Canonical Duality-Triality: Complete Sets of Analytical Solutions to a Class of Challenging Problems in Nonconvex Analysis and Complex Systems

David Yang Gao

1. Unified Modeling and Challenging Problems in Nonlinear Analysis/Elasticity
2. Canonical Duality Principle and General Analytic Solution
3. Triality and Global Extremal Solutions

Canonical Duality-Triality Theory


A methodological theory comprises mainly

1. Canonical dual transformation
   Unified Modeling
2. Complementary-Dual Principle
   Unified Solution
3. Triality Theory
   Identify both global and local extrema
   Design powerful algorithms
   Unified understanding complexities

\[ \min = \max \]
\[ \min = \max \]
Unified Modeling in Complex Systems

\[ A(u) = f \]

**input** \( f \) \[ \rightarrow \] \[ A(u) = f \] \[ \rightarrow \] **out puts** \( u \)

1. \( A(u) \) is a potential operator (odd order) \( A(u) = \partial W(u) \) \( \Rightarrow \)
   \[ \min P(u) = W(u) - (u, f) \]

2. \( A(u) \) is a non potential operator (even order)

Least squares \( \Rightarrow \) \( \min P(u) = \| A(u) - f \|^2 \)

In ether case, \( A(u) = f \) can be written in

**U(u) = (u, f)** subjective function,
action-reaction duality \( u^* = \partial U = f \)

\( W(v) : \) Objective function

\[ W(Qv) = W(v) \ \forall Q^T = Q^{-1}, \ \det Q = 1 \]

\( v^* = \partial W(v) \) Constitutive duality

\( D: \) Linea (differential) operator

\( \delta P(u) = 0 \Rightarrow \) Euler-Lagrange eqn:

\[ A(u) = D^* \partial W(Du) = f \]

\[ u^3 + au^2 + bu = f \]

\[ u^2 = f \]

\[ W(u) \]

\[ \nu^* = \partial U = f \]

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Unified Model in Mathematical Physics

System:
\[ Au = f \]

Dynamic system:
\[ v = Du \]
\[ Au = D^*mDu = f \Rightarrow \]
\[ v^* = m v \]
\[ u^* = D^*v^* \]

\[ \min P(u) = W(Du) - (u, u^*) \]

\[ f \rightarrow \]
\[ \text{input} \rightarrow \]
\[ u \rightarrow \text{output} \]

\[ D^* = -\frac{d}{dt} \]
\[ Au = -m u'' = f \]
\[ f = u^* \leftrightarrow (u^*, u) \rightarrow u \]
\[ D^* = -\text{div} \]
\[ -\Delta u = f \]
\[ D = \text{grad} \]

\[ \sigma \leftrightarrow (\sigma : \varepsilon) \rightarrow \varepsilon \]
\[ \sigma = H\varepsilon \]

\[ (p ; Du) = \int p u' dt = \int -p' u dt = (D^*p , v ) \]
Buckling Phenomenon

1. Euler-Bernoulli Elastic Beam Model (1750)

\[ 0 = u_{xxxx} + \lambda u_{xx} - au_x^2u_{xx} + f - \rho u_{tt} \Rightarrow \text{Chaos} \]

Total Potential: \[ P(u) = \int \frac{1}{2} [u_{xx}^2 - \lambda u_x^2] \, dx \]

If \( \lambda < \lambda_c = \min \int u_{xx}^2 \, dx / \int u_x^2 \, dx \Rightarrow \text{convex } P(u) \Rightarrow \text{pre-buckling} \)

If \( \lambda > \lambda_c \Rightarrow \text{Concave } P(u) \Rightarrow \text{crush} \)

von Karman’s nonlinear plate is linear in 1-D!

