Damping by heat conduction in the Timoshenko system: Fourier and Cattaneo are the same

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The Fourier law

In 1807, the French mathematical physicist Joseph Fourier proposed a constitutive relation of the heat flux of the form

\[ q(x, t) = -\kappa \nabla \theta(x, t), \]  

(1)

where \( x \) stands for the material point, \( t \) is the time, \( q \) is the heat flux, \( \theta \) is the temperature, \( \nabla \) is the gradient operator and \( \kappa \) is the thermal conductivity of the material, which is a thermodynamic state property. Equation (1) together with the energy equation of the heat conduction

\[ \rho \theta_t + \varrho \text{div} q = 0 \]  

(2)

yields the classical heat transport equation (of parabolic type)

\[ \rho \theta_t - \kappa \varrho \Delta \theta = 0, \]  

(3)

that allows an infinite speed for thermal signals.
The Cattaneo law

To overcome the drawback in the Fourier law, a number of modifications of the basic assumption on the relation between the heat flux and the temperature have been made, such as: Cattaneo law, Gurtin & Pipkin theory, Jeffreys law, Green & Naghdi theory and others. The common feature of these theories is that all lead to hyperbolic differential equation and permit transmission of heat flow as thermal waves at finite speed.

The Cattaneo law

\[ \tau q_t + q + \kappa \nabla \theta = 0, \quad (\tau > 0, \text{relatively small}) \]  

was proposed by Cattaneo in 1948. It is perhaps the most obvious, the most widely accepted and simplest generalization of Fourier’s law that gives rise to a finite speed of propagation of heat.
When the Fourier law (1) is replaced by the Cattaneo law (4) for the heat conduction, the equations of thermoelasticity become purely hyperbolic. Indeed, from the energy balance law

$$\rho \theta_t + \varrho \text{div}q = 0 \quad (5)$$

and (4), we obtain the telegraph equation

$$\rho \theta_{tt} - \frac{\varrho \kappa}{\tau} \Delta \theta + \frac{\rho}{\tau} \theta_t = 0, \quad (6)$$

which is a hyperbolic equation and predicts a finite signal speed equals to $\left(\frac{\varrho \kappa}{\rho \tau}\right)^{1/2}$. 

B. Said-Houari
Single-phase-lagging model

Note that the Cattaneo constitutive relation (4) is actually a first-order approximation of a more general constitutive relation (single-phase-lagging model).

\[ q(x, t + \tau) = -\kappa \nabla \theta(x, t). \]  

(7)

The relation (7) states that the temperature gradient established at a point \( x \) at time \( t \) gives rise to a heat flux vector at \( x \) at a later time \( t + \tau \). The delay time \( \tau \) is interpreted as the relaxation time due to the fast-transient effects of thermal inertia (or small-scale effects of heat transport in time) and is called the phase-lag of the heat flux. It has been confirmed by many experiments that the Cattaneo law generates a more accurate prediction than the classical Fourier law. However, some studies show that the Cattaneo constitutive relation has only taken account of the fast-transient effects, but not the micro-structural interactions.
The coupling of the heat conduction with elasticity

- **Racke (2002)** has shown that the norm of the difference between the solution \( U = (u, u_t, \theta, q) \) of the classical thermoelastic system

\[
\begin{align*}
    u_{tt} - \alpha u_{xx} + \beta \theta_x &= 0, \\
    \theta_t + q_x + \delta u_{tx} &= 0, \\
    q + \kappa \theta_x &= 0,
\end{align*}
\]

and the solution \( \hat{U} = (\hat{u}, \hat{u}_t, \hat{\theta}, \hat{q}) \) of the second sound thermoelastic model

\[
\begin{align*}
    \hat{u}_{tt} - \alpha \hat{u}_{xx} + \beta \hat{\theta}_x &= 0, \\
    \hat{\theta}_t + \hat{q}_x + \delta \hat{u}_{tx} &= 0, \\
    \tau \hat{q}_t + \hat{q} + \kappa \hat{\theta}_x &= 0,
\end{align*}
\]

can be estimated as \( \|U(t,x) - \hat{U}(t,x)\|_2 \leq C \tau^2 \). This last estimate means that the difference will go to zero as \( \tau \to 0 \).
Coupling of the heat conduction with the Timoshenko system

The Timoshenko system consists of a coupled system of two wave equations describing the transverse vibration of a beam and it ignores damping effects of any nature. Specifically, Timoshenko derived the following system

\[
\begin{align*}
\rho \varphi_{tt} &= (K(\varphi_x - \psi))_x, \quad \text{in } (0, L) \times (0, +\infty), \\
I_\rho \psi_{tt} &= (EI\psi_x)_x + K(\varphi_x - \psi), \quad \text{in } (0, L) \times (0, +\infty),
\end{align*}
\]

where \( t \) is time, \( x \) is the coordinate along the beam length. The function \( \varphi = \varphi(t, x) \) is the transverse displacement of the beam from an equilibrium state and \( \psi = \psi(t, x) \) is the rotation angle of the filament of the beam. The coefficients \( \rho, I_\rho, E, I \) and \( K \) are, respectively, the density, the polar moment of inertia of a cross section, Young’s modulus of elasticity, the moment of inertia of a cross section, and the shear modulus.
Some Results

- **Rivera and Racke (2002)** → Heat conduction through Fourier’s law
  \[
  \begin{aligned}
  \rho_1 \varphi_{tt} - \sigma(\varphi_x, \psi)_x &= 0, \quad \text{in } (0, L) \times (0, +\infty), \\
  \rho_2 \psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + \gamma \theta_x &= 0, \quad \text{in } (0, L) \times (0, +\infty), \\
  \rho_3 \theta_t - k\theta_{xx} + \gamma \psi_{tx} &= 0, \quad \text{in } (0, L) \times (0, +\infty)
  \end{aligned}
  \]
  → Exponential decay results for the linearized system and a non exponential stability result for the case of different wave speeds.

