1| + |\xi_2|)^{p-1-\alpha} |\xi_1 - \xi_2|^\alpha \quad (7)$$

$$(a(y, \xi_1) - a(y, \xi_2), \xi_1 - \xi_2) \geq c_2(1 + |\xi_1| + |\xi_2|)^{p-\beta} |\xi_1 - \xi_2|^\beta, \quad (8)$$

for $\mu_\delta$ a.e. $y \in \mathbb{R}^N$ and any $\xi_1, \xi_2 \in \mathbb{R}^N$.

We will treat two cases:

$$a(x, Du_\delta) \quad \text{and} \quad a_\delta(x, Du_\delta).$$

We Suppose that

$$a(x, \varphi) \in (C_0^\infty(\Omega))^N \quad \text{for all} \ \varphi \in (C_0^\infty(\Omega))^N.$$ 

As $\delta$ is small enough, we obtain the following results:
Lemma

There exist a subsequence of \( u_\delta \) still denoted by \( u_\delta \) and \( \bar{a} \in (L^q(\Omega, d\mu))^N \) such that \( a(x, Du_\delta) \xrightarrow{\delta \to 0} \bar{a} \).

Proof.

\[
|a(x, Du_\delta)| \leq c_1 (1 + |Du_\delta|)^{p-1-\alpha} |Du_\delta|^{\alpha} \leq c_1 (1 + |Du_\delta|)^{p-1} \leq 2^{p-1} c_1 (1 + |Du_\delta|^{p-1})
\]

which gives that \( a(x, Du_\delta) \) is bounded in \((L^q(\Omega, d\mu_\delta))^N \) since \((Du_\delta)\) is bounded in \((L^p(\Omega, d\mu_\delta))^N\). By proposition 3 there exist a subsequence (still denoted by \( \delta \)) such that: there exists a \( \bar{a} \in (L^q(\Omega, d\mu))^N \) such that \( a(x, Du_\delta) \xrightarrow{\delta \to 0} \bar{a} \) i.e :

\[
\lim_{\delta \to 0} \int_{\Omega} (a(x, Du_\delta), \varphi) d\mu_\delta = \int_{\Omega} (\bar{a}, \varphi) d\mu
\]

for all \( \varphi \in (C^\infty_0(\Omega))^N \). \( \square \)

Lemma

Assume that

\[
\int_{\Omega} (\bar{a} - a(x, \varphi), z - \varphi) d\mu \geq 0, \forall \varphi \in (C^\infty_0(\Omega))^N
\]

for some \( \bar{a} \) and \( z \) belong to \((L^q(\Omega, d\mu))^N \) and \((L^p(\Omega, d\mu))^N \) respectively. Then, the inequality holds for every \( \varphi \in (L^p(\Omega, d\mu))^N \).
Theorem 1:

Assume that $a$ satisfies the conditions (6)-(7)-(8). Let $(Du_\delta)$ be a bounded sequence in $(L^p(\Omega, d\mu_\delta))^N$ which converges weakly to $z \in (L^p(\Omega, d\mu))^N$ and assume that $a(x, Du_\delta)$ converges weakly to $\bar{a} \in (L^q(\Omega, d\mu))^N$. Then

$$\liminf_{\delta \to 0} \int_{\Omega} (a(x, Du_\delta), Du_\delta) d\mu_\delta \geq \int_{\Omega} (\bar{a}, z) d\mu.$$

If equality holds, i.e:

$$\liminf_{\delta \to 0} \int_{\Omega} (a(x, Du_\delta), Du_\delta) d\mu_\delta = \int_{\Omega} (\bar{a}, z) d\mu,$$

then

$$\bar{a} = a(x, z) \text{ for } \mu \text{ a.e } x \in \Omega.$$
Assume the contradiction, i.e.

$$\liminf_{\delta \to 0} \int_{\Omega} (a(x, Du_\delta), Du_\delta) d\mu_\delta < \int_{\Omega} (\bar{a}, z) d\mu.$$ 

Then there exists a positive constant $C > 0$ such that

$$\liminf_{\delta \to 0} \int_{\Omega} (a(x, Du_\delta), Du_\delta) d\mu_\delta = \int_{\Omega} (\bar{a}, z) d\mu - C. \quad (9)$$