2. Nonlinear Beam (Gao, 1996)

\[ P(u) = \int \frac{1}{2} [u_{xx}^2 - \lambda u_x^2 + \frac{1}{6} a u_x^4] \, dx - \int f u \, dx \]

Total Action: \[ \int \left[ \frac{1}{2} \rho u_t^2 - P(u) \right] \, dt \Rightarrow \text{min?} \]

Nonconvexity leads to chaos
New Phenomena in Chaos (Gao, 2005)

Tri-Chaos: life of beam vibration

Meta-Chaos: sign for changing
Unified nonlinear problems in $\mathbb{R}^\infty$

Min $P(u) = W(Du) + \frac{1}{2} (u, Qu) - (u, f)$, $W = \int \frac{1}{2} (|e|^2 - a)^2$

$Qu + D^* \partial W(Du) = f$

1. Duffing: $Q = - \partial_{tt}$, $D = I$
   
   $- u_{,tt} + \left( \frac{1}{2} u^2 - a \right) u = f$

2. Landau-Ginzburg: $Q = - \Delta$, $D = I$
   
   $- \Delta u + \left( \frac{1}{2} |u|^2 - a \right) u = f$

3. Cahn-Hillard: $Q = \Delta + \text{curl curl}$, $D = I$
   
   $\Delta u + \text{curl curl} u + \left( \frac{1}{2} |u|^2 - a \right) u = f$

4. Nonlinear Gordon: $Q = - \partial_{tt} + \Delta$, $D = I$
   
   $- u_{tt} + \Delta u + \left( \frac{1}{2} |u|^2 - a \right) u = f$

5. Nonlinear Schrodinger: $Q = \partial_{tt} - i \partial_x$, $D = I$
   
   $u_{tt} - i u_x + \left( \frac{1}{2} |u|^2 - a \right) u = f$

6. Large Deformed Beam (Gao, 1996): $Q = - \partial_{tt} + \partial_{xxxx}$, $D = \partial_x$
   
   $u(t, x) = q(t) \sin \omega x$

$\Rightarrow$ Duffing eqn
Classification of Nonlinearities

In nonlinear analysis, a PDE $A(u, \partial u) = f$ is called

1. Semi-Linear: $A$ is nonlinear in $u$ but linear in $e = \partial u$

2. Quasi-Linear: $A$ is linear in the highest order of $\partial u$

3. Fully Nonlinear: $A$ is nonlinear in the highest order of $\partial u$

This classification does not make sense in global analysis!

Min $P(u) = W(Du) - (u, f) \Rightarrow A(u, \partial u) = f$

convex $P(u) \Rightarrow$ monotone $A \Rightarrow$ easy problem

Nonconvex $P(u) \Rightarrow$ non-monotone $A \Rightarrow$ NP-hard problem

Canonical Problem: Min $P(u) = V(\Lambda(u)) - (u, f)$

Geometrically nonlinear: $\Lambda(u)$ is nonlinear (far from equilibrium)

Physically nonlinear: $\partial V(e)$ is nonlinear (canonical duality)

Fully Nonlinear: both geometrically and physically nonlinear
Convexity vs Nonconvexity

Convex

Nonconvex

Static systems ➔ Buckling
Dynamical systems ➔ Chaos

Computer Science ➔ NP-Hardness
Challenges and New Methodology

\((\text{VP}): \min P(u) = \int [W(\nabla u) - uf]d\Omega\)

\(\delta P(u) = 0 \Rightarrow (\text{PDE}): -\text{div } \partial W(\nabla u) = f\)

Nonconvex \(W(e) \Rightarrow \) non monotone \(e^* = \partial W(e)\)

Exam: \(W = \frac{1}{2} k \left( e^2 - \lambda \right)^2 = \frac{1}{2} k \left( \varepsilon - \lambda \right)^2\)

\(\delta P(u) = 0 \Rightarrow [k u_x( u_x^2 - \lambda )]_x = 0\)

Three \(u_x = \{ 0, \pm \sqrt{\lambda} \} \Rightarrow \infty\) solutions

J. Ball: outstanding open problems …,

Basic Reason: \(e = \nabla u\) is not a strain measure

Canonical Transformation: \(\forall\) objective \(W(e),\) \(\exists\) an objective measure \(\varepsilon = \Lambda(e) = e^2\)

and a convex \(V(\varepsilon) \Rightarrow W(e) = V(\Lambda(e))\) and \(\varepsilon^* = \partial V(\varepsilon)\) is one-to-one (canonical duality)!

