Asymptotic stability for solitons of the Gross-Pitaevskii and Landau-Lifshitz equations

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Introduction

1. The Gross-Pitaevskii equation

The Gross-Pitaevskii equation writes as

$$i\partial_t \Psi + \Delta\Psi + \Psi(1 - |\Psi|^2) = 0,$$

for a function $\Psi : \mathbb{R}^N \times \mathbb{R} \to \mathbb{C}$.

The equation is Hamiltonian. Its Hamiltonian is the so-called Ginzburg-Landau energy, which is given by

$$E_{\mathrm{GP}}(\Psi) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla\Psi|^2 + \frac{1}{4}\int_{\mathbb{R}^N} (1 - |\Psi|^2)^2.$$

In the sequel, we restrict the analysis to the Hamiltonian framework of finite energy solutions.

The equation is also dispersive. Its dispersion relation is equal to

$$\omega_{\mathrm{GP}}^2 = |k|^4 + 2|k|^2.$$

The norm of the group velocity is at least equal to the sound speed $c_s = \sqrt{2}$.

When the map Ψ does not vanish, it may be lifted as

$$\Psi = \sqrt{1-\eta}\,\exp i\varphi.$$

The functions η and $v := -\nabla\varphi$ are solutions of the hydrodynamical Gross-Pitaevskii equation

$$\begin{cases} \partial_t \eta = -2\mathrm{div}\Big((1-\eta)v\Big), \\ \partial_t v = -\nabla\Big(\eta - |v|^2 - \frac{\Delta\eta}{2(1-\eta)} - \frac{|\nabla\eta|^2}{4(1-\eta)^2}\Big), \end{cases}$$

which is related to the compressible Euler equation.

2. The Landau-Lifshitz equation

The Landau-Lifshitz equation

$$\partial_t m + m \wedge \Big(\Delta m - \lambda m_3 e_3 \Big) = 0, \tag{LL}$$

describes the dynamics of the magnetization $m = (m_1, m_2, m_3) : \mathbb{R}^N \times \mathbb{R} \to \mathbb{S}^2$ in a ferromagnetic material.

The vector $e_3 = (0, 0, 1)$ and the parameter λ give account of the anisotropy of the material. In the sequel, we focus on the easy-plane anisotropy along the plane $x_3 = 0$ by setting

$$\lambda = 1.$$

The Landau-Lifshitz equation is Hamiltonian. Its Hamiltonian is the Landau-Lifshitz energy, which is equal to

$$E_{\mathsf{LL}}(m) := \frac{1}{2}\int_{\mathbb{R}^N} \Big(|\nabla m|^2 + m_3^2\Big).$$

The equation is also dispersive. The dispersion relation is given by the formula

$$\omega_{\mathsf{LL}}^2 = |k|^4 + |k|^2.$$

This relation is almost identical to the dispersion relation of the Gross-Pitaevskii equation.

When the map $\check{m} := m_1 + im_2$ does not vanish, it may be lifted as

$$\check{m} = \sqrt{1 - m_3^2}\, \exp i\varphi.$$

The functions $v := m_3$ and $w := \nabla\varphi$ solve the hydrodynamical Landau-Lifshitz equation

$$\begin{cases} \partial_t v = -\mathrm{div}\Big((1 - v^2)w\Big), \\ \partial_t w = -\nabla\Big(v - v|w|^2 - \dfrac{\Delta v}{1 - v^2} - \dfrac{v|\nabla v|^2}{(1 - v^2)^2}\Big). \end{cases} \tag{HLL}$$

which is similar to the hydrodynamical Gross-Pitaevskii equation.

3. Travelling-wave solutions

Travelling waves are special solutions of the form

$$m(x,t) = m_c(x_1 - ct, \ldots, x_N).$$

Their profile m_c is solution to the nonlinear elliptic equation

$$\Delta m_c + |\nabla m_c|^2 m_c + [m_c]_3^2 m_c - [m_c]_3 e_3 + c m_c \wedge \partial_1 m_c = 0.$$

In dimension $N = 1$, travelling-wave solutions are called dark solitons. They are explicitly given by the following formulae.