- **Fernández Sare and Racke (2009)** → Heat conduction through Cattaneo’s law
  \[
  \begin{aligned}
  \rho_1 \varphi_{tt} - K(\varphi_x + \psi)_x &= 0 \\
  \rho_2 \psi_{tt} - b\psi_{xx} + K(\varphi_x + \psi) + \gamma \theta_x &= 0 \\
  \rho_3 \theta_t + \kappa q_x + \gamma \psi_{tx} &= 0 \\
  \tau_0 q_t + \delta q + \kappa \theta_x &= 0,
  \end{aligned}
  \]
  → No exponential decay even if the wave speeds are equal.

And additional damping of the form \( \int_0^\infty g(t - s)\psi_{xx}(s)ds \) in the left-hand side of (9) is not enough to stabilize the system exponentially.
The Cauchy problem (only few results)

- **Ide, Haramoto & Kawashima (2008)** investigated the problem

\[
\begin{aligned}
\varphi_{tt}(t,x) - (\varphi_x - \psi)_x(t,x) &= 0, & (t,x) &\in \mathbb{R}^+ \times \mathbb{R}, \\
\psi_{tt}(t,x) - a^2 \psi_{xx}(t,x) - (\varphi_x - \psi)(t,x) + \lambda \psi_t(t,x) &= 0, & (t,x) &\in \mathbb{R}^+ \times \mathbb{R}, \\
(\varphi, \varphi_t, \psi, \psi_t)(0,x) &= (\varphi_0, \varphi_1, \psi_0, \psi_1), & x &\in \mathbb{R},
\end{aligned}
\]  

(10)

where \(a\) and \(\lambda\) are positive constants. They proved that if \(a = 1\), then solutions of (10) decay as

\[
\left\| \partial_x^k U(t) \right\|_2 \leq C \left(1 + t\right)^{-1/4 - k/2} \left\| U_0 \right\|_1 + C e^{-ct} \left\| \partial_x^k U_0 \right\|_2,
\]  

(11)

where \(U = (\varphi_x - \psi, \varphi_t, a\psi_x, \psi_t)^T\). If on the other hand, \(a \neq 1\), then decay property of system (10) is of **regularity-loss** type and solutions decay as

\[
\left\| \partial_x^k U(t) \right\|_2 \leq C \left(1 + t\right)^{-1/4 - k/2} \left\| U_0 \right\|_1 + C \left(1 + t\right)^{-l/2} \left\| \partial_x^{k+l} U_0 \right\|_2.
\]  

(12)

The parameters \(k\) and \(l\) in (11) and (12) are non-negative integers, and \(C\) and \(c\) are positive constants.
Racke & Said-Houari (2011) analyzed the semilinear problem

\begin{align}
\varphi_{tt} - (\varphi_x - \psi)_x &= 0, & (t, x) &\in \mathbb{R}^+ \times \mathbb{R}, \\
\psi_{tt} - a^2\psi_{xx} - (\varphi_x - \psi) + \lambda\psi_t &= f(\psi), & (t, x) &\in \mathbb{R}^+ \times \mathbb{R}, \\
(\varphi, \varphi_t, \psi, \psi_t)(0, x) &= (\varphi_0, \varphi_1, \psi_0, \psi_1), & x &\in \mathbb{R},
\end{align}

where $f(\psi(t, x)) = |\psi(t, x)|^p$ with $p > 1$. For the linear case (i.e. $f = 0$), they improved the decay results obtained in (11) so that, for initial data $U_0 \in H^s(\mathbb{R}) \cap L^{1, \gamma}(\mathbb{R})$ with a suitably large $s$ and $\gamma \in [0, 1]$, solutions decay faster than those given in (11) and (12). Further, we analyzed the asymptotic behavior of the semilinear problem (13) with the power type nonlinearity $|\psi|^p$ with $p > 12$. 
The Cauchy problem: The coupling with the Fourier law

Said-Houari & Kasimov (2012)

We considered the Fourier-Timoshenko system:

\[
\begin{align*}
\varphi_{tt} - (\varphi_x - \psi)_x &= 0, \\
\psi_{tt} - a^2 \psi_{xx} - (\varphi_x - \psi) + \lambda \psi_t + \beta \theta_x &= 0, \\
\theta_t - \theta_{xx} + \beta \psi_{tx} &= 0,
\end{align*}
\]

where \( t \in (0, \infty) \) denotes the time variable and \( x \in \mathbb{R} \) is the space variable, the functions \( \varphi \) and \( \psi \) denote the displacements of the elastic material, the function \( \theta \) is the temperature difference and \( a, \lambda \) and \( \beta \) are certain positive constants. Initial conditions are of the following form,

\[
\begin{align*}
\varphi(., 0) &= \varphi_0(x), & \varphi_t(., 0) &= \varphi_1(x), & \psi(., 0) &= \psi_0(x), \\
\psi_t(., 0) &= \psi_1(x), & \theta(., 0) &= \theta_0(x).
\end{align*}
\]
Let us first write system (73)-(74) as a first order (in time) system of the form

\[
\begin{cases}
U_t + AU_x + LU = BU_{xx}, \\
U(x, 0) = U_0.
\end{cases}
\]  

(16)

where \(A, L, B\) are matrices and \(U\) is a solution vector identified below. To obtain system (16), we introduce the following variables:

\[v = \varphi_x - \psi, \quad u = \varphi_t, \quad z = a\psi_x, \quad y = \psi_t.\]

Consequently, system (73) can be rewritten into the following first order system of hyperbolic-parabolic type

\[
\begin{align*}
vt - ux + y &= 0, \\
u_t - vx &= 0, \\
z_t - ay_x &= 0, \\
y_t - az_x - v + \lambda y + \beta \theta_x &= 0, \\
\theta_t - \theta_{xx} + \beta y_x &= 0,
\end{align*}
\]

(17)

