A logistic problem in the two-dimensional sphere

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A facility location problem
A modern classical subject on Optimization

Choose $N$ points in the 2 dimensional sphere $\mathbb{S}$ in such a way that the average temperature in $\mathbb{S}$ is the greatest possible.

Here, the temperature $u = u(x, t)$ is assumed to satisfy the standard diffusion equation $u_t = \Delta u - \lambda$, where $\lambda$ is a cooling rate, constant in $\mathbb{S}$, and the heat sources are assumed to be “infinite”.
Recall that $\mathbb{S}$ is the Riemann sphere, that is the sphere of radius $1/2$ centered at $(0, 0, 1/2)$. Let

$$F_q : \mathbb{S} \setminus \{q\} \rightarrow \mathbb{R}$$

$$p \mapsto \log \|p - q\|^{-1}$$
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$$F_q : \mathbb{S} \setminus \{q\} \rightarrow \mathbb{R}$$

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The (Riemannian) Laplacian of this function is constant:

$$\Delta F_q(p) = 2 \quad \forall \, p \in \mathbb{S} \setminus \{q\}. $$
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After some assumptions on the rate of growth near the sources, the following stationary solution is found:

$$u(x) = u(x, t) = \frac{\lambda}{2N} \sum_{i=1}^{N} \log \|x_i - x\|^{-1} + u_0.$$
Points satisfying some “extremal” property

A frequent choice

Look for $N$ points $x_1, \ldots, x_N$ in the sphere $\mathbb{S}$ such that the logarithmic energy (aka logarithmic potential)

$$E(x_1, \ldots, x_N) = \log \prod_{i<j} \|x_i - x_j\|^{-1} = - \sum_{i<j} \log \|x_i - x_j\|$$

is minimized.
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is minimized.

A set of $N$ points in $S$ minimizing $E$ (i.e. maximizing the product of their mutual distances) is called a set of **Elliptic Fekete Points**.
Elliptic Fekete points
Early works by Fekete, Szegö, Whyte, Hille, Tsuji, etc.

For $X = (x_1, \ldots, x_N) \in \mathbb{S}^N$ where $x_i \in \mathbb{S}$, $1 \leq i \leq N$, elliptic Fekete points minimize the logarithmic energy

$$E(X) = E(x_1, \ldots, x_N) = \log \prod_{i<j} \|x_i - x_j\|^{-1} = -\sum_{i<j} \log \|x_i - x_j\|$$

Equivalently, they maximize the product of their mutual distances.
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Equivalently, they maximize the product of their mutual distances. Let

$$m_N = \min\{E(X) : X \in \mathbb{S}^N\}.$$ 

Smale’s 7th problem: can one find $X \in \mathbb{S}^N$ such that

$$E(X) - m_N \leq c \log N?$$
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“Can one find” means...
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$$E(X) - m_N \leq c \log N?$$

“Can one find” means...can one describe a polynomial time algorithm (BSS model)?
The difficulty of Smale’s 7th problem.

1 The value of \( m_N \) is not known, even to \( O(N) \).

**Theorem (Elkies, Wagner, Rakhmanov–Saff–Zhou, Dubickas, Brauchart, Sandiers–Serfaty, Bétermin)**

For the radius 1/2 sphere, we have:

\[
m_N = \frac{N^2}{4} - \frac{N \ln(N)}{4} + C_N N,
\]

where

\[-0.4593423... \leq \lim_{N \to \infty} C_N \leq -0.40217...\]

**Conjecture [Brauchart, Hardin, Saff; Serfaty et al]:**

\[
\lim_{N \to \infty} C_N = 2 \log 2 + \frac{1}{2} \log \frac{1}{3} + 3 \log \frac{\sqrt{\pi}}{\Gamma(1/3)} = -0.40217...
\]
The difficulty of Smale’s 7th problem.

We need to solve a global minimization problem, not just a local minimization problem. Moreover, usual minimization algorithms will likely fall into “traps”: experiments find many local minima of $E$. 