By monotonicity we have that

$$\int_{\Omega} (a(x, Du_\delta) - a(x, \varphi), Du_\delta - \varphi) d\mu_\delta \geq 0, \forall \varphi \in (C_0^\infty(\Omega))^N.$$

Using (9) on the corresponding term and passing to the limit when $\delta \to 0$ in this inequality. Its left-hand-side contains four terms. We have

$$\int_{\Omega} (\bar{a} - a(x, \varphi), z - \varphi) d\mu \geq C, \forall \varphi \in (C_0^\infty(\Omega))^N.$$

By density and continuity (see lemma 4) this inequality holds also for any $\varphi \in (L^p(\Omega, d\mu))^N$. Let $t \phi = z - \varphi$, with $t > 0$. Then after dividing by $t$ we get

$$\int_{\Omega} (\bar{a} - a(x, z - t\phi), \phi) d\mu \geq \frac{C}{t}.$$

Letting $t \to 0$ gives

$$\int_{\Omega} (\bar{a} - a(x, z), \phi) d\mu \geq \infty.$$

Repeating the procedure for $t < 0$ implies that

$$\int_{\Omega} (\bar{a} - a(x, z), \phi) d\mu \leq -\infty.$$

We have clearly reached a contradiction.
If equality holds we can repeat the procedure above:

\[
\int_\Omega (a(x, Du_\delta) - a(x, \varphi), Du_\delta - \varphi) d\mu_\delta \geq 0, 
\]

for every \( \varphi \in (C_0^\infty(\Omega))^N \). Let us pass to the limit in this inequality. Its left-hand side contains four terms. We have:

\[
\lim_{\delta \to 0} \int_\Omega (a(x, Du_\delta), \varphi) d\mu_\delta = \int_\Omega (\bar{a}, \varphi) d\mu.
\]

\[
\lim_{\delta \to 0} \int_\Omega (a(x, \varphi), \varphi) d\mu_\delta = \int_\Omega (a(x, \varphi), \varphi) d\mu.
\]

\[
\lim_{\delta \to 0} \int_\Omega (a(x, \varphi), Du_\delta) d\mu_\delta = \int_\Omega (a(x, \varphi), z) d\mu.
\]

\[
\liminf_{\delta \to 0} \int_\Omega (a(x, Du_\delta), Du_\delta) d\mu_\delta = \int_\Omega (\bar{a}, z) d\mu.
\]

As a result, we obtain the inequality:

\[
\int_\Omega (\bar{a} - a(x, \varphi), z - \varphi) d\mu \geq 0, \quad \forall \varphi \in (C_0^\infty(\Omega))^N.
\]

By density and continuity this inequality holds for any \( \varphi \in (L^p(\Omega, d\mu))^N \). Put \( \varphi = z - t\phi, \forall \phi \in (L^p(\Omega, d\mu))^N \).
For $t > 0$ we get that
\[
\int_{\Omega} (\bar{a} - a(x, z - t\phi), \phi) d\mu \geq 0.
\]
(10)

For $t < 0$ we obtain that
\[
\int_{\Omega} (\bar{a} - a(x, z - t\phi), \phi) d\mu \leq 0.
\]
(11)

By taking (10) and (11) into account and letting $t$ tend to 0 we find that
\[
\int_{\Omega} (\bar{a} - a(x, z), \phi) d\mu = 0, \forall \phi \in (L^P(\Omega, d\mu))^N.
\]

This implies that $\bar{a} = a(x, z)$ for $\mu$ a.e. $x \in \Omega$. This ends the proof.
Theorem 2:

Assume that $a$ satisfies the conditions (6)-(7)-(8). Let $(Du_\delta)$ be a bounded sequence in $(L^p(\Omega, d\mu))^N$ which converges weakly to $z \in (L^p(\Omega, d\mu))^N$, $a(x, Du_\delta)$ converges weakly to $\bar{a} \in (L^q(\Omega, d\mu))^N$, and $|Du_\delta|^{p-2} Du_\delta$ converges weakly to $v \in (L^q(\Omega, d\mu))^N$. If

$$\lim_{\delta \to 0} \int_\Omega (a(x, Du_\delta), Du_\delta) d\mu_\delta = \int_\Omega (\bar{a}, z) d\mu,$$  \quad (12)

then the sequence $Du_\delta$ is strongly convergent to $z$.

