Faceting space(time): Regge’s view of geometry

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Curved surfaces as ‘simple’ models of curved spacetimes in Einstein’s General Relativity
The curvature of a *generic* smooth surface is perceived through its embedding into the 3D Euclidean space.

Looking at different regions three types of Gauss model geometries can be recognized:

- The **Euclidean plane** is flat, i.e. its curvature is zero.
- The **saddle surface** (negative Gauss curvature).
- The **surface of a sphere** (positive Gauss curvature).
Principal curvatures are defined through ‘extrinsic properties’ of the surface, which is bent as seen in the ambient 3D space.

Glimpse definition
In every point consider the tangent plane and the normal vector to the surface. (Any pair of) normal planes intersect the surface in curved lines. By resorting to the notion of osculating circle, the curvature of these embedded curves is evaluated (in the point). CASES:
• $> 0$ and equal to $1/r$
• $< 0$ and equal to $-1/r$
• $= 0$

$r$: radius of the osculating circle
(Th.) There are only two distinct and mutually orthogonal principal directions in each point of an embedded surface, or: every direction is principal

Principal curvatures (modulus)

\[ K_1 = \frac{1}{r_1} \]
\[ K_2 = \frac{1}{r_2} \]

\((r_1, r_2 : \text{radii of the osculating circles})\)

Saddle surface: the principal curvatures have opposite sign

\[ K_1 = + \frac{1}{r_1} \quad K_2 = - \frac{1}{r_2} \]

Sphere of radius \(r\):

\[ K_1 = K_2 = \frac{1}{r} > 0 \]

All principal curvatures are equal in each point

Plane: limiting case of the sphere

\[ r \to \infty \quad (K_1 = K_2 = 0) \]
Gauss curvature & the *theorema egregium*

**Definition**: in every point $p$ of a smooth surface $S$ the Gauss curvature is the **product** of the two principal curvatures:

$$K(p) = K_1(p) \cdot K_2(p) \quad [\text{dimension: } 1/\text{area}]$$

*Th egregium*: the Gauss curvature of a surface $S$ actually depends only on an **intrinsic** quantity, the first fundamental form or **metric** of $S$, and its first and second derivatives.

The **total Gauss curvature** of $S$ (compact) is the integral over $S$ of the local Gauss curvature $K(p)$

$$K(S) = \int K(p) \, d\sigma$$

$d\sigma$ is the area element (first fundamental form), to be expressed in suitable parametric coordinates $(u,v)$, with $K(p)=K(u,v)$ a smooth function.
Ex: evaluation of the total Gauss curvature of a sphere $S$ of radius $r$

- Don’t mind of the equation of the sphere in the Euclidean space $(x,y,z)$
- Just recall that all directions are principal and

$$K(p) = \frac{1}{r} \left( \frac{1}{r} \right)$$ for every $p$ in $S$

In the previous formula $K(S) = \int K(p) \, d\sigma$:

- $K(p)$ is a constant in each $p$ of $S$
- The area $\int d\sigma$ amounts to: $4\pi \times$ (square of the radius $r$)

Then $K(S) = 4\pi$ (independent of $r$)
The geometry of surfaces and a perspective on Riemannian (or Einstein-type) geometries

- The **total** Gaussian curvature of any (connected, compact, oriented) surface is a **topological invariant**, *i.e.* does not depend either on local metric details (*cf.* the ex. of the sphere). This is the content of the Gauss-Bonnet theorem (the sphere $S$ has no holes)
  \[
  \text{Total Gaussian curvature} = 2\pi [2 - 2 \text{ (number of holes)}].
  \]

- **Message from Th. Egregium:** Riemannian manifolds of any dimension $D$ are spaces endowed with an intrinsic metric (metric tensor) from which the ‘curvature’ can be evaluated at each point **with no need of embeddings** them into ‘ambient’ spaces

- Einstein’s spacetimes of General Relativity are 4-dimensional (pseudo) Riemannian manifolds where the analog of Gauss curvature $K$ are tensorial quantities, the sectional curvatures ($\ldots$). Very roughly:

- **Einstein’s equations:** spacetime curvature $= \text{content of mass-energy}$
Such Gauss and Riemann geometries seem so complicated to be defined and studied: is there any simplest way to visualize and evaluate the curvature, of course in an intrinsic way?