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[RSZ] For each $N$, the number of local minimal values found

[BCEG] For $N=87$, distribution of the local minima for two different distributions of random $N$-tuples
Approximation to 1000 elliptic Fekete points by Bendito, Carmona, Encinas, Gesto, Gómez, Mouriño, Sánchez
We do know some things
Separation distance for the sphere of radius 1/2

• Theorem (Toth, Habicht– van der Waerden)
  For the Tammes problem (maximize separation distance)
  \[ d_{\text{sep}}(X_{\text{Tammes}}) \approx \frac{1.9046...}{\sqrt{N}}. \]

• Theorem (Rakhmanov–Saff–Zhou, Dubickas, Dragnev)
  For the elliptic Fekete points,
  \[ \frac{1}{\sqrt{N - 1}} \leq d_{\text{sep}}(X_{\text{Fekete}}) \leq \frac{1.9046...}{\sqrt{N}}. \]
We do know some things

Baricenter [Bergersen-Boal-Palffy Muhoray], [Dragnev-Legg-Townsend]. True for any critical point of \( \mathcal{E} \). [Brauchart] for the discrepancy, [Leopardi] for the comparison to \( s \)-energy.

Let \( x_1, \ldots, x_N \) be a set of elliptic Fekete points.

- The baricenter of \( x_1, \ldots, x_N \) is the center of the sphere.
- For each \( i \),
  \[
  \sum_{j \neq i} \frac{x_i - x_j}{\|x_i - x_j\|^2} = 2(N - 1)x_i, \quad \sum_{j \neq i} \|x_i - x_j\|^2 = \frac{N}{2}.
  \]
- Spherical cap discrepancy \( cN^{-1/4} \) (see later).
- Asymptotically optimal \( s \)-energy in relative error (see later).
We do know some things
Baricenter [Bergersen-Boal-Palffy Muhoray], [Dragnev-Legg-Townsend]. True for any critical point of $\mathcal{E}$. [Brauchart] for the discrepancy, [Leopardi] for the comparison to $s$–energy.

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- Spherical cap discrepancy $cN^{-1/4}$ (see later).
- Asymptotically optimal $s$–energy in relative error (see later).
Theorem (Main)

Let \( X = (x_1, \ldots, x_N) \in \mathbb{S}^N \) a configuration of distinct points and let \( \delta \in (0, 1) \) be such that \( d_R(x_i, x_j) \geq 2 \arcsin \sqrt{\delta/N} \quad \forall i \neq j \). Let

\[
B_i = \left\{ x \in \mathbb{S}: d_R(x, x_i) \leq \arcsin \sqrt{\delta/N} \right\}.
\]

Then

\[
\mathcal{E}(X) = C_1(N, \delta) + \frac{N}{2} \sum_{i=1}^{N} \int_{x \in \bigcup B_j} \log \|x - x_i\|^{-1} \, dx \quad (1)
\]

where

\[
C_1(N, \delta) = -\frac{N^2}{4} + \frac{N}{4} \log \frac{\delta}{N} - \frac{N(N-1)(N-\delta)}{4\delta} \log \left( 1 - \frac{\delta}{N} \right)
\]
Recall our facility location problem

Choose $N$ points in the 2 dimensional sphere $\mathbb{S}$ in such a way that the average temperature in the whole $\mathbb{S}$ is the greatest possible.

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After some assumptions on the rate of growth near the sources, the following stationary solution is found:

$$u(x) = u(x, t) = \frac{\lambda}{2N} \sum_{i=1}^{N} \log \|x_i - x\|^{-1} + u_0.$$
Corollary

The following problems are equivalent:

1. The $N$–tuple $X = (x_1, \ldots, x_N)$ is a set of elliptic Fekete points.

2. For any $\delta \in (0, 1)$ and $r = \arcsin \sqrt{\delta/N}$ such that $d_R(x_i, x_j) \geq 2r$ for $i \neq j$, heat sources located at the points $x_1, \ldots, x_N$ maximize the average temperature out of a safety radius $r$ around the sources.
An immediate consequence
Elliptic Fekete Points vs. Heat sources

Corollary

The following problems are equivalent:

- The \( N \)-tuple \( X = (x_1, \ldots, x_N) \) is a set of elliptic Fekete points.
- For any \( \delta \in (0, 1) \) and \( r = \arcsin \sqrt{\frac{\delta}{N}} \) such that \( d_R(x_i, x_j) \geq 2r \) for \( i \neq j \), heat sources located at the points \( x_1, \ldots, x_N \) maximize the average temperature out of a safety radius \( r \) around the sources.