For $k > 0$, we have:

$$\lim_{\delta \to 0} \int_\Omega (a(x, Du_\delta), Du_\delta) d\mu_\delta = \limsup_{\delta \to 0} \int_\Omega k |Du_\delta|^p d\mu_\delta - \limsup_{\delta \to 0} \int_\Omega k |Du_\delta|^p d\mu_\delta$$

$$+ \lim_{\delta \to 0} \int_\Omega (a(x, Du_\delta), Du_\delta) d\mu_\delta$$

$$= \limsup_{\delta \to 0} \int_\Omega k |Du_\delta|^p d\mu_\delta + \liminf_{\delta \to 0} \int_\Omega -k |Du_\delta|^p d\mu_\delta$$

$$+ \lim_{\delta \to 0} \int_\Omega (a(x, Du_\delta), Du_\delta) d\mu_\delta.$$

This together with (12) give

$$\limsup_{\delta \to 0} \int_\Omega k |Du_\delta|^p d\mu_\delta + \liminf_{\delta \to 0} \int_\Omega (a(x, Du_\delta) - kDu_\delta |Du_\delta|^{p-2}, Du_\delta) d\mu_\delta$$

$$= \int_\Omega (kv, z) d\mu + \int_\Omega (\bar{a} - kv, z) d\mu.$$
Monotonicity and convergence

First Case:

For \( k > 0 \) sufficiently small the function \( a(x, Du) - kDu |Du|^{p-2} \) satisfies the conditions in Theorem 1 therefore

\[
\liminf_{\delta \to 0} \int_{\Omega} (a(x, Du) - kDu |Du|^{p-2}, Du) d\mu_\delta \geq \int_{\Omega} (\bar{a} - kv, z) d\mu. \tag{13}
\]

This and (12) imply

\[
\limsup_{\delta \to 0} \int_{\Omega} |Du_\delta|^p d\mu_\delta \leq \int_{\Omega} (v, z) d\mu. \tag{14}
\]

The function \( Du_\delta \) satisfies the conditions in Theorem 1 which implies that

\[
\liminf_{\delta \to 0} \int_{\Omega} |Du_\delta|^p d\mu_\delta = \liminf_{\delta \to 0} \int_{\Omega} (Du_\delta |Du_\delta|^{p-2}, Du_\delta) d\mu_\delta \geq \int_{\Omega} (v, z) d\mu. \tag{15}
\]

By (14) and (15) it follows that

\[
\lim_{\delta \to 0} \int_{\Omega} |Du_\delta|^p d\mu_\delta = \lim_{\delta \to 0} \int_{\Omega} (Du_\delta |Du_\delta|^{p-2}, Du_\delta) d\mu_\delta = \int_{\Omega} (v, z) d\mu
\]

Hence it follows by Theorem 1 that \( v = |z|^{p-2} z \). Then, \( Du_\delta \) converges strongly to \( z \).
Theorem 3:

Assume that \( a \) satisfies the conditions (6)-(7)-(8). Let \( (Du_{\delta}) \) be a bounded sequence in \( (L^p(\Omega, d\mu_{\delta}))^N \) which converges strongly to \( z \in (L^p(\Omega, d\mu))^N \) and assume that \( a(x, Du_{\delta}) \) converges weakly to \( \bar{a} \in (L^q(\Omega, d\mu))^N \). Then

\[
\bar{a} = a(x, z) \text{ for } \mu \text{ a.e } x \in \Omega.
\]

Proof.

In the same spirit of the proof of Theorem 1 we show that

\[
\bar{a} = a(x, z) \text{ for } \mu \text{ a.e } x \in \Omega.
\]
Lemma

There exists a subsequence and $a_0 \in (L^q(\Omega, d\mu))^N$ such that $a_\delta(x, Du) \xrightarrow{\delta \to 0} a_0$.