System (75) is equivalent to system (16) with

\[
U = \begin{pmatrix} v \\ u \\ z \\ y \\ \theta \end{pmatrix}, \quad A = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a & 0 \\ 0 & 0 & -a & 0 & \beta \\ 0 & 0 & 0 & \beta & 0 \end{pmatrix}, \quad L = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},
\]

\[
B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.
\]
Fourier law: $U_0 \in H^s(\mathbb{R}) \cap L^1(\mathbb{R})$

**Theorem 1**

Let $s$ be a nonnegative integer and assume that $U_0 \in H^s(\mathbb{R}) \cap L^1(\mathbb{R})$. Then the solution $U$ of problem (16) satisfies the following decay estimates:

- **When** $a = 1$,
  
  $$\| \partial_x^k U(t) \|_2 \leq C (1 + t)^{-1/4-k/2} \| U_0 \|_{L^1} + C e^{-ct} \| \partial_x^k U_0 \|_2. \quad (18)$$

- **When** $a \neq 1$,
  
  $$\| \partial_x^k U(t) \|_2 \leq C (1 + t)^{-1/4-k/2} \| U_0 \|_{L^1} + C (1 + t)^{-l/2} \| \partial_x^{k+l} U_0 \|_2, \quad (19)$$

where $k$ and $l$ are non-negative integers satisfying $k + l \leq s$ and $C, c$ are two positive constants.
The Fourier law: \( U_0 \in H^s(\mathbb{R}) \cap L^{1,\gamma}(\mathbb{R}) \)

**Theorem 2**

Let \( \gamma \in [0, 1] \). Let \( s \) be a nonnegative integer and assume that \( U_0 \in H^s(\mathbb{R}) \cap L^{1,\gamma}(\mathbb{R}) \). Then the solution \( U \) of problem (16) satisfies the following decay estimates:

- **When** \( a = 1 \), we have
  \[
  \| \partial_x^k U(t) \|_2 \leq C (1 + t)^{-1/4 - k/2 - \gamma/2} \| U_0 \|_{L^{1,\gamma}} + C e^{-ct} \| \partial_x^k U_0 \|_2 \\
  + C (1 + t)^{-1/4 - k/2} \left\| \int_{\mathbb{R}} U_0(x) \, dx \right\|.
  \]  
  \( (20) \)

- **When** \( a \neq 1 \), we have
  \[
  \| \partial_x^k U(t) \|_2 \leq C (1 + t)^{-1/4 - k/2 - \gamma/2} \| U_0 \|_{L^{1,\gamma}} + C (1 + t)^{-1/2} \| \partial_x^{k+1} U_0 \|_2 \\
  + C (1 + t)^{-1/4 - k/2} \left\| \int_{\mathbb{R}} U_0(x) \, dx \right\|.
  \]  
  \( (21) \)
The Cauchy problem: The coupling with the Cattaneo law

The second problem we investigate in the present work is the following Timoshenko system in thermoelasticity of second sound

\[
\begin{aligned}
\varphi_{tt} - (\varphi_x - \psi)_x &= 0, \\
\psi_{tt} - a^2 \psi_{xx} - (\varphi_x - \psi) + \beta \theta_x + \lambda \psi_t &= 0, \\
\theta_t + \kappa q_x + \beta \psi_{tx} &= 0, \\
\tau q_t + \delta q + \kappa \theta_x &= 0.
\end{aligned}
\]  

(22)

where \( t \in (0, \infty) \) and \( x \in \mathbb{R} \) and \( \gamma, \tau, \delta, \kappa, \lambda \) and \( \beta \) are positive constants and the following initial conditions are assumed,

\[
\begin{aligned}
\varphi(., 0) &= \varphi_0(x), & \varphi_t(., 0) &= \varphi_1(x), & \psi(., 0) &= \psi_0(x), \\
\psi_t(., 0) &= \psi_1(x), & \theta(., 0) &= \theta_0(x), & q(., 0) &= q_0(x).
\end{aligned}
\]  

(23)
By introducing the following variables:

\[ v = \varphi_x - \psi, \quad u = \varphi_t, \quad z = a\psi_x, \quad y = \psi_t, \quad w = \tau_0 q. \]

system (22) can be rewritten as the following first order hyperbolic system

\[
\begin{align*}
    v_t - u_x + y &= 0, \\
    u_t - v_x &= 0, \\
    z_t - ay_x &= 0, \\
    y_t - az_x - v + \lambda y + \beta \theta_x &= 0, \\
    \theta_t + \frac{\kappa}{\tau} w_x + \beta y_x &= 0, \\
    w_t + \frac{\delta}{\tau} w + \kappa \theta_x &= 0
\end{align*}
\] (24)

and the initial conditions (31) take the form

\[ (v, u, z, y, \theta, w) (x, 0) = (v_0, u_0, z_0, y_0, \theta_0, w_0). \] (25)

System (24)-(35) is equivalent to

\[
\begin{align*}
    U_t + AU_x + LU &= 0, \\
    U (x, 0) &= U_0.
\end{align*}
\] (26)

where \( A \) is a real symmetric matrix and \( L \) is non-negative definite matrix, with

\[
U = \begin{pmatrix}
    v \\
    u \\
    z \\
    y \\
    \theta \\
    w
\end{pmatrix}, \quad A = \begin{pmatrix}
    0 & 1 & 0 & 0 & 0 & 0 \\
    1 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & a & 0 & 0 \\
    0 & 0 & a & 0 & -\beta & 0 \\
    0 & 0 & 0 & -\beta & 0 & \kappa/\tau \\
    0 & 0 & 0 & 0 & \kappa & 0
\end{pmatrix}, \quad L = \begin{pmatrix}
    0 & 0 & 0 & 1 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 \\
    -1 & 0 & 0 & \lambda & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & \delta/\tau & 0
\end{pmatrix}. \] (27)

and \( U_0 = (v_0, u_0, z_0, y_0, \theta_0, w_0)^T. \)
System (26) can be seen as a particular case of a general hyperbolic system of balance laws. We point out that Shizuta & Kawashima introduced the so-called algebraic condition (SK), namely