Moreover... the total average temperature in \( \mathbb{S} \) is independent of the location of the sources.
Spherical cap discrepancy
Definition and theorems by Beck, Aistleitner, Brauchart, Dick, Gotz

\[ D_C(X) = \sup_{x \in \mathbb{S}, r \in [0, \pi/2]} \left| \frac{\#(i : x_i \in B(x, r))}{N} - \frac{\text{Area}(B(x, r))}{\pi} \right|. \]
Spherical cap discrepancy
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\[ D_C(X) = \sup_{x \in \mathbb{S}, r \in [0, \pi/2]} \left| \#(i : x_i \in B(x, r)) \frac{N}{\pi} - \frac{\text{Area}(B(x, r))}{\pi} \right|. \]

\[ cN^{-3/4} \leq \min_X D_C(X) \leq CN^{-3/4} \log N. \]
Spherical cap discrepancy
Definition and theorems by Beck, Aistleitner, Brauchart, Dick, Gotz

\[
D_C(X) = \sup_{x \in \mathbb{S}, r \in [0, \pi/2]} \left| \frac{\#(i : x_i \in B(x, r))}{N} - \frac{\text{Area}(B(x, r))}{\pi} \right|.
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\[cN^{-3/4} \leq \min_X D_C(X) \leq CN^{-3/4} \log N.\]

If \(X\) is a set of Elliptic Fekete points,

\[D_C(X) \leq O(N^{-1/4}).\]
Theorem

Let $X \in \mathbb{S}^N$, $N \geq 2$ and let $\delta \in (0, 1)$ be such that

$$d_R(x_i, x_j) \geq 2 \arcsin \sqrt{\frac{\delta}{N}} \text{ for } i \neq j.$$  

Then,

$$\mathcal{E}(X) \leq m_N + \frac{N^2}{4} D_C(X) \log \frac{N}{2\delta} + \frac{N \log(8\pi\delta)}{4}.$$
Spherical cap discrepancy
This follows from our main theorem

**Theorem**

Let $X \in \mathbb{S}^N$, $N \geq 2$ and let $\delta \in (0, 1)$ be such that $d_R(x_i, x_j) \geq 2 \arcsin \sqrt{\delta/N}$ for $i \neq j$. Then,

$$E(X) \leq m_N + \frac{N^2}{4} D_C(X) \log \frac{N}{2\delta} + \frac{N \log(8\pi \delta)}{4}.$$

Wagner proved a similar result for the case that we have the unit circle instead of $\mathbb{S}$. 
Spherical cap discrepancy
This follows from our main theorem

Theorem
Let \( X \in \mathbb{S}^N, N \geq 2 \) and let \( \delta \in (0, 1) \) be such that
\( d_R(x_i, x_j) \geq 2 \arcsin \sqrt{\delta/N} \) for \( i \neq j \). Then,

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\mathcal{E}(X) \leq m_N + \frac{N^2}{4} D_C(X) \log \frac{N}{2\delta} + \frac{N \log(8\pi \delta)}{4}.
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Wagner proved a similar result for the case that we have the unit circle instead of \( \mathbb{S} \).

This is somehow a reciprocal of the known result that Elliptic Fekete points have small discrepancy [Brauchart] and are well–separated [Rakhmanov–Saff–Zhou, Dubickas, Dragnev].
Theorem

Let $X \in \mathbb{S}^N$, $N \geq 2$ and let $\delta \in (0, 1)$ be such that $d_R(x_i, x_j) \geq 2 \arcsin \sqrt{\delta/N}$ for $i \neq j$. Then,

$$\mathcal{E}(X) \leq m_N + \frac{N^2}{4} D_C(X) \log \frac{N}{2\delta} + \frac{N \log(8\pi\delta)}{4}.$$ 

So... Small discrepancy plus not too small separation implies small energy.
Relation to other sets of points
A consequence of the last theorem

Corollary

Fix $s \in (0, 2)$. If $X_N$ minimizes the Riesz $s$–energy

$$\sum_{1 \leq i < j \leq N} \| (X_N)_i - (X_N)_j \|^{-s}$$

for $N \geq 2$, then $\lim_{N \to \infty} \mathcal{E}(X_N)/m_N = 1$.

This is a reciprocal to a result by P. Leopardi.
Relation to other sets of points
Schiffmann; Sloan, Womersley; Marzo, Ortega–Cerdá, Weymar

(non–elliptic) Fekete points: points $x_j$ which maximise $\det \phi_i(x_j)$ where $\phi_i$ form an o.n. basis of spherical harmonics of given degree.