Canonical problem:

\((\text{P}): \min P(u) = \int [V(\Lambda(Du)) - uf]d\Omega\)
Analytical Solutions to “Fully Nonlinear” Problems

(P): \[ \min P(u) = \int \kappa [1 + u'^2]^{1/2} \, dx - \int f u \, dx \]

s.t. \[ u(0) = 0 = u(1) \]

\[ \delta P(u) = 0 \Rightarrow A(u) = -\kappa [u'(1 + u'^2)^{-1/2}]' = f \]

Convex \[ W(e) = \kappa [1 + e^2]^{1/2} \quad \text{W}^*(\sigma) = [\kappa^2 - \sigma^2]^{1/2} \]

(Pd): \[ \max P^d(\sigma) = \int [\kappa^2 - \sigma^2]^{1/2} \, dx \]

s.t. \[ -\sigma' = f, \quad |\sigma| < \kappa \]

\[ \sigma(x) = \int f(t) \, dt + c = g(x) + c \]

\[ P^d_\sigma(c) := P^d(\sigma(c)) = \int I \sqrt{\kappa^2 - (g(x) + c)^2} \, dx \rightarrow \max \forall c \in \mathcal{S}_\alpha^c. \]

\[ u' = e = \partial W^*(\sigma) \]

\[ \bar{u}(x) = \int_0^x \frac{g(t) + c}{\sqrt{\kappa^2_0 - (g(t) + c)^2}} \, dt, \]

Einstein’s special relativity:

\[ W(v) = -m_0 (1 - v^2/c^2)^{1/2} \]

Analytic solution (Gao, 2000)
**Canonical Duality-Triality:**

**Nonlinear Differential Eqn ⇒ Algebraic Eqn**

\((\mathbf{P})\): \( \min P(u) = \int [W(u_x) - u f] \, dx, \quad W(e) = \frac{1}{2} \left( \frac{1}{2} e^2 - 1 \right)^2 \)

Diff. eqn: \[ \left( \frac{1}{2} u_x^2 - 1 \right) u_x = f \Rightarrow \text{non-unique } u_x \text{ at each } x \]

\( \varepsilon = \frac{1}{2} e^2 \Rightarrow \mathbf{W}(\varepsilon) = V(\varepsilon) = \frac{1}{2} (\varepsilon - 1)^2 \quad \sigma = \partial V(\varepsilon) = \varepsilon - 1 \)

\((\mathbf{P}^d)\): \( \max P^d(\sigma) = - \int \left[ \frac{1}{2} \tau^2 \sigma^{-1} + \frac{1}{2} \sigma^2 + \sigma \right] \, dx \)

s.t. \( \tau_x = f(x) \Rightarrow \tau(x) = \int f(x) \, dx \)

\( \partial P^d(\sigma) = 0 \Rightarrow \text{Algebraic eqn:} \)

\[ 2 \sigma^2 (\sigma + 1) = \tau^2(x) \]

\( \sigma_3 \leq -\frac{2}{3} \leq \sigma_2 \leq 0 \leq \sigma_1 \)

**Thm (G. 1996):** For each critical point \( \sigma_i \) (i=1,2,3),

\[ u_i = - \int_x \tau(t) \sigma^{-1}_i(t) \, dt \] is a critical solution of \((\mathbf{P})\) and

\[ P(u_1) = \min P(u) = \max_{\sigma > 0} P^d(\sigma) = P^d(\sigma_1) \]

\[ P(u_2) = \min P(x) = \min_{\sigma < 0} P^d(\sigma) = P^d(\sigma_2) \]

\[ P(u_3) = \max P(x) = \max_{\sigma < 0} P^d(\sigma) = P^d(\sigma_3) \]

Newton method:

\[ \left[ \sigma(u_x^k) u_x^{k+1} \right]_x = f \]
Complete solutions for
(P): \( \text{min} \left\{ \int \left[ \frac{1}{2} (u_t^2 - 1)^2 - uf \right] \, dt = \int g(u) \, dt \right\} \)

Nonlinear ODE: \( [\sigma (u_t) u_t]_t = f(t) \)

Canonical dual:
\( \sigma^2 (\sigma + 1) = \frac{1}{2} \tau^2 \)
\( \tau(t) = \int -f(t) \, dt \)

Newton method
\( [\sigma(u_t^k) u_t^{k+1}]_t = f(t) \)
which only produces smooth numerical “solutions”!