Lemma. *There exist non-constant solitons with speed c if and only if*

$$|c| < 1.$$

Up to translations, rotations around the axis x_3, and the orthogonal symmetry with respect to the plane $x_3 = 0$, they are given by the formulae

$$[m_c]_1(x) = \frac{c}{\cosh\left(\sqrt{1-c^2}x\right)}, \quad [m_c]_2(x) = \tanh\left(\sqrt{1-c^2}x\right),$$

and

$$[m_c]_3(x) = \frac{\sqrt{1-c^2}}{\cosh\left(\sqrt{1-c^2}x\right)}.$$

When $c \neq 0$, the soliton

$$m_c = \Big(\sqrt{1 - [m_c]_3^2} \cos(\varphi_c), \sqrt{1 - [m_c]_3^2} \sin(\varphi_c), [m_c]_3 \Big),$$

can be identified in the hydrodynamical formulation with the pair

$$Q_c := \big(v_c, w_c \big) = \big([m_c]_3, \partial_x \varphi_c \big),$$

where

$$v_c(x) = \frac{(1 - c^2)^{\frac{1}{2}}}{\cosh\big((1 - c^2)^{\frac{1}{2}} x\big)},$$

and

$$w_c(x) = \frac{c \, v_c(x)}{1 - v_c(x)^2}.$$

4. Integrability by the inverse scattering method

In dimension $N = 1$, the Gross-Pitaevskii equation (Zakharov-Shabat [73]) and the Landau-Lifshitz equation (Sklyanin [79]) are integrable by means of the inverse scattering method.

Given a solution Ψ of the Gross-Pitaevskii equation, the operators

$$L_\Psi = \begin{pmatrix} i(1+\sqrt{3})\partial_x & \overline{\Psi} \\ \Psi & i(1-\sqrt{3})\partial_x \end{pmatrix},$$

and

$$B_\Psi = \begin{pmatrix} -\sqrt{3}\partial_{xx}^2 + \frac{|\Psi|^2-1}{\sqrt{3}+1} & i\partial_x\overline{\Psi} \\ -i\partial_x\Psi & -\sqrt{3}\partial_{xx}^2 + \frac{|\Psi|^2-1}{\sqrt{3}-1} \end{pmatrix},$$

satisfy

$$\partial_t L_{\Psi(t)} = i[L_{\Psi(t)}, B_{\Psi(t)}].$$

The spectral characteristics of the operators $L_{\Psi(t)}$ are conserved along the flow. The solution $\Psi(t)$ can be recovered using the spectral characteristics of the operator $L_{\Psi(t)}$.

This provides an (at least formal) description of the long-time dynamics, which is governed by the propagation of a train of solitons plus a dispersive part (see Gérard-Zhifei Zhang [08] and Cuccagna-Jenkins [16]).

I. The Cauchy problem

1. In the original framework

The energy space is defined as

$$\mathcal{E}_{\mathrm{GP}}(\mathbb{R}) = \Big\{ \psi : \mathbb{R} \to \mathbb{C}, \text{ s.t. } \partial_x \psi \in L^2(\mathbb{R}) \text{ and } 1 - |\psi|^2 \in L^2(\mathbb{R}) \Big\}.$$

Theorem (Zhidkov [01]).
Let $\Psi^0 \in \mathcal{E}_{\mathrm{GP}}(\mathbb{R})$. There exists a unique global solution $\Psi \in \mathcal{C}^0(\mathbb{R}, \mathcal{E}_{\mathrm{GP}}(\mathbb{R}))$ to (GP) *with initial datum Ψ^0. The Ginzburg-Landau energy is conserved along the flow.*