Proof.

$$|a_\delta(x, Du)| \leq c_1(1 + |Du|)^{p-1-\alpha} |Du|^\alpha \leq c_1(1 + |Du|)^{p-1} \leq 2^{p-1}c_1(1 + |Du|^{p-1})$$

which gives that $a_\delta(x, Du)$ is bounded in $(L^q(\Omega, d\mu_\delta))^N$ since $(Du)$ is bounded in $(L^p(\Omega, d\mu_\delta))^N$. By proposition 3 there exist a subsequence (still denoted by $\delta$) such that: There exist a $a_0 \in (L^q(\Omega, d\mu))^N$ such that $a_\delta(x, Du) \xrightarrow{\delta \to 0} a_0$ i.e:

$$\lim_{\delta \to 0} \int_{\Omega} (a_\delta(x, Du), \varphi) d\mu_\delta = \int_{\Omega} (a_0, \varphi) d\mu$$

for all $\varphi \in (C_0^\infty(\Omega))^N$. 

\qed
Theorem 4:
Assume that \( a_\delta \) satisfies the conditions (6)-(7)-(8) and converges weakly to \( a_0 \) in sens (5). Then the limite \( a_0 \) also satisfies the conditions (6)-(7)-(8).

For all \( \varphi \in (C_0^\infty (\Omega))^N \):

\[
0 = \lim_{\delta \to 0} \int_\Omega (a_\delta (x, 0), \varphi) d\mu_\delta = \int_\Omega (a_0 (x, 0), \varphi) d\mu.
\]

For \( \varphi \in C_0^\infty (\Omega) \) and for all \( \xi_1, \xi_2 \in \mathbb{R}^n \), we have:

\[
\left| \int_\Omega (a_\delta (x, \xi_1) - a_\delta (x, \xi_2), \varphi(\xi_1 - \xi_2)) d\mu_\delta \right| \leq \int_\Omega |(a_\delta (x, \xi_1) - a_\delta (x, \xi_2)| |\varphi(\xi_1 - \xi_2)| d\mu_\delta
\]

\[
\leq c_1 \int_\Omega (1 + |\xi_1| + |\xi_2|)^{p-1-\alpha} |\xi_1 - \xi_2|^\alpha |\varphi(\xi_1 - \xi_2)| d\mu_\delta
\]

Passing here to the limit we obtain the inequality

\[
\left| \int_\Omega (a_0 (x, \xi_1) - a_0 (x, \xi_2), \varphi(\xi_1 - \xi_2)) d\mu \right| \leq c_1 \int_\Omega (1 + |\xi_1| + |\xi_2|)^{p-1-\alpha} |\xi_1 - \xi_2|^\alpha |\varphi(\xi_1 - \xi_2)| d\mu
\]
which shows that
\[ \left| (a_0(x, \xi_1) - a_0(x, \xi_2) \right| \leq c_1 (1 + |\xi_1| + |\xi_2|)^{p-1-\alpha} |\xi_1 - \xi_2|^{\alpha}, \]
for \( \mu \) a.e. \( x \in \Omega \) and any \( \xi_1, \xi_2 \in \mathbb{R}^N \).

Finally, we have for all \( \varphi \in C_0^\infty(\Omega), \varphi \geq 0 \):
\[
\int_\Omega (a_\delta(x, \xi_1) - a_\delta(x, \xi_2), \varphi(\xi_1 - \xi_2)) d\mu_\delta = \int_\Omega (a_\delta(x, \xi_1) - a_\delta(x, \xi_2), (\xi_1 - \xi_2)) \varphi d\mu_\delta \\
\geq c_2 \int_\Omega (1 + |\xi_1| + |\xi_2|)^{p-\beta} |\xi_1 - \xi_2|^\beta \varphi d\mu_\delta
\]
Passing here to the limit we obtain inequality
\[
\int_\Omega (a_0(x, \xi_1) - a_0(x, \xi_2), (\xi_1 - \xi_2)) \varphi d\mu \geq c_2 \int_\Omega (1 + |\xi_1| + |\xi_2|)^{p-\beta} |\xi_1 - \xi_2|^\beta \varphi d\mu
\]
for every \( \varphi \in C_0^\infty(\Omega), \varphi \geq 0 \), which complete the proof.
Lemma

There exists a subsequence and \( a_1 \in (L^q(\Omega, \, d\mu))^N \) such that \( a_\delta(x, Du_\delta) \xrightarrow{\delta \to 0} a_1 \).