Analytic Sol. \( u_i(t) = \int \sigma_i^{-1} \tau(t) \, dt \)

\( \sigma_3(t) \leq \sigma_2(t) \leq 0 \leq \sigma_1(t) \)
ALL Newton-type methods fail to converge
Complete Solutions to 3-D “Fully Nonlinear” PDE

(P): \[ \min P(u) = \int [W(\nabla u) - uf] \, d\Omega \]

Deformation gradient: \[ F = \nabla u + I \]
Cauchy Strain tensor: \[ C = F^T F \]
St. Venant-Kichhoff: \[ V(C) = \frac{1}{2} C : H : C = \mu \text{tr} (CC) + \frac{1}{2} \lambda (\text{tr} C)^2 \]

2nd Piola-Kichhoff stress: \[ T = H^{-1} : C \]

Pure Complementary Energy (G. 1999)

(Pd): \[ \max P^d(T) = \int [\frac{1}{2} \text{tr} (\tau T^{-1} \tau) - V^*(T)] \, d\Omega \], s.t. \[ -\text{div} \tau = f \]

Thm: For a given \( \tau \) s.t. \(-\text{div} \tau = f\), \( (Pd) \) has only one \( T \geq 0 \), at most nine \( T_i < 0 \), and 17 indefinite \( T_i \) at each point \( x \).

Each \( T_i \Rightarrow \) analytical solution

\[ u_i(x) = \int \tau T_i^{-1} \, d\Omega \] \( i = 1, \ldots, 24 \).
Nonsmooth Elasto-Plastic Problem (Gao, 2000)

\((\mathcal{P})\): \(\min P(u) = \int [V(e(u_x)) - u f] dx\)

Canonical strain: \(e = \Lambda(u) = \frac{1}{2} u_x^2\)

Nonsmooth strain energy:

\[ V(e) = \begin{cases} 
\frac{1}{2} k_1 e^2 & \text{if } e \leq e_c \\
\frac{1}{2} k_1 e_c^2 + \frac{1}{2} k_2 (e - e_c) + \sigma_p (e - e_c) & \text{if } e > e_c
\end{cases} \]

\(\sigma \in \partial V(e) \Rightarrow \text{discontinuous constitutive law}\)

Fenchel Trans: \(E^*(\sigma) = \sup \{(e; \sigma) - E(e)\}\)
Understand Chaos

Nonlinear G. Beam:

\[ u_{tt} - u_{xxxx} + \left[ (\lambda - \frac{1}{2} u_x^2) u_x \right]_x = f \]

Euler Iteration:

\[ F(\sigma(u^k)) u^{k+1} = f \]

Why periodic three ⇒ Chaos  Reason: Nonconvexity (double-well)
Canonical Dual Control Against Chaos

Non-convex Dynamical System:
\[
q_{tt} + \nu q_t = q(\lambda - \frac{1}{2} q^2) + f(t)
\]
\[
\Pi(q) = \int e^{\nu t} \left[ \frac{1}{2} q_t^2 - \frac{1}{2} \left( \frac{1}{2} q^2 - \lambda \right)^2 + qf \right] dt
\]
\[
\Pi^d(p, \sigma) = \int e^{\nu t} \left[ \frac{(p_t + f)^2}{2\sigma} - \left( \frac{p^2}{2} - \frac{\sigma^2}{2} \right) \right] dt
\]
\[
\delta \Pi^d(p, \sigma) = 0 \Rightarrow \text{Dual Duffing System (DAE)}:
\]
\[
2\sigma^2(\sigma + \lambda) = (f - p_t - \nu q)^2
\]
\[
\left[ (p_t + \nu q - f) \sigma^{-1} \right]_t + p = 0
\]

Chaotic & Bifurcation Criterion:
\[
\beta(t) = (f - p_t - \nu q)^2
\]
If \( \beta(t) > \beta_o = (2/3 \lambda)^3 \) \( \Rightarrow \) one \( \sigma(t) > 0 \)
If \( \beta(t) < \beta_o \) \( \Rightarrow \) three \( \sigma_3 \leq \sigma_2 \leq 0 \leq \sigma_1 \)

Fig. Dual feedback control of a smart beam
Variation \((\mathbb{R}^\infty) \Rightarrow \) Optimization \((\mathbb{R}^n)\)

\[
\begin{align*}
\inf P(u) &= W(Du) - (u, f) \\
u &= \{u^i(x)\}^d
\end{align*}
\]

\[
W = \int \frac{1}{2} H u_{xx}^2 \, dx \quad \text{(E-B)}
\]

\[
+ \int \frac{1}{2} \alpha (|u_x|^2 - \lambda)^2 \, dx \quad \text{(G, 1996)}
\]

\[
\begin{align*}
\min P(x) &= W(Dx) - (x, f) \\
x &= \{x^i\}^d
\end{align*}
\]

\[
W = \frac{1}{2} (Dx)^T H(Dx)
\]

\[
= \frac{1}{2} x^T A x \quad \text{(A=} D^T HD \text{ )}
\]

\[
+ \sum \frac{1}{2} \alpha_{kp} (|x_k - x_p|^2 - \lambda_{kp}^2)^2
\]

Database analysis
Sensor localization

Continuum mech.