$$(SK) \forall \xi \in \mathbb{R} - \{0\}, \quad \text{Ker}(L) \cap \{\text{eigenvectors of } (\xi A)\} = \{0\},$$

which is satisfied in many examples and sufficient to establish a general result of global existence for small perturbations of constant-equilibrium state. Our system (26) satisfies the condition (SK), but the general theory on the dissipative structure established by Shizuta & Kawashima (1985) is not applicable since the matrices $A$ and $L$ are not real symmetric. Beauchard & Zuazua (2011) have recently shown that the condition (SK) is equivalent to the classical Kalman rank condition in control theory for the pair $(A, L)$, that is

$$\text{rk}[L, \tilde{A}(i\xi)L, ..., \tilde{A}(i\xi)^{N-1}L] = N,$$

where $\tilde{A}(i\xi) = i\xi A$ is an $N \times N$ matrix.
Let $s$ be a nonnegative integer and assume that $U_0 \in H^s(\mathbb{R}) \cap L^1(\mathbb{R})$. Then the solution $U = (\hat{v}, \hat{u}, \hat{z}, \hat{y}, \hat{\theta}, \hat{w})'$ of problem (26) satisfies the following decay estimates:

$$\left\| \partial_x^k U(t) \right\|_2 \leq C (1 + t)^{-1/4-k/2} \left\| U_0 \right\|_{L^1} + C (1 + t)^{-l/2} \left\| \partial_x^{k+l} U_0 \right\|_2,$$

where $k$ and $l$ are non-negative integers satisfying $k + l \leq s$ and $C$, $c$ are positive constants.
Theorem 4 \((U_0 \in H^s(\mathbb{R}) \cap L^{1,\gamma}(\mathbb{R}))\)

Let \(\gamma \in [0, 1]\), let \(s\) be a nonnegative integer, and assume that \(U_0 \in H^s(\mathbb{R}) \cap L^{1,\gamma}(\mathbb{R})\). Then the solution \(U = (\hat{v}, \hat{u}, \hat{z}, \hat{y}, \hat{\theta}, \hat{w})'\) of problem (26) satisfies the following decay estimates:

\[
\left\| \partial_x^k U(t) \right\|_2 \leq C (1 + t)^{-1/4-k/2-\gamma/2} \left\| U_0 \right\|_{L^{1,\gamma}} + C (1 + t)^{-l/2} \left\| \partial_x^{k+l} U_0 \right\|_2 \\
+ C (1 + t)^{-1/4-k/2} \left| \int_{\mathbb{R}} U_0(x) \, dx \right|
\]

(29)

where \(k\) and \(l\) are non-negative integers satisfying \(k + l \leq s\) and \(C\) and \(c\) are two positive constants.
We consider the system

\[
\begin{align*}
\varphi_{tt} - (\varphi_x - \psi)_x &= 0, \\
\psi_{tt} - a^2 \psi_{xx} - (\varphi_x - \psi) + \delta \theta_x &= 0, \\
\theta_t + q_x + \delta \psi_{tx} &= 0, \\
\tau q_t + \beta q + \theta_x &= 0,
\end{align*}
\]

where \( t \in (0, \infty) \), \( x \in \mathbb{R} \), and \( a, \tau, \delta, \beta, \beta \) are positive constants. System (30) is supplied with the following initial conditions:

\[
\begin{align*}
\varphi(x, 0) &= \varphi_0(x), & \varphi_t(x, 0) &= \varphi_1(x), & \psi(x, 0) &= \psi_0(x), \\
\psi_t(x, 0) &= \psi_1(x), & \theta(x, 0) &= \theta_0(x), & q(x, 0) &= q_0(x).
\end{align*}
\]
To rewrite the system as a first-order system, we define new variables, as follows:

$$v = \varphi_x - \psi, \quad u = \varphi_t, \quad z = a\psi_x, \quad y = \psi_t, \quad w = q.$$  \hspace{1cm} (32)

Then, (30) takes the form:

$$\begin{cases}
    v_t - u_x + y = 0, \\
    u_t - v_x = 0, \\
    z_t - ay_x = 0, \\
    y_t - az_x - v + \delta\theta_x = 0, \\
    \theta_t + w_x + \delta y_x = 0, \\
    w_t + \frac{\beta}{\tau}w + \frac{1}{\tau}\theta_x = 0
\end{cases}$$ \hspace{1cm} (33)

and, if we denote

$$U(x, t) = (v, u, z, y, \theta, w)^T,$$ \hspace{1cm} (34)

the initial conditions can then be written as

$$U(x, 0) = U_0(x) = (v_0, u_0, z_0, y_0, \theta_0, w_0)^T,$$

where $v_0 = \varphi_{0,x} - \psi_0$, $u_0 = \psi_1$, $z_0 = a\psi_{0,x}$, $y_0 = \psi_1$, $w_0 = q_0$. 
Theorem 5 (Cattaneo model \( \tau \neq 0 \))

Let \( s \) be a nonnegative integer and let \( \alpha = (\tau - 1) (1 - a^2) - \tau \delta^2 \).

Assume that \( U_0 \in H^s(\mathbb{R}) \cap L^1(\mathbb{R}) \). Then, the solution, \( U \), of problem (33) satisfies the following decay estimates:

- if \( \alpha = 0 \),
\[
\left\| \partial_x^k U(t) \right\|_2 \leq C (1 + t)^{-1/12 - k/6} \left\| U_0 \right\|_1 + C e^{-ct} \left\| \partial_x^k U_0 \right\|_2 ; \quad (36)
\]