Proof.

\[
|a_\delta(x, Du_\delta)| \leq c_1(1 + |Du_\delta|)^{p-1-\alpha} |Du_\delta|^{\alpha} \leq c_1(1 + |Du_\delta|)^{p-1} \leq 2^{p-1} c_1(1 + |Du_\delta|^{p-1})
\]

which gives that \( a_\delta(x, Du_\delta) \) is bounded in \((L^q(\Omega, \, d\mu_\delta))^N \) since \((Du_\delta)\) is bounded in \((L^p(\Omega, \, d\mu_\delta))^N \). By proposition 3 there exists a subsequence (still denoted by \( \delta \)) such that: There exist a \( a_1 \in (L^q(\Omega, \, d\mu))^N \) such that \( a_\delta(x, Du_\delta) \xrightarrow{\delta \to 0} a_1 \) i.e:

\[
\lim_{\delta \to 0} \int_\Omega (a_\delta(x, Du_\delta), \varphi) \, d\mu_\delta = \int_\Omega (a_1, \varphi) \, d\mu
\]

for all \( \varphi \in (C_0^\infty(\Omega))^N \). \( \square \)
Theorem 5:

Assume that $a_\delta$ satisfies the conditions (6)-(7)-(8). Let $(Du_\delta)$ be a bounded sequence in $(L^p(\Omega, d\mu_\delta))^N$ which converges strongly to $z \in (L^p(\Omega, d\mu))^N$ and assume that $a_\delta(x, Du_\delta)$ converges weakly to $a_1 \in (L^q(\Omega, d\mu))^N$. Then,

$$ a_1 = a_0(x, z) $$

where $a_0$ is the limit of $a_\delta$ in sens (5).

By the monotonicity of $a_\delta$ we have

$$ \int_\Omega (a_\delta(x, Du_\delta) - a_\delta(x, \varphi), Du_\delta - \varphi)d\mu_\delta \geq 0, $$

for every $\varphi \in (C^\infty_0(\Omega))^N$. Let us pass to the limit in this inequality. Its left-hand side contains four terms. We have:

$$ \lim_{\delta \to 0} \int_\Omega (a_\delta(x, Du_\delta), \varphi)d\mu_\delta = \int_\Omega (a_1, \varphi)d\mu, $$

$$ \lim_{\delta \to 0} \int_\Omega (a_\delta(x, \varphi), \varphi)d\mu_\delta = \int_\Omega (a_0(x, \varphi), \varphi)d\mu, $$

$$ \lim_{\delta \to 0} \int_\Omega (a_\delta(x, \varphi), Du_\delta)d\mu_\delta = \int_\Omega (a_0(x, \varphi), z)d\mu, $$

and

$$ \lim_{\delta \to 0} \int_\Omega (a_\delta(x, Du_\delta), Du_\delta)d\mu_\delta = \int_\Omega (a_1, z)d\mu. $$
As a result, we obtain the inequality
\[
\int_{\Omega} (a_1 - a_0(x, \varphi), z - \varphi) d\mu \geq 0, \quad \forall \varphi \in (C_0^\infty(\Omega))^N.
\]

By density and continuity this inequality holds for any \( \varphi \in (L^p(\Omega, d\mu))^N \). Put \( \varphi = z - t\phi \), \( \forall \phi \in (L^p(\Omega, d\mu))^N \). For \( t > 0 \) we get that
\[
\int_{\Omega} (a_1 - a_0(x, z - t\phi), \phi) d\mu \geq 0. \tag{16}
\]

For \( t < 0 \) we obtain that
\[
\int_{\Omega} (a_1 - a_0(x, z - t\phi), \phi) d\mu \leq 0 \tag{17}
\]

By taking (16) and (17) into account and letting \( t \) tend to 0 we find that
\[
\int_{\Omega} (a_1 - a_0(x, z), \phi) d\mu = 0, \quad \forall \phi \in (L^p(\Omega, d\mu))^N .
\]

Which implies that \( a_1 = a_0(x, z) \) for \( \mu \) a.e. \( x \in \Omega \). The proof is complete.
Example

\( N = p = 2 \). Let

\[ I = \{ x = (x_1, x_2) / a \leq x_1 \leq b ; \ x_2 = 0 \} \]

be a segment in \( \mathbb{R}^2 \), and suppose that a bounded domain \( \Omega \) contains \( I \). For any sufficiently small \( \delta > 0 \) consider the bar

\[ I_\delta := \{ x = (x_1, x_2) / a < x_1 < b ; \ -\delta < x_2 < \delta \} \subset \Omega. \]