(See Zhou-Guo [84], Sulem-Sulem-Bardos [86], Zhou-Guo-Tan [91], Chang-Shatah-Uhlenbeck [00], McGahagan [04] and Nahmod-Shatah-Vega-Zeng [06])

2. In the hydrodynamical framework

The non-vanishing space is defined as

$$\mathcal{NV}(\mathbb{R}) = \Big\{(\eta, v) \in H^1(\mathbb{R}) \times L^2(\mathbb{R}), \text{ s.t. } \max_{x \in \mathbb{R}} \eta(x) < 1\Big\}.$$

Theorem (Mohamad [12]).
Let $(\eta^0, v^0) \in \mathcal{NV}(\mathbb{R})$. There exists a maximal time $T_{\text{max}} > 0$ and a unique solution $(\eta, v) \in \mathcal{C}^0([0, T_{\text{max}}), \mathcal{NV}(\mathbb{R}))$ to (HGP) *with initial datum (η^0, v^0). The maximal time T_{max} is characterized by*

$$\lim_{t \to T_{\text{max}}} \max_{x \in \mathbb{R}} \eta(x, t) = 1 \quad \text{if} \quad T_{\text{max}} < +\infty.$$

The Ginzburg-Landau energy and the momentum are conserved along the flow.

(See de Laire-G. [15])

II. Orbital stability of (trains of) solitons

1. Minimizing nature of solitons

For $c \neq 0$, the soliton Q_c is a minimizer of the variational problem

$$E_{\mathrm{min}}(\mathfrak{p}_c) = \inf \Big\{ E_{\mathrm{LL}}(v), v : \mathbb{R} \to \mathbb{C} \text{ s.t. } P_{\mathrm{LL}}(v) = \mathfrak{p}_c \Big\}.$$

The speed c is related to the momentum $\mathfrak{p}_c$ by the formula

$$\mathfrak{p}_c = \arctan\Big(\frac{\sqrt{1-c}}{c} \Big).$$

The minimizing energy is equal to

$$E_{\mathrm{min}}(\mathfrak{p}_c) = 2\sqrt{1-c^2}.$$

2. Orbital stability of a well-prepared train of solitons

Let $\mathfrak{a} \in \mathbb{R}^N$, $\mathfrak{c} \in (-1, 1)^N$ and $\mathfrak{s} \in \{\pm 1\}^N$. A train of solitons is defined as a perturbation of a sum of solitons

$$S_{\mathfrak{c},\mathfrak{a},\mathfrak{s}}(x) := \sum_{j=1}^{N} s_j \, Q_{c_j}(x - a_j) = \sum_{j=1}^{N} s_j \Big(v_{c_j}(x - a_j), \, w_{c_j}(x - a_j) \Big).$$

The train is well-prepared when the speeds are ordered according to the positions, i.e. if

$$c_1 < \cdots < c_N,$$

when

$$a_1 < \cdots < a_N.$$

Theorem (de Laire-G. [15]).
Let $\mathfrak{c}^0 \in (-1,1)^N$, $\mathfrak{s}^0 \in \{\pm 1\}^N$, *and* $\mathfrak{v}^0 = (v^0, w^0) \in \mathcal{NV}(\mathbb{R})$ *such that*

$$c_1^0 < \cdots < 0 < \cdots < c_N^0.$$

There exist four positive numbers α^*, ν^*, A^* *and* L^*, *such that, if*

$$\left\| \mathfrak{v}^0 - S_{\mathfrak{c}^0,\mathfrak{a}^0,\mathfrak{s}^0} \right\|_{H^1 \times L^2} = \alpha^0 < \alpha^*,$$

