Asymptotic stability for solitons of the
Gross-Pitaevskii and Landau-Lifshitz equations

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Introduction

1. The Gross-Pitaevskii equation

The Gross-Pitaevskii equation writes as

\[ i\partial_t \Psi + \Delta \Psi + \Psi(1 - |\Psi|^2) = 0, \]

for a function \( \Psi : \mathbb{R}^N \times \mathbb{R} \to \mathbb{C} \).

The equation is Hamiltonian. Its Hamiltonian is the so-called Ginzburg-Landau energy, which is given by

\[ E_{GP}(\Psi) := \frac{1}{2} \int_{\mathbb{R}^N} |\nabla \Psi|^2 + \frac{1}{4} \int_{\mathbb{R}^N} (1 - |\Psi|^2)^2. \]

In the sequel, we restrict the analysis to the Hamiltonian framework of finite energy solutions.
The equation is also dispersive. Its dispersion relation is equal to

\[ \omega_{\text{GP}}^2 = |k|^4 + 2|k|^2. \]

The norm of the group velocity is at least equal to the sound speed \( c_s = \sqrt{2} \).

When the map \( \Psi \) does not vanish, it may be lifted as

\[ \Psi = \sqrt{1 - \eta} \exp i\varphi. \]

The functions \( \eta \) and \( v := -\nabla \varphi \) are solutions of the hydrodynamical Gross-Pitaevskii equation

\[
\begin{cases}
\partial_t \eta = -2 \text{div} ((1 - \eta)v), \\
\partial_t v = -\nabla \left( \eta - |v|^2 - \frac{\Delta \eta}{2(1-\eta)} - \frac{|\nabla \eta|^2}{4(1-\eta)^2} \right),
\end{cases}
\]

which is related to the compressible Euler equation.
2. The Landau-Lifshitz equation

The Landau-Lifshitz equation

\[ \partial_t m + m \wedge (\Delta m - \lambda m_3 e_3) = 0, \]  

(LL)

describes the dynamics of the magnetization \( m = (m_1, m_2, m_3) : \mathbb{R}^N \times \mathbb{R} \to S^2 \) in a ferromagnetic material.

The vector \( e_3 = (0, 0, 1) \) and the parameter \( \lambda \) give account of the anisotropy of the material. In the sequel, we focus on the easy-plane anisotropy along the plane \( x_3 = 0 \) by setting \( \lambda = 1. \)
The Landau-Lifshitz equation is Hamiltonian. Its Hamiltonian is the Landau-Lifshitz energy, which is equal to

\[ E_{LL}(m) := \frac{1}{2} \int_{\mathbb{R}^N} \left( |\nabla m|^2 + m_3^2 \right). \]

The equation is also dispersive. The dispersion relation is given by the formula

\[ \omega_{LL}^2 = |k|^4 + |k|^2. \]

This relation is almost identical to the dispersion relation of the Gross-Pitaevskii equation.
When the map $\tilde{m} := m_1 + im_2$ does not vanish, it may be lifted as

$$\tilde{m} = \sqrt{1 - m_3^2} \exp i\varphi.$$ 

The functions $v := m_3$ and $w := \nabla \varphi$ solve the hydrodynamical Landau-Lifshitz equation

$$\begin{cases}
\partial_t v &= -\text{div}((1 - v^2)w), \\
\partial_t w &= -\nabla \left( v - v |w|^2 - \frac{\Delta v}{1 - v^2} - \frac{v|\nabla v|^2}{(1 - v^2)^2} \right). 
\end{cases} \quad (\text{HLL})$$

which is similar to the hydrodynamical Gross-Pitaevskii equation.
3. Travelling-wave solutions

Travelling waves are special solutions of the form

\[ m(x, t) = m_c(x_1 - ct, \ldots, x_N). \]

Their profile \( m_c \) is solution to the nonlinear elliptic equation

\[
\Delta m_c + \left| \nabla m_c \right|^2 m_c + [m_c]^3 m_c - [m_c]_3 e_3 + cm_c \wedge \partial_1 m_c = 0.
\]

In dimension \( N = 1 \), travelling-wave solutions are called dark solitons. They are explicitly given by the following formulae.
Lemma. There exist non-constant solitons with speed $c$ if and only if
\[ |c| < 1. \]

Up to translations, rotations around the axis $x_3$, and the orthogonal symmetry with respect to the plane $x_3 = 0$, they are given by the formulae
\[
[m_c]_1(x) = \frac{c}{\cosh(\sqrt{1 - c^2}x)}, \quad [m_c]_2(x) = \tanh(\sqrt{1 - c^2}x),
\]
and
\[
[m_c]_3(x) = \frac{\sqrt{1 - c^2}}{\cosh(\sqrt{1 - c^2}x)}.
\]
When $c \neq 0$, the soliton

$$m_c = \left( \sqrt{1 - [m_c]^2} \cos(\varphi_c), \sqrt{1 - [m_c]^2} \sin(\varphi_c), [m_c]_3 \right),$$

can be identified in the hydrodynamical formulation with the pair

$$Q_c := (v_c, w_c) = ([m_c]_3, \partial_x \varphi_c),$$

where

$$v_c(x) = \frac{(1 - c^2)^{\frac{1}{2}}}{\cosh((1 - c^2)^{\frac{1}{2}}x)},$$

and

$$w_c(x) = \frac{c v_c(x)}{1 - v_c(x)^2}.$$
4. Integrability by the inverse scattering method

In dimension $N = 1$, the Gross-Pitaevskii equation (Zakharov-Shabat [73]) and the Landau-Lifshitz equation (Sklyanin [79]) are integrable by means of the inverse scattering method.