Structural mech.

Network flow
Transportation
TSP, TTP Problems
Complete Solutions for Post-Buckled Beam

\[- u_{tt} + u_{xxxx} - \left[ \left( \frac{1}{2} u_x^2 - f \right) u_x \right]_x = q \]
What is Chaos and its global optimal solution

Nonlinear dynamic system $u' = f(t, u), \quad u(t_0) = u_0$

What kind of nonlinear $f \Rightarrow$ Chaotic $u(t)$?

$u^2 - a = 0 \Rightarrow a > 0,$ two real solutions, $a < 0,$ no real solution

FDM for $t \in [0, T]$ 

$u(t) \Rightarrow x \in \mathbb{R}^n,$

$x_{k+1} = x_k + h f(t_k, x_k), \quad k = 1, \ldots, n-1$

Least squares $\Rightarrow$ nonconvex global optimization problem

$\min \| x_{k+1} - x_k - h f(t_k, x_k) \|^2,$ $x \in \mathbb{R}^n$

Logistic equation in population growth (Ruan and Gao 2012-2014)

$u_t = ru (1 - k^{-1}u) - C(t)$

$P(x) = \frac{1}{2} \sum_{k=1}^{200} \| x_k - rx_{k-1} (1 - x_{k-1}) \|^2,$

Li-Zhou-Gao:
Nonlinear Dynamics, 2014
Navier-Stokes equation (Euler $\nu = 0$):

$$u_t + u \cdot \nabla u + \nabla p = f + \nu \Delta u, \quad \text{div} \ u = 0$$

Weak form + (FEM, FDM) $\Rightarrow$ Nonlinear quadratic algebraic system

Least squares $\Rightarrow$ nonconvex global optimization problem

$$(\mathcal{P})$: \ \min \ \Sigma \| N(u_k) - f(t_k, x_k) \|^2$$

![Stable trajectory of logistic map](image)
Finite Element Simulations
for Large Deformation Hyper-Elastic Digraph (Ionita-Gao, 2001)
Nonconvex, nonsmooth, neoconservative dynamical problem

Diaphragm Montabert p=12.20 kPa | 15 Jan 2001

Mooney-Rivlin,
A=4250 kPa, B=0, ν=0.49
NIKE3Dm
A novel canonical dual computational approach for prion AGAAAAGA amyloid fibril molecular modeling


\[
\min \sum_{i=1}^{N} \sum_{j=1, j<i}^{N} \left( \frac{1}{\tau_{ij}^6} - \frac{1}{\tau_{ij}^3} \right)
\]

\[
\tau_{ij} = (x_{3i-2} - x_{3j-2})^2 + (x_{3i-1} - x_{3j-1})^2 + (x_{3i} - x_{3j})^2
\]

Lennard-Jones Potential

\[
V(r) = 4\epsilon \left[ \left( \frac{\sigma}{r} \right)^{12} - \left( \frac{\sigma}{r} \right)^6 \right]
\]

![Graph showing repulsive and attractive forces with the Lennard-Jones potential equation.](image)
Applications to Discrete Systems

Given $a_i$, $e_{ik}$ and $d_{ij}$, find $x_i \in \mathbb{R}^d$, such that

$$|x_i - x_j|^2 = 2d_{ij}^2 \quad i, j \in \mathcal{N}$$

$$|x_i - a_k| \leq e_{ik} \quad i, k \in \mathcal{B}$$

($P$): $\min P(x) = \sum \frac{1}{2} \alpha_{ij} (\frac{1}{2} |D_i x_j|^2 - d_{ij}^2)^2$

s.t. $x \in \mathbb{R}^{d \times n}$, $D_i x_j = x_i - x_j$

NP–complete problem even $d=1$!