- if \( \alpha \neq 0 \),
\[
\left\| \partial_x^k U(t) \right\|_2 \leq C (1 + t)^{-1/12 - k/6} \left\| U_0 \right\|_1 + C (1 + t)^{-l/6} \left\| \partial_x^{k+l} U_0 \right\|_2 , \quad (37)
\]

where \( k \) and \( l \) are non-negative integers satisfying \( k + l \leq s \) and \( C \) and \( c \) are two positive constants.
Taking the Fourier transform of (33), we obtain the following ODE system:

\begin{align*}
\hat{v}_t - i\xi \hat{u} + \hat{y} &= 0, \quad (38a) \\
\hat{u}_t - i\xi \hat{v} &= 0, \quad (38b) \\
\hat{z}_t - ai\xi \hat{y} &= 0, \quad (38c) \\
\hat{y}_t - ai\xi \hat{z} - \hat{v} + \delta i\xi \hat{\theta} &= 0, \quad (38d) \\
\hat{\theta}_t + i\xi \hat{w} + \delta i\xi \hat{y} &= 0, \quad (38e) \\
\hat{w}_t + \frac{\beta}{\tau} \hat{w} + \frac{1}{\tau} i\xi \hat{\theta} &= 0, \quad (38f)
\end{align*}

together with initial data, written in terms of the solution vector, \( \hat{U}(\xi, t) = (\hat{v}, \hat{u}, \hat{z}, \hat{y}, \hat{\theta}, \hat{w})^T \), as

\[ \hat{U}(\xi, 0) = \hat{U}_0(\xi). \quad (39) \]

**Proposition 1**

Let \( \hat{U}(\xi, t) \) be a solution of (38)-(39) and \( \alpha = (\tau - 1) \left(1 - a^2\right) - \tau \delta^2 \). Then, there exist two positive constants, \( C \) and \( c \), such that, for all \( t \geq 0 \):

\[ \left| \hat{U}(\xi, t) \right|^2 \leq C \left| \hat{U}(\xi, 0) \right|^2 e^{-c \rho_1(\xi)}, \quad \rho_1(\xi) = \frac{\xi^6}{(1 + \xi^2)^3}, \quad \text{if} \quad \alpha = 0, \quad (40) \]

\[ \left| \hat{U}(\xi, t) \right|^2 \leq C \left| \hat{U}(\xi, 0) \right|^2 e^{-c \rho_2(\xi)}, \quad \rho_2(\xi) = \frac{\xi^6}{(1 + \xi^2)^6}, \quad \text{if} \quad \alpha \neq 0. \quad (41) \]
Define the following energy functional:

\[ E(t) = \frac{1}{2}(|\hat{v}|^2 + |\hat{u}|^2 + |\hat{z}|^2 + |\hat{y}|^2 + |\hat{\theta}|^2 + \tau |\hat{w}|^2). \] (42)

Then,

\[ c_1 |\hat{U}(\xi, t)|^2 \leq E(\xi, t) \leq c_2 |\hat{U}(\xi, t)|^2, \] (43)

where \( c_1 = \min(1/2, \tau/2) \) and \( c_2 = \max(1/2, \tau/2) \), and

\[ \frac{d}{dt} E(t) = -\beta \tau |\hat{w}|^2, \quad \forall t \geq 0. \] (44)
In the following lemma, we show a dissipative term of $|\hat{v}|^2$.

**Lemma 7**

The functional

$$G(t) = \text{Re}\left[ -\tau a\hat{z}^* \hat{u} + \left( \tau + \frac{1}{\delta^2} - \frac{a^2}{\delta^2} \right) \delta \hat{\theta}^* \hat{u} - \tau \hat{v} \hat{y}^* \right] + \frac{1 - a^2}{\delta} \text{Re} \left( -\tau \hat{v}^* \hat{w} \right)$$

(45)

satisfies

$$\frac{d}{dt} G(t) + \tau |\hat{v}|^2 - \tau |\hat{y}|^2 = \frac{1 - a^2}{\delta} \beta \text{Re} \left( \hat{w} \hat{v}^* \right) + \frac{1 - a^2}{\delta} \text{Re} \left( \tau \hat{w}^* \hat{y} \right)$$

$$+ \alpha \text{Re} \left( i\xi \hat{y} \hat{u}^* \right) + \frac{\alpha}{\delta} \text{Re} \left( i\xi \hat{w} \hat{u}^* \right),$$

(46)

where

$$\alpha = (\tau - 1)(1 - a^2) - \tau \delta^2.$$  

(47)
Lemma 8

Let $K(t)$ be defined by

$$K(t) = \text{Re} \left( i \xi \hat{u}^* \hat{v} + a i \xi \hat{z} \hat{y} - \delta a i \xi \hat{z}^* \hat{w} \right).$$

(48)

Then, for any $\epsilon_1, \epsilon_2 > 0$,

$$\frac{d}{dt} K(t) + (1 - \epsilon_1) \xi^2 |\hat{u}|^2 + (a^2 - \epsilon_2) \xi^2 |\hat{z}|^2$$

$$\leq C(\epsilon_1, \epsilon_2) (1 + \xi^2) \left( |\hat{v}|^2 + |\hat{y}|^2 \right) + C(\epsilon_2) (1 + \xi^2) |\hat{w}|^2.$$

(49)
Lemma 9

The following inequalities hold true:

\[
\frac{d}{dt} \text{Re} \left( i \xi \hat{w}^* \hat{\theta} \right) + \left( \frac{1}{\tau} - \epsilon_3 \right) \xi^2 |\hat{\theta}|^2 \leq C \left( \epsilon_3, \epsilon'_3 \right) (1 + \xi^2) |\hat{w}|^2 + \epsilon'_3 \frac{\xi^4}{1 + \xi^2} |\hat{y}|^2
\]  

(50)

and

\[
\frac{d}{dt} \text{Re} \left( i \xi \hat{\theta}^* \hat{y} \right) + (\delta - \epsilon_4) \xi^2 |\hat{y}|^2 \leq C \left( \epsilon'_4 \right) (1 + \xi^2) |\hat{\theta}|^2 + \epsilon'_4 \frac{\xi^4}{1 + \xi^2} |\hat{z}|^2 \\
+ \epsilon'_4 \xi^2 |\hat{v}|^2 + C \left( \epsilon_4 \right) \xi^2 |\hat{w}|^2,
\]  

(51)

where \( \epsilon_3, \epsilon'_3, \epsilon_4, \text{ and } \epsilon'_4 \) are arbitrary positive constants.
To prove Proposition 1, we consider the two cases, $\alpha = 0$ and $\alpha \neq 0$, separately.