Given a solution $\Psi$ of the Gross-Pitaevskii equation, the operators

$$L_\Psi = \begin{pmatrix} i(1 + \sqrt{3})\partial_x & \bar{\Psi} \\ \Psi & i(1 - \sqrt{3})\partial_x \end{pmatrix},$$

and

$$B_\Psi = \begin{pmatrix} -\sqrt{3}\partial_{xx} + \frac{|\Psi|^2 - 1}{\sqrt{3}+1} & i\partial_x \bar{\Psi} \\ -i\partial_x \Psi & -\sqrt{3}\partial_{xx} + \frac{|\Psi|^2 - 1}{\sqrt{3}-1} \end{pmatrix},$$

satisfy

$$\partial_t L_\Psi(t) = i[L_\Psi(t), B_\Psi(t)].$$
The spectral characteristics of the operators $L_{\Psi(t)}$ are conserved along the flow. The solution $\Psi(t)$ can be recovered using the spectral characteristics of the operator $L_{\Psi(t)}$.

This provides an (at least formal) description of the long-time dynamics, which is governed by the propagation of a train of solitons plus a dispersive part (see Gérard-Zhifei Zhang [08] and Cuccagna-Jenkins [16]).
I. The Cauchy problem

1. In the original framework

The energy space is defined as

\[ \mathcal{E}_{GP}(\mathbb{R}) = \{ \psi : \mathbb{R} \to \mathbb{C}, \text{s.t. } \partial_x \psi \in L^2(\mathbb{R}) \text{ and } 1 - |\psi|^2 \in L^2(\mathbb{R}) \}. \]

**Theorem (Zhidkov [01]).**

Let \( \Psi^0 \in \mathcal{E}_{GP}(\mathbb{R}) \). There exists a unique global solution \( \Psi \in C^0(\mathbb{R}, \mathcal{E}_{GP}(\mathbb{R})) \) to (GP) with initial datum \( \Psi^0 \). The Ginzburg-Landau energy is conserved along the flow.

(See Zhou-Guo [84], Sulem-Sulem-Bardos [86], Zhou-Guo-Tan [91], Chang-Shatah-Uhlenbeck [00], McGahagan [04] and Nahmod-Shatah-Vega-Zeng [06])
2. In the hydrodynamical framework

The non-vanishing space is defined as

$$\mathcal{NV}(\mathbb{R}) = \left\{ (\eta, v) \in H^1(\mathbb{R}) \times L^2(\mathbb{R}), \text{ s.t. } \max_{x \in \mathbb{R}} \eta(x) < 1 \right\}.$$ 

**Theorem (Mohamad [12]).**

Let $$(\eta^0, v^0) \in \mathcal{NV}(\mathbb{R})$$. There exists a maximal time $T_{\text{max}} > 0$ and a unique solution $$(\eta, v) \in C^0([0, T_{\text{max}}), \mathcal{NV}(\mathbb{R}))$$ to (HGP) with initial datum $$(\eta^0, v^0)$$. The maximal time $T_{\text{max}}$ is characterized by

$$\lim_{t \to T_{\text{max}}} \max_{x \in \mathbb{R}} \eta(x, t) = 1 \quad \text{if} \quad T_{\text{max}} < +\infty.$$ 

The Ginzburg-Landau energy and the momentum are conserved along the flow.

(See de Laire-G. [15])
II. Orbital stability of (trains of) solitons

1. Minimizing nature of solitons

For $c \neq 0$, the soliton $Q_c$ is a minimizer of the variational problem

$$E_{\text{min}}(p_c) = \inf \left\{ E_{\text{LL}}(v), v : \mathbb{R} \rightarrow \mathbb{C} \text{ s.t. } P_{\text{LL}}(v) = p_c \right\}.$$ 

The speed $c$ is related to the momentum $p_c$ by the formula

$$p_c = \arctan \left( \frac{\sqrt{1 - c^2}}{c} \right).$$

The minimizing energy is equal to

$$E_{\text{min}}(p_c) = 2\sqrt{1 - c^2}.$$
2. Orbital stability of a well-prepared train of solitons

Let $a \in \mathbb{R}^N$, $c \in (-1, 1)^N$ and $s \in \{\pm 1\}^N$. A train of solitons is defined as a perturbation of a sum of solitons

$$S_{c,a,s}(x) := \sum_{j=1}^{N} s_j Q_{c_j}(x - a_j) = \sum_{j=1}^{N} s_j \left( v_{c_j}(x - a_j), w_{c_j}(x - a_j) \right).$$

The train is well-prepared when the speeds are ordered according to the positions, i.e. if

$$c_1 < \cdots < c_N,$$

when

$$a_1 < \cdots < a_N.$$
**Theorem** (de Laine-G. [15]).