Canonical Dual Trans. (Cauchy-type strain)

$$\varepsilon = \{ \varepsilon_{ij} \} = \{ \frac{1}{2} |D_i x_j|^2 \} \geq 0$$

$$W(Dx) = E(\varepsilon(x)) = \sum \frac{1}{2} \alpha_{ij} (\varepsilon_{ij} - d_{ij}^2)^2$$ convex

3D Tensegrity graph

**Biotensegrity**

Sensor localization

Tensegrity in a cell
Example (Ruan-Gao, 2014) Perturbation for Sensor Network $d = 2, n = 50, a = 4$

Figure 6: Computed locations information of 50 sensors and 4 anchors
Topology Optimization

Min $P(u, \rho)$

s.t. $\rho(x) \in \{0, 1\}, \ x \in \Omega, \ u(x) \in U$

Finite Element Method:
$\Omega = \bigcup \Omega_e, \ u(x) = N(x) q, \ x \in \Omega_e$

Mixed Integer Programming
Min $\{ P(q, \rho) \mid \rho \in \{0, 1\}^m, \ q \in \mathbb{R}^n \}$

Topology Optimization provides the most challenging (NP-Hard) problems in global optimization and computer science!

Canonical transformation: $\varepsilon_i = \rho_i (\rho_i - 1) = 0$
Problems that can be solved

$$\begin{align*}
(P) : \quad & \min \left\{ \Pi(x) := \frac{1}{2} x^T A x - f^T x + W(x) + T(x) \mid x \in \mathbb{R}^n \right\} \\
W(x) & := \sum_{i=1}^{r} \frac{\alpha_i}{2} \left( \frac{1}{2} x^T B_i x - b_i^T x + c_i \right)^2 , \\
T(x) & := \frac{1}{\beta} \log \left[ 1 + \sum_{i=1}^{P} \exp \left( \beta \left( \frac{1}{2} x^T Q_i x - q_i^T x + d_i \right) \right) \right] , \\
P(c) & := \frac{1}{2} \sum_{p=1}^{P} \left( \sum_{i=1}^{N} w_i e^{-\frac{\|x_p-c_i\|^2}{2\alpha^2}} - y_p \right)^2 + \frac{1}{2} \beta \|c\|^2 - f \|c\| .
\end{align*}$$

Benchmark Problems:

1. Rosenbrock function
2. Lennard-Jones potential minimization
3. Three Hump Camel Back Problem
4. Goldstein-Price Problem
5. $2^n$ order polynomials minimizations
6. Canonical functions ... New math–Nonlinear space
Part I  Symmetry in Convex Systems
   1. Mono-duality in static systems
   2. Bi-duality in dynamical systems

Part II  Symmetry Breaking:
   Triality Theory in Nonconvex Systems
   3. Tri-duality in nonconvex systems
   4. Multi-duality and classifications of general systems

Part III  Duality in Canonical Systems
   5. Duality in geometrically linear systems
   6. Duality in finite deformation systems
   7. Applications, open problems and concluding remarks

Recent papers:
Unified Global Optimization

Discrete optimization

- Combinatorial Optim. Integer Programming
- Mixed Integer Optim. Supply Chain Process
- Nonconvex/nonsmooth Variational/V.I. Analysis

Continuous Optimization

- Canonical Duality-Triality Theory

Combinatorial Algebra

- Graph, lattice, fuzzy max-plus algebra
- FEM, FDM, FVM, SDP, Meshless, Wavelet, SIP

Numerical Analysis

International Society of Global Optimization

www.ISoGOp.org
Discussion and Open Problems

1. To model complex phenomena within a unified framework
2. To understand and identify NP-hard problems
3. To solve general global optimization/computational sciences problems
4. Nonlinear PDEs $\Rightarrow$ Nonlinear algebraic equations $\Rightarrow$ All solutions
5. To understand and solve nonlinear (chaotic) dynamic systems
6. To check, verify and predict model-theory-event
7. Unified understanding natural phenomena

Open Problems: $\mathbf{(P)}$ is NP-Hard if $(\mathbf{P}^d)$ has no solution in $S_a^+$

$u(x) \geq 0 \Rightarrow$ nonconvex V.I. + free boundary

$f > f_c$ s.t. $u(x) = 0$ for certain $x$

Thanks!