Corollary

For every $L \geq 2$, let $X_L = (x_1, \ldots, x_{\pi L})$ be a set of (non–elliptic) Fekete points. Then

$$\lim_{L \rightarrow \infty} \frac{\mathcal{E}(X_L)}{\min(\mathcal{E})} = 1.$$
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I do not want to cheat you: this is **not** so good.
The critical set of $\mathcal{E}$
Laplacian and Maximum Principle

Recall: for $X = (x_1, \ldots, x_N) \in \mathbb{S}^N$, where $\mathbb{S}^N$ has the product Riemannian structure,

$$\mathcal{E}(x_1, \ldots, x_N) = \log \prod_{i < j} \|x_i - x_j\|^{-1} = -\sum_{i < j} \log \|x_i - x_j\|.$$ 

Then:

- $\Delta \mathcal{E} = 2N(N - 1)$. 

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Then:

- $\Delta \mathcal{E} = 2N(N - 1)$.
- Thus, there exist no local maxima of $\mathcal{E}$. 
A “Nash equilibrium” is a tuple $X = (x_1, \ldots, x_N)$ such that $E(X)$ cannot be improved if only one of the $x_i$ is moved.
The critical set of $\mathcal{E}$
A concept from game theory

A "Nash equilibrium" is a tuple $X = (x_1, \ldots, x_N)$ such that $\mathcal{E}(X)$ cannot be improved if only one of the $x_i$ is moved. This is also called a componentwise minimum.
The critical set of $\mathcal{E}$

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A corollary from our main result:

Corollary

Let $X$ be a Nash equilibrium of $\mathcal{E}$. Then,

$$\mathcal{E}(X) \leq \frac{N^2}{4}.$$  \hspace{1cm} (optimal in the first order term)
Comparison of the end–game starting at...
Random eigenvalues, random zeros of B-W polynomials, uniform points in $S$

The distribution of the values of $E$ at the end–points of the gradient flow seem not to depend on the distribution of the initial data.
The distribution of the values of $\mathcal{E}$ at the end–points of the gradient flow seem not to depend on the distribution of the initial data. **Which is clearly impossible.**
Other aspects of the problem
Locating good points in the sphere is studied in other contexts

- Packing and covering radius.
- One component plasma Coulomb gases at zero temperature.
- Quadrature formulas (spherical $N$–designs), spherical harmonics and interpolation.
- Applications to Biology and Crystallography.
- Implications in Number Theory and Arakelov Theory.

and many others.
Computability of elliptic Fekete points
There exists a simply exponential Turing machine for Smale’s 7th Problem

Theorem (B. 2013)

There exists a Turing machine which, on input $N \in \{2, 3, \ldots \}$, outputs $X = (x_1, \ldots, x_N) \in \mathbb{S}^N \cap \mathbb{Q}^{3N}$ satisfying

$$\mathcal{E}(X) \leq m_N + \frac{1}{18},$$

and such that the running time is simply exponential in $N$. More precisely: $\text{polynomial}(N) \cdot (11N)^{36N}$. 
Elliptic Fekete points and the condition number of polynomials

Shub and Smale’s condition number

Let \( f : \mathbb{C} \rightarrow \mathbb{C} \) be a polynomial of degree \( N \) and let \( \zeta \in \mathbb{C} \) be a zero of \( f \). Let

\[
\mu(f, \zeta) = \frac{N^{1/2}(1 + \|\zeta\|^2)^{\frac{N-2}{2}}}{|f'(\zeta)|} \|f\|_{B-W}.
\]

This is the condition number, which actually controls the sensibility of the zero \( \zeta \) to perturbations of \( f \). Let \( \mu(f) = \max(\mu(f, \zeta) : f(\zeta) = 0) \).

Theorem (Shub–Smale)

For every polynomial \( f \), we have \( \mu(f) \geq 1 \). For random \( f \), with probability at least \( 1/2 \) we have \( \mu(f) \leq N \).
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$$\mu(f) = \max(\mu(f, \zeta) : f(\zeta) = 0).$$

**Theorem (Shub–Smale)**

*For every polynomial $f$, we have $\mu(f) \geq 1$. For random $f$, with probability at least $1/2$ we have $\mu(f) \leq N$.***
Elliptic Fekete points and the condition number of polynomials

Best conditioned polynomials

So, for many polynomials, $\mu(f) \leq N$. 
So, for many polynomials, $\mu(f) \leq N$. Can we find one $f$ with that property? **not easy!** even changing $N$ to $N^c$, $c$ a constant.