**Case 1. $\alpha = 0$.**

In this case, (46) takes the form

$$
\frac{dG(t)}{dt} + (\tau - \epsilon_0) |\hat{v}|^2 \leq C(\epsilon_0) \left( |\hat{w}|^2 + |\hat{y}|^2 \right). 
$$

(52)

Define the functional

$$
L_1(t) = \frac{\xi^2}{1 + \xi^2} \left\{ \gamma_0 \xi^2 G(\xi, t) + \frac{\xi^2}{1 + \xi^2} K(\xi, t) + \gamma_4 \text{Re} \left( i \xi \hat{\theta}^* \hat{y} \right) \right\} 
$$

$$
+ \gamma_3 \text{Re} \left( i \xi \hat{w}^* \hat{\theta} \right) + N \left( 1 + \xi^2 \right) E(\xi, t), 
$$

(53)

where $N, \gamma_0, \ldots, \gamma_4$ are positive constants to be fixed later. Taking the derivative of $L(\xi, t)$ with respect to $t$ and using (52), (49), (51, and (50), we get Consequently, we deduce that there exists a positive constant, $\eta > 0$, such that

$$
\frac{d}{dt} L_1(t) + \eta Q_1(t) \leq 0, \quad \forall t \geq 0, 
$$

(54)

where

$$
Q_1(t) = \frac{\xi^4}{1 + \xi^2} |\hat{v}|^2 + \frac{\xi^4}{1 + \xi^2} |\hat{y}|^2 + \frac{\xi^6}{(1 + \xi^2)^2} \left( |\hat{\theta}|^2 + |\hat{\phi}|^2 \right) + \xi^2 |\hat{\theta}|^2 + \left( 1 + \xi^2 \right) |\hat{w}|^2. 
$$

(55)

It is not difficult to see now that, for all $t \geq 0$ and for all $\xi \in \mathbb{R}$, we have

$$
Q_1(t) \geq \frac{\xi^6}{(1 + \xi^2)^2} \left( |\hat{\theta}|^2 + |\hat{\phi}|^2 + |\hat{\theta}|^2 + |\hat{\phi}|^2 + |\hat{\phi}|^2 + |\hat{\phi}|^2 \right) 
$$

$$
= \frac{\xi^6}{(1 + \xi^2)^2} |\hat{U}(\xi, t)|^2. 
$$

(56)
On the other hand we can show that

$$\beta_1 \left( 1 + \xi^2 \right) E(t) \leq L_1(t) \leq \beta_2 \left( 1 + \xi^2 \right) E(t).$$  \hspace{1cm} (57)

Combining (43), (54), (56), and (57), we deduce that there exists a positive constant, $\eta_1 > 0$, such that for all $t \geq 0$, we have

$$\frac{d}{dt} L_1(t) \leq -\eta_1 \frac{\xi^6}{(1 + \xi^2)^3} L_1(t).$$  \hspace{1cm} (58)

Integrating (58) over $t$ and using once again (43) and (57), we deduce (40).

**Case 2.** $\alpha \neq 0$.

In this case, the two last terms on the right-hand side of (46) can be estimated as

$$\alpha \text{Re} \left( i \xi \hat{y} \hat{u}^* \right) + \frac{\alpha}{\delta} \text{Re} \left( i \xi \hat{w} \hat{u}^* \right) \leq \epsilon'_0 \frac{\xi^2}{1 + \xi^2} |\hat{u}|^2 + C \left( \epsilon'_0 \right) \left( 1 + \xi^2 \right) \left( |\hat{y}|^2 + |\hat{w}|^2 \right).$$  \hspace{1cm} (59)

Inserting (59) into (46) and applying Young’s inequality as before, we find that (52) takes the form

$$\frac{dG(t)}{dt} + \left( \tau - \epsilon_0 \right) |\hat{v}|^2 \leq C \left( \epsilon'_0, \epsilon_0 \right) \left( 1 + \xi^2 \right) \left( |\hat{v}|^2 + |\hat{y}|^2 \right) + \epsilon'_0 \frac{\xi^2}{1 + \xi^2} |\hat{u}|^2.$$  \hspace{1cm} (60)

Now, we define

$$L_2(t) = \frac{\xi^2}{(1 + \xi^2)^2} \left\{ \lambda_0 \frac{\xi^2}{1 + \xi^2} G(\xi, t) + \lambda_1 \frac{\xi^2}{(1 + \xi^2)^2} K(\xi, t) + \lambda_4 \text{Re} \left( i \xi \hat{y}^* \hat{y} \right) \right\} + \lambda_3 \text{Re} \left( i \xi \hat{w}^* \hat{w} \right) + M \left( 1 + \xi^2 \right)^2 E(\xi, t),$$

where $M, \epsilon_j, \epsilon'_d, \lambda_j$, with $j = 1, \ldots, 4$ and $d = 0, 3, 4$, are positive constants to be fixed later.
Taking the time derivative of $L_2$, we obtain

$$
\frac{d}{dt} L_2 (t) + \lambda Q_2 (t) \leq 0, \quad \forall t \geq 0,
$$

for some $\lambda > 0$, where

$$
Q_2 = \frac{\xi^4}{(1 + \xi^2)^3} |\dot{\nu}|^2 + \frac{\xi^4}{(1 + \xi^2)^2} |\dot{\nu}|^2 + \frac{\xi^6}{(1 + \xi^2)^4} \left( |\dot{\nu}|^2 + |\dot{z}|^2 \right) + \xi^2 |\dot{\theta}|^2 + \left( 1 + \xi^2 \right)^2 |\hat{w}|^2
$$

$$
\geq \frac{\xi^6}{(1 + \xi^2)^4} |\dot{U}(\xi, t)|^2.
$$

On the other hand, there exist two positive constants, $\beta_3$ and $\beta_4$, such that, for all $t \geq 0$,

$$
\beta_3 \left( 1 + \xi^2 \right)^2 E(t) \leq L_2(t) \leq \beta_4 \left( 1 + \xi^2 \right)^2 E(t).
$$