for positions $\mathfrak{a}^0 \in \mathbb{R}^N$ *such that*

$$\min \left\{ a_{k+1}^0 - a_k^0, \ 1 \leq k \leq N-1 \right\} = L^0 > L^*,$$

then there exists a unique solution $\mathfrak{v} \in \mathcal{C}^0(\mathbb{R}_+, \mathcal{NV}(\mathbb{R}))$ *of* (HLL) *with initial datum* $\mathfrak{v}^0$*, as well as* N *functions* $a_k \in \mathcal{C}^1(\mathbb{R}_+, \mathbb{R})$*, with* $a_k(0) = a_k^0$*, such that*

$$\left| a_k'(t) - c_k^0 \right| \leq A^* \Big(\alpha^0 + e^{-\nu^* L^0} \Big),$$

and

$$\Big\| \mathfrak{v}(\cdot, t) - S_{\mathfrak{c}^0, \mathfrak{a}(t), \mathfrak{s}^0} \Big\|_{H^1 \times L^2} \leq A^* \Big(\alpha^0 + e^{-\nu^* L^0} \Big),$$

for any non-negative number t*.*

(See Zhiwu Lin [02], Béthuel-G.-Saut-Smets [08], Gérard-Zhifei Zhang [08], and Béthuel-G.-Smets [14])

The proof relies on arguments developed by Martel-Merle-Tsai [02, 06] in the spirit of the strategy by Weinstein [85] and Grillakis-Shatah-Strauss [87] for a single soliton.

III. Asymptotic stability of solitons

1. In the hydrodynamical framework

Theorem (Béthuel-G.-Smets [15]).
Let $0 < |c| < \sqrt{2}$ *and* $(\eta^0, v^0) \in \mathcal{NV}(\mathbb{R})$. *There exists a number* $\alpha^* > 0$ *such that, if*

$$\left\|\eta^0 - \eta_c\right\|_{H^1} + \left\|v^0 - v_c\right\|_{L^2} < \alpha^*,$$

then, there exist a unique solution $(\eta, v) \in \mathcal{C}^0(\mathbb{R}, \mathcal{NV}(\mathbb{R}))$ *to* (HGP) *with initial datum* (η^0, v^0), *as well as a speed* $0 < |c^*| < \sqrt{2}$ *and a position function* $a \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$ *such that*

$$\Big(\eta(\cdot + a(t), t), v(\cdot + a(t), t)\Big) \rightharpoonup Q_{c^*} \quad \text{in} \quad H^1(\mathbb{R}) \times L^2(\mathbb{R}),$$

and

$$a'(t) \to c^*,$$

as $t \to +\infty$.

(See Bahri [16, 16])

2. In the original framework

Theorem (Béthuel-G.-Smets [15], G.-Smets [15]).
Let $c \in (-\sqrt{2}, \sqrt{2})$ *and* $\Psi^0 \in \mathcal{E}(\mathbb{R})$. *There exists a number* $\delta^* > 0$ *such that, if*

$$d_c(\Psi^0, U_c) < \delta^*,$$

then, there exist a number $c^* \in (-\sqrt{2}, \sqrt{2})$, *and two functions* $a \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$ *and* $\theta \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$ *such that the solution* Ψ *to* (GP) *with initial datum* Ψ^0 *satisfies*

$$e^{-i\theta(t)}\Psi(\cdot + a(t), t) \to U_{c^*} \quad \text{in} \quad L^\infty_{\text{loc}}(\mathbb{R}),$$

$$e^{-i\theta(t)}\partial_x\Psi(\cdot + a(t), t) \rightharpoonup \partial_x U_{c^*} \quad \text{in} \quad L^2(\mathbb{R}),$$

$$a'(t) \to c^*,$$

and

$$\theta'(t) \to 0,$$

as $t \to +\infty$.

3. Sketch of the proof of asymptotic stability

The proof relies on arguments developed by Martel-Merle [00, 01, 05, 08] for the Korteweg-de Vries equation. It splits into three main steps :

1. constructing a limit profile,

2. establishing its smoothness and exponential decay,

3. proving that a smooth localized solution in the neighborhood of a soliton is a soliton.

Thank you very much !