**Theorem (Shub–Smale)**

Let $x_1, \ldots, x_N \in S$ be a set of elliptic Fekete points. Let $z_1, \ldots, z_N \in \mathbb{C}$ be the preimage of $x_1, \ldots, x_N$ under the stereographic projection. Let $f$ be the polynomial which has zeros $z_1, \ldots, z_N$. Then,

$$\mu(f) \leq \sqrt{N(N + 1)}.$$
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$$\mu(f) \leq \sqrt{N(N + 1)}.$$  

*Experiments suggest $\mu(f) \leq \sqrt{N}/2$.***
Let $x_1, \ldots, x_N \in \mathbb{S}$ and associated $z_1, \ldots, z_N \in \mathbb{C}$. Let

$$f(Z) = (Z - z_1) \cdots (Z - z_N).$$
Elliptic Fekete points and the condition number of polynomials
Condition number, logarithmic potential and Bombieri–Weyl norm

Let \( x_1, \ldots, x_N \in S \) and associated \( z_1, \ldots, z_N \in \mathbb{C} \). Let

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\]

Then,

\[
\mathcal{E}(x_1, \ldots, x_N) = \frac{1}{2} \sum_{i=1}^{N} \ln \mu(f, z_i) + \frac{N}{2} \ln \left( \prod_{i=1}^{N} \sqrt{1 + |z_i|^2} / \|f\| \right) - \frac{N}{4} \ln N,
\]
Elliptic Fekete points and the condition number of polynomials

Condition number, logarithmic potential and Bombieri–Weyl norm

\[ \mathcal{E}(x_1, \ldots, x_N) = \frac{1}{2} \sum_{i=1}^{N} \ln \mu(f, z_i) + \frac{N}{2} \ln \left( \prod_{i=1}^{N} \frac{\sqrt{1 + |z_i|^2}}{\|f\|} \right) - \frac{N}{4} \ln N, \]

\[ f(z) = \prod_{i=1}^{N} (z - z_i) \]

\[ \left( \prod_{i=1}^{N} \| g_i \| \right) / \| f \| \]
Elliptic Fekete points and the condition number of polynomials

Condition number, logarithmic potential and Bombieri–Weyl norm

\[ \mathcal{E}(x_1, \ldots, x_N) = \min \left( \frac{1}{2} \sum_{i=1}^{N} \ln \mu(f, z_i) + \frac{N}{2} \ln \left( \prod_{i=1}^{N} \frac{\sqrt{1 + |z_i|^2}}{\|f\|} \right) - \frac{N}{4} \ln N, \right) \]
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Condition number, logarithmic potential and Bombieri–Weyl norm
Back to the Problems about Smale’s 7th Problem

Minimum value vs. Expected value

- Recall we have said
  \[ m_N = -\frac{N^2}{4} - \frac{N \ln(N)}{4} + O(N) \]

- Expectation if we choose \( x_1, \ldots, x_N \) just randomly and uniformly in \( \mathbb{S} \):
  \[ \mathbb{E}_{\text{uniform}} = -\frac{N^2}{4} - \frac{N}{4}. \]
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**Theorem (Armentano-B.-Shub)**

*Expectation for points coming from the zeros of random polynomials (Bombieri–Weyl distribution):*

\[ E_{B-W} = -\frac{N^2}{4} - \frac{N \ln(N)}{4} - \frac{N}{4}. \]
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\[ m_N = \frac{N^2}{4} - \frac{N \ln(N)}{4} + O(N) \]

\[ E_{\text{uniform}} = \frac{N^2}{4} - \frac{N}{4} \]

\[ E_{\text{B-W}} = \frac{N^2}{4} - \frac{N \ln(N)}{4} - \frac{N}{4} \]
Roots of a randomly chosen (B–W) polynomial

Just one try
Roots of a random polynomial vs. random points

Maybe 4 or 5 tries
• Enhancing the relation between the logarithmic energy and the condition number of polynomials.
• Investigating the relation with eigenvalues of random matrices.
• Investigating the structure of the critical set of $\mathcal{E}$.
• Juan González Criado del Rey: investigating the topological properties of the set of minimizers of Tammes problem.
Japanese art and spherical points

Thank you for your attention.