Combining (43), (62), (63), and (64), we obtain (41), which completes the proof of Proposition 1.
Proof of Theorem 5

The proof is based on the pointwise estimates in Proposition 1. First, let us assume that $\alpha = 0$. Then,

$$
\rho_1(\xi) \geq \frac{1}{8} \xi^6, \quad \text{for } |\xi| \leq 1,
$$

$$
\rho_1(\xi) \geq \frac{1}{8}, \quad \text{for } |\xi| \geq 1.
$$

(65)

Applying the Plancherel theorem and making use of the inequality in (40), we obtain

$$
\left\| \partial_x^k U(t) \right\|^2_2 = \int_{\mathbb{R}} |\xi|^{2k} \left| \hat{U}(\xi, t) \right|^2 d\xi
\leq C \int_{\mathbb{R}} |\xi|^{2k} e^{-c\rho_1(\xi)t} \left| \hat{U}(\xi, 0) \right|^2 d\xi
= C \int_{|\xi| \leq 1} |\xi|^{2k} e^{-c\rho_1(\xi)t} \left| \hat{U}(\xi, 0) \right|^2 d\xi + C \int_{|\xi| \geq 1} |\xi|^{2k} e^{-c\rho_1(\xi)t} \left| \hat{U}(\xi, 0) \right|^2 d\xi
= I_1 + I_2.
$$

(66)

That is, the integral is split into its low-frequency part, $I_1$, and its high-frequency part, $I_2$. For $I_1$, we find, by using the first inequality in (65), that

$$
I_1 \leq C \int_{|\xi| \leq 1} |\xi|^{2k} e^{-\frac{1}{8} \xi^6 t} \left| \hat{U}(\xi, 0) \right|^2 d\xi \leq C \left\| \hat{U}_0 \right\|^2_{L^\infty} \int_{|\xi| \leq 1} |\xi|^{2k} e^{-\frac{1}{8} \xi^6 t} d\xi.
$$

(67)

Then, using the inequality

$$
\int_0^1 |\xi|^{2k} e^{-\frac{1}{8} \xi^6 t} d\xi \leq (1 + t)^{-1/6 - k/3},
$$

we estimate

$$
I_1 \leq \left\| U_0 \right\|^2_1 C (1 + t)^{-1/6 - k/3}.
$$

(68)
For $I_2$, using the second inequality in (65), we find
\begin{equation}
I_2 \leq C \int_{|\xi| \geq 1} |\xi|^{2k} e^{-\frac{1}{8} t} |\hat{U}(\xi, 0)|^2 d\xi \leq C e^{-\frac{1}{8} t} \|\partial_x^k U_0\|^2_2.
\end{equation}

Therefore, estimate (36) holds by adding (68) and (69). Now, we assume that $\alpha \neq 0$. Then,
\begin{equation}
\begin{cases}
\rho_2(\xi) \geq \frac{1}{64} \xi^6, & \text{for } |\xi| \leq 1, \\
\rho_2(\xi) \geq \frac{1}{64} \xi^{-6}, & \text{for } |\xi| \geq 1.
\end{cases}
\end{equation}

As before, using (41), we find that
\begin{equation}
\left\|\partial_x^k U(t)\right\|^2_2 \leq C \int_{|\xi| \leq 1} |\xi|^{2k} e^{-c\rho_2(\xi)t} |\hat{U}(\xi, 0)|^2 d\xi + C \int_{|\xi| \geq 1} |\xi|^{2k} e^{-c\rho_2(\xi)t} |\hat{U}(\xi, 0)|^2 d\xi
= J_1 + J_2.
\end{equation}

The low-frequency part, $J_1$, is estimated as $I_1$ and results in
\begin{equation}
J_1 \leq \|U_0\|^2_1 C (1 + t)^{-1/6 - k/3}.
\end{equation}

The high-frequency part, $J_2$, using the second inequality in (70), can be estimated as
\begin{equation}
J_2 \leq C \int_{|\xi| \geq 1} |\xi|^{2k} e^{-\frac{1}{64} \xi^{-6} t} |\hat{U}(\xi, 0)|^2 d\xi.
\end{equation}

Using the inequality
\begin{equation}
\sup_{|\xi| \geq 1} \left\{ |\xi|^{-2l} e^{-c\xi^{-6} t} \right\} \leq C (1 + t)^{-l/3},
\end{equation}
we obtain
\begin{equation}
\begin{aligned}
J_2 & \leq C \sup_{|\xi| \geq 1} \left\{ |\xi|^{-2l} e^{-c\xi^{-6} t} \right\} \int_{|\xi| \geq 1} |\xi|^{2(k+l)} |\hat{U}(\xi, 0)|^2 d\xi \\
& \leq C (1 + t)^{-l/3} \left\|\partial_x^{k+l} U_0\right\|^2_2.
\end{aligned}
\end{equation}
The Timoshenko–Fourier without mechanical damping

We consider the model

\[
\begin{align*}
\varphi_{tt} - (\varphi_x - \psi)_x &= 0, \\
\psi_{tt} - a^2 \psi_{xx} - (\varphi_x - \psi) + \delta \theta_x &= 0, \\
\theta_t - \theta_{xx} + \delta \psi_{tx} &= 0,
\end{align*}
\]

These equations are supplemented with appropriate initial conditions,

\[
\begin{align*}
\varphi(x,0) &= \varphi_0(x), & \varphi_t(x,0) &= \varphi_1(x), & \psi(x,0) &= \psi_0(x), \\
\psi_t(x,0) &= \psi_1(x), & \theta(x,0) &= \theta_0(x).
\end{align*}
\]

As before, we introduce the following new variables:

\[ v = \varphi_x - \psi, \quad u = \varphi_t, \quad z = a \psi_x, \quad y = \psi_t. \]