Let \( c^0 \in (-1, 1)^N \), \( s^0 \in \{\pm 1\}^N \), and \( v^0 = (v^0, w^0) \in \mathcal{N}_N(\mathbb{R}) \) such that

\[
c_1^0 < \cdots < 0 < \cdots < c_N^0.
\]

There exist four positive numbers \( \alpha^*, \nu^*, A^* \) and \( L^* \), such that, if

\[
\|v^0 - S_{c^0, a^0, s^0}\|_{H^1 \times L^2} = \alpha^0 < \alpha^*,
\]

for positions \( a^0 \in \mathbb{R}^N \) such that

\[
\min \left\{ a_{k+1}^0 - a_k^0, \; 1 \leq k \leq N - 1 \right\} = L^0 > L^*.
\]
then there exists a unique solution \( v \in C^0(\mathbb{R}_+, \mathcal{N}\mathcal{V}(\mathbb{R})) \) of (HLL) with initial datum \( v^0 \), as well as \( N \) functions \( a_k \in C^1(\mathbb{R}_+, \mathbb{R}) \), with \( a_k(0) = a_k^0 \), such that

\[
\left| a_k'(t) - c_k^0 \right| \leq A^* \left( \alpha^0 + e^{-\nu^*L^0} \right),
\]

and

\[
\| v(\cdot, t) - S_{c^0, a(t), s^0} \|_{H^1 \times L^2} \leq A^* \left( \alpha^0 + e^{-\nu^*L^0} \right),
\]

for any non-negative number \( t \).

(See Zhiwu Lin [02], Béthuel-G.-Saut-Smets [08], Gérard-Zhifei Zhang [08], and Béthuel-G.-Smets [14])

The proof relies on arguments developed by Martel-Merle-Tsai [02, 06] in the spirit of the strategy by Weinstein [85] and Grillakis-Shatah-Strauss [87] for a single soliton.
III. Asymptotic stability of solitons

1. In the hydrodynamical framework

Theorem (Béthuel-G.-Smets [15]).
Let $0 < |c| < \sqrt{2}$ and $(\eta^0, v^0) \in \mathcal{N} \mathcal{V}(\mathbb{R})$. There exists a number $\alpha^* > 0$ such that, if

$$\|\eta^0 - \eta_c\|_{H^1} + \|v^0 - v_c\|_{L^2} < \alpha^*,\$$

then, there exist a unique solution $(\eta, v) \in C^0(\mathbb{R}, \mathcal{N} \mathcal{V}(\mathbb{R}))$ to (HGP) with initial datum $(\eta^0, v^0)$, as well as a speed $0 < |c^*| < \sqrt{2}$ and a position function $a \in C^1(\mathbb{R}, \mathbb{R})$ such that

$$\left(\eta(\cdot + a(t), t), v(\cdot + a(t), t)\right) \to Q_{c^*} \text{ in } H^1(\mathbb{R}) \times L^2(\mathbb{R}),$$

and

$$a'(t) \to c^*,\$$

as $t \to +\infty$.

(See Bahri [16, 16])
2. In the original framework

**Theorem** (Béthuel-G.-Smets [15], G.-Smets [15]).

Let $c \in (-\sqrt{2}, \sqrt{2})$ and $\Psi^0 \in \mathcal{E}(\mathbb{R})$. There exists a number $\delta^* > 0$ such that, if

$$d_c(\Psi^0, U_c) < \delta^*,$$

then, there exist a number $c^* \in (-\sqrt{2}, \sqrt{2})$, and two functions $a \in C^1(\mathbb{R}, \mathbb{R})$ and $\theta \in C^1(\mathbb{R}, \mathbb{R})$ such that the solution $\Psi$ to (GP) with initial datum $\Psi^0$ satisfies

$$e^{-i\theta(t)} \psi(\cdot + a(t), t) \to U_{c^*} \quad \text{in} \quad L^\infty_{\text{loc}}(\mathbb{R}),$$

$$e^{-i\theta(t)} \partial_x \psi(\cdot + a(t), t) \to \partial_x U_{c^*} \quad \text{in} \quad L^2(\mathbb{R}),$$

$$a'(t) \to c^*,$$

and

$$\theta'(t) \to 0,$$

as $t \to +\infty$. 
3. Sketch of the proof of asymptotic stability

The proof relies on arguments developed by Martel-Merle [00, 01, 05, 08] for the Korteweg-de Vries equation. It splits into three main steps:

1. constructing a limit profile,

2. establishing its smoothness and exponential decay,

3. proving that a smooth localized solution in the neighborhood of a soliton is a soliton.
Thank you very much!