Denote by \( \mu_\delta \) the measure in \( \Omega \), concentrated and uniformly distributed on \( I_\delta \):

\[ \mu_\delta (dx) = \frac{\chi_{I_\delta}(x)}{2\delta(b - a)} \, dx_1 \, dx_2. \]

The family \( \mu_\delta \) converges weakly, as \( \delta \to 0 \), to a measure

\[ \mu (dx) = \frac{1}{(b - a)} \, dx_1 \times \delta(x_2) \]

where \( \delta(z) \) stands for the Dirac mass at zero.

Consider the operator \( a_\delta \) with the following form:

\[ a_\delta(x, Du_\delta) = \left( \frac{1 + \delta|x|}{2 + \delta|x|} \right) Du_\delta. \]

We can prove that \( a_\delta \) satisfies suitable assumptions of uniform strict monotonicity and uniform Lipschitz-continuity and \( a_\delta(x, 0) = 0 \) for a.e. \( x \in \mathbb{R}^2 \).
Example

Suppose that $Du_\delta$ is strongly convergent to $z$.

$$
limit_{\delta \to 0} \int_{\Omega} (a_\delta(x, \varphi), \varphi) d\mu_\delta = \lim_{\delta \to 0} \int_{\Omega} ((\frac{1 + \delta |x|}{2 + \delta |x|}) \varphi, \varphi) d\mu_\delta
$$

$$= \int_{\Omega} (\frac{1}{2} \varphi, \varphi) d\mu, \forall \varphi \in (C^\infty_0(\Omega))^2.$$

This implies that

$$a_0(x, \xi) = \frac{1}{2} \xi$$

and

$$|a_0(x, \xi_1) - a_0(x, \xi_2)| = \frac{1}{2} |\xi_1 - \xi_2|,$$

for a.e. $x \in \mathbb{R}^2$ and for every $\xi_1, \xi_2 \in \mathbb{R}^2$.

Finally, the hypothesis of Theorem 5 are satisfied then

$$\lim_{\delta \to 0} \int_{\Omega} (a_\delta(x, Du_\delta), \varphi) d\mu_\delta = \int_{\Omega} (a_1, \varphi) d\mu$$

$$= \int_{\Omega} (a_0(x, z), \varphi) d\mu$$

$$= \int_{\Omega} (\frac{1}{2} z, \varphi) d\mu, \forall \varphi \in (C^\infty_0(\Omega))^2.$$

Mohamed Mamchaoui
Department of Mathematics, Faculty of Sciences
University Abou Bakr Belkaid Tlemcen, Algeria
mamchaoui@mail.univ-tlemcen.dz

Basque Center for Applied Mathematics, Bilba
October 11, 2014 26 / 31
Theorem 6:

Assume that $a$ satisfies the conditions (6)-(7)-(8). Let $(Du_{\epsilon, \delta})$ be a bounded sequence in $(L^p(\Omega, d\mu_{\epsilon, \delta}))^N$ which converges weakly to $z_{\epsilon} \in (L^p(\Omega, d\mu))^N$ and assume that $a(\frac{x}{\epsilon}, Du_{\epsilon, \delta})$ converges weakly to $\bar{a}_{\epsilon} \in (L^q(\Omega, d\mu_{\epsilon}))^N$. If

$$\liminf_{\delta \to 0} \int_{\Omega} (a(\frac{x}{\epsilon}, Du_{\epsilon, \delta}), Du_{\epsilon, \delta}) d\mu_{\epsilon, \delta} = \int_{\Omega} (\bar{a}_{\epsilon}, z_{\epsilon}) d\mu_{\epsilon} ,$$

then the diagram

$$
\begin{align*}
a(\frac{x}{\epsilon}, Du_{\epsilon, \delta}) &\quad \delta \to 0 \quad a(\frac{x}{\epsilon}, Du_{\epsilon}) \\
\epsilon &\quad \downarrow \quad \downarrow \\
b(Du_0, \delta) &\quad \delta \to 0 \quad b(Du_0)
\end{align*}
$$

is commutative.
Proof.