System (73) can then be rewritten as

\[
\begin{align*}
v_t - u_x + y &= 0, \\
u_t - v_x &= 0, \\
z_t - ay_x &= 0, \\
y_t - az_x - v + \delta \theta_x &= 0, \\
\theta_t - \theta_{xx} + \delta \psi_x &= 0,
\end{align*}
\]

and initial conditions (74), using \( V(x, t) = (v, u, z, y, \theta)^T \) to denote the solution vector, take the form

\[ V(x, 0) = V_0(x) = (v_0, u_0, z_0, y_0, \theta_0), \]

where \( v_0 = \varphi_{0,x} - \psi_0, \ u_0 = \psi_1, \ z_0 = a \psi_{0,x}, \) and \( y_0 = \psi_1. \)
Theorem 10

Let \( s \) be a nonnegative integer and assume that \( V_0 \in H^s(\mathbb{R}) \cap L^1(\mathbb{R}) \). Then, the solution, \( V \), of problem (75)-(76) satisfies the following decay estimates:

- when \( a = 1 \),
  \[
  \| \partial_x^k V(t) \|_2 \leq C (1 + t)^{-1/12 - k/6} \| V_0 \|_1 + Ce^{-ct} \| \partial_x^k V_0 \|_2 ; \quad (77)
  \]

- when \( a \neq 1 \),
  \[
  \| \partial_x^k V(t) \|_2 \leq C (1 + t)^{-1/12 - k/6} \| V_0 \|_1 + C(1 + t)^{-l/2} \| \partial_x^{k+l} V_0 \|_2 , \quad (78)
  \]

where \( k \) and \( l \) are non-negative integers satisfying \( k + l \leq s \), and \( C \) and \( c \) are two positive constants.
Taking the Fourier transform of (75), we obtain

\[ \hat{\nu}_t - i \xi \hat{u} + \hat{\gamma} = 0, \]  
\[ \hat{u}_t - i \xi \hat{\nu} = 0, \]  
\[ \hat{z}_t - a i \xi \hat{\gamma} = 0, \]  
\[ \hat{\gamma}_t - a i \xi \hat{z} - \hat{\nu} + \delta i \xi \hat{\theta} = 0, \]  
\[ \hat{\theta}_t + \xi^2 \hat{\theta} + \delta i \xi \hat{\gamma} = 0, \] 

with the initial data written in terms of \( \hat{V}(\xi, t) = (\hat{\nu}, \hat{u}, \hat{z}, \hat{\gamma}, \hat{\theta})^T \):

\[ \hat{V}(\xi, 0) = \hat{V}_0(\xi, 0) = (\hat{\nu}_0, \hat{u}_0, \hat{z}_0, \hat{\gamma}_0, \hat{\theta}_0)^T. \] 

**Proposition 2**

Let \( \hat{V}(\xi, t) \) be the solution of (79)-(80). Then, there exist two positive constants, \( \check{C} \) and \( \check{c} \), such that, for all \( t \geq 0 \), the following estimates hold true:

if \( a = 1 \),

\[ \left| \hat{V}(\xi, t) \right|^2 \leq \check{C} \left| \hat{V}(\xi, 0) \right|^2 e^{-c e_1(\xi)} , \quad e_1(\xi) = \frac{\xi^6}{(1 + \xi^2)(1 + \xi^2 + \xi^4)}; \]  

if \( a \neq 1 \),

\[ \left| \hat{V}(\xi, t) \right|^2 \leq \check{C} \left| \hat{V}(\xi, 0) \right|^2 e^{-c e_2(\xi)} , \quad e_2(\xi) = \frac{\xi^6}{(1 + \xi^2)^2(1 + \xi^2 + \xi^4)}. \]
We proved here that there is essentially no difference between the Timoshenko-Cattaneo model and the Timoshenko-Fourier model in terms of decay rates and regularity assumptions on the initial data. In the regularity-loss case and in both systems, the decay rate is obtained only for low-order derivatives, but not for all derivatives up to order $s$. Indeed, for the Cattaneo model, we see from (37) that our estimates can be obtained for all $k$ satisfying $0 \leq k \leq [s/2 − 1/4] = s_1$. For the Fourier model, we find from (78) that our estimates hold for $0 \leq k \leq [3s/4 − 1/8] = s_2$. We see that $s_1 < s_2$ which is the only difference between the two models.
By comparing our present result with other known results on this problem, we see that in bounded domains and for appropriate assumptions on the coefficients, the following models all lead to an exponential decay rate:

- only linear mechanical damping $\psi_t$ present as in (10).
- only viscoelastic damping present, i.e., $\int_0^t g(s)\psi_{xx}(t-s)ds$ instead of $\psi_t$ in (10) with an exponentially decaying $g(s))$;
- Fourier heat conduction with no mechanical damping.
- Cattaneo heat conduction with no mechanical damping.

However, this is not true for the Cauchy problem. Indeed, with frictional damping as in (10), the decay rate of the $L^2$-norm is $(1 + t)^{-1/4}$ (Ide & Kawashima), with the viscoelastic damping, the decay rate is $(1 + t)^{-1/8}$ (Liu & Kawashima), whereas as we have shown here, the decay rate with heat dissipation alone is $(1 + t)^{-1/12}$. As we see, the heat dissipation is slowest of all. The fact that both Fourier and Cattaneo models lead to the same decay rate smears the distinction between them that prior studies have highlighted.
As in Theorem 2, it can be shown that if one assumes that 
\[ \int_\mathbb{R} U_0(x) dx = 0, \]
then the decay rate in Theorem 5 can be improved. Nevertheless, the total decay rate is still polynomial. We recall that the assumption \( \int_\mathbb{R} U_0(x) dx = 0 \) is equivalent to \( \hat{U}_0(0) = 0 \). An exponential decay rate can be obtained by assuming that \( \hat{U}_0(\xi) = 0 \) for \( |\xi|^2 \leq \alpha_0 \), with some \( \alpha_0 > 0 \).

The decay rates can be further improved by assuming that the higher momenta of the initial data are zeros.
Thank you for your attention