From Theorem 1 it follows that

$$\bar{a}_\epsilon = a(\frac{x}{\epsilon}, z_\epsilon) \text{ for } \mu \text{ a.e } x \in \Omega.$$ 

We have for all $\varphi \in (C_0^\infty(\Omega))^N$

$$\lim_{\delta \to 0} \left( \lim_{\epsilon \to 0} \int_{\Omega} (a(\frac{x}{\epsilon}, Du_\epsilon, \delta), \varphi) d\mu_{\epsilon, \delta} \right) = \lim_{\delta \to 0} \int_{\Omega} (b(Du_0, \delta), \varphi) dx.$$

Since $z_\epsilon = Du_\epsilon$, we obtain:

$$\lim_{\epsilon \to 0} \left( \lim_{\delta \to 0} \int_{\Omega} (a(\frac{x}{\epsilon}, Du_\epsilon, \delta), \varphi) d\mu_{\epsilon, \delta} \right) = \lim_{\epsilon \to 0} \int_{\Omega} (\bar{a}_\epsilon, \varphi) d\mu_{\epsilon}.$$

$$= \lim_{\epsilon \to 0} \int_{\Omega} (a(\frac{x}{\epsilon}, Du_\epsilon), \varphi) d\mu_{\epsilon}.$$

$$= \int_{\Omega} (b(Du_0), \varphi) dx.$$
Theorem 7:

Assume that $a_\delta$ satisfies the conditions (6)-(7)-(8). Let $(Du_\epsilon, \delta)$ be a bounded sequence in $(L^p(\Omega, d\mu_{\epsilon, \delta}))^N$ which converges strongly to $z_\epsilon \in (L^p(\Omega, d\mu_{\epsilon}))^N$ and assume that $a_\delta(\frac{x}{\epsilon}, Du_\epsilon, \delta)$ converges weakly to $a_1^\epsilon \in (L^q(\Omega, d\mu_{\epsilon}))^N$. Then the diagram

\[
\begin{array}{ccc}
\overset{\delta \to 0}{a_\delta(\frac{x}{\epsilon}, Du_\epsilon, \delta)} & \overset{\epsilon \to 0}{\longrightarrow} & \overset{\delta \to 0}{a_0(\frac{x}{\epsilon}, Du_\epsilon)} \\
\overset{\epsilon}{\downarrow} & & \overset{\epsilon}{\downarrow} \\
0 & \overset{\delta \to 0}{\longrightarrow} & 0 \ \\
\end{array}
\]

is commutative.
Proof.

From Theorem 5 it follows that

\[ a_1^\epsilon = a_0(\frac{x}{\epsilon}, z_\epsilon) \text{ for } \mu \text{ a.e } x \in \Omega. \]

We have for all \( \varphi \in \left( C_0^\infty(\Omega) \right)^N \)

\[
\lim_{\delta \to 0} \left( \lim_{\epsilon \to 0} \int_{\Omega} \left( a_\delta(\frac{x}{\epsilon}, Du_\epsilon, \delta), \varphi \right) d\mu_{\epsilon, \delta} \right) = \lim_{\delta \to 0} \int_{\Omega} \left( b(Du_0, \delta), \varphi \right) dx.
\]

Since \( z_\epsilon = Du_\epsilon \), we obtain:

\[
\lim_{\epsilon \to 0} \left( \lim_{\delta \to 0} \int_{\Omega} \left( a_\delta(\frac{x}{\epsilon}, Du_\epsilon, \delta), \varphi \right) d\mu_{\epsilon, \delta} \right) = \lim_{\epsilon \to 0} \int_{\Omega} \left( a_1^\epsilon, \varphi \right) d\mu_{\epsilon}.
\]

\[
= \lim_{\epsilon \to 0} \int_{\Omega} \left( a_0(\frac{x}{\epsilon}, Du_\epsilon), \varphi \right) d\mu_{\epsilon}.
\]

\[
= \int_{\Omega} \left( b(Du_0), \varphi \right) dx.
\]
\[
\begin{align*}
a(\frac{x}{\varepsilon}, Du_{\varepsilon, \delta}) & \xrightarrow{\delta \to 0} a(\frac{x}{\varepsilon}, z_{\varepsilon}) \\
b(Du^*, \delta) & \xrightarrow{\delta \to 0} b(z^*)
\end{align*}
\]