Regge’s view of geometry:
move from ‘smooth’ Riemannian geometric objects to deal with suitable ‘discretizations’ or approximations

Too drastic?
tetrahedron

octahedron

cube

icosahedron

dodecahedron

...then have a look here
Platonic solids and the geometry of surfaces

A Platonic polyhedron (or solid) is a portion of the Euclidean 3-space bounded by a collection of regular polygons of a same type. Each edge shares 2 polygons and in each vertex constant numbers of polygons and edges concur.

The collections of polygons bounding all of the Platonic solids are regular tessellations of the sphere $S$, the ‘model surface’ uniquely characterized by total Gaussian curvature $= 4\pi$.

Check: (1) (combinatorial Gauss-Bonnet formula)
(2) Direct evaluation through Regge’s prescription→→
“Heisenberg amava distinguere tra fisici democritei e fisici platonici sostenendo che i primi cercano di ricostruire le simmetrie nascoste nelle particelle e la teoria unificata tramite l’accurato esame di tutti i dati empirici in loro possesso, i secondi fanno invece discendere la verità da alti principi teorici.

Sotto questo aspetto, Fermi era un democriteo e Einstein un platonico.

La distinzione non è netta. I grandi fisici hanno partecipato ad ambedue le nature. [nota al testo: “Riconosco un fondo di verità in quanto asseriva Heisenberg, anche perché la fisica rimane sempre più divisa tra sperimentali e fenomenologi da un lato, e fisici di stampo teorico-matematico dall’altro, ma la separazione non riscuote le mie simpatie quando è sintomo di scarsa comunicazione e ristrettezza di orizzonti”]

Se devo scegliere mi dichiarerò platonico

If I have to choose, I shall be Platonic
General Relativity without Coordinates.

I. Krueger

Department of Mathematics, Brown University, Providence, R.I.

(Received 17 October 1950)

Summary. — In this paper we develop an approach to the theory of Riemannian manifolds which avoids the use of coordinates. General spaces are approximated by finite-dimensional spaces of 'polyhedra'.

Hence the achievement of the procedure we may lift the possibility of converting into a simplified model the essential features of geometry like Wheeler's wormhole and a deeper geometrical insight.

1. - Polyhedra.

In this section we shall first describe our approach for the simple case of 4-dimensional manifolds (surfaces). Following Alexandroff (1) we develop the theory of intrinsic curvature in polyhedra. A general surface is then considered as the limit of a suitable sequence of polyhedra with an increasing number of faces. A rigorous definition of limit is not given here since it would involve a treatment of the topology of the space of all polyhedra and this would carry us too far. It is to be expected, however, that any surface can be arbitrarily approximated, as closely as desired, by a suitable polyhedron. The approximation will be such if we look at the details in the picture but an observer looking at the inside details only will find it quite satisfactory. On any surface we can define an integral Gaussian curvature by carrying out curvature experiments with geodesic triangles.

Let $s$ be the sum of a triangle and let $a, b, c$ be its internal angles. If the geometry inside the triangle is not constant we have in general $s = a + b + c - \pi$.

A. Is a discretization procedure of spaces (or even spacetimes): a D-dimensional Riemannian manifold is replaced by a collection of **D-dimensional blocks**, whose intrinsic geometry (metric) is Euclidean (flat)

B. The overall geometry of any such ‘skeleton’ is characterized intrinsically by assigning:
   - The collection of the edge lengths of the blocks
   - The glueing rules for assembling the blocks

C. The curvature turns out to be ‘concentrated’ into sub-blocks of dimension (D-2), the *hinges* or *bones*
In Regge lattices (or spaces)

the elementary building blocks are simplices

O-simplex (vertex)
1-simplex (edge)
2-simplex (triangle)
3-simplex (solid tetrahedron)
(4-simplex, ecc.)

- in order to get rigid dissections \((\text{item B.})\) → →
- then the collection of the edge lengths suffices to specify the intrinsic geometry (no need of independent assignments of angles)
- As for glueing rules: \textit{cf.} M Carfora’s talk
In Regge lattices

the simplicial blocks are not equilateral, in general: they are suitable to model complicated geometries.

**NB** The edge lengths replace the metric tensor encoding the degrees of freedom of the ‘gravitational field’

DYNAMICAL LATTICES
Regge calculus

does not provide in a straightforward way the analog of Einstein’s eqs.

• The basic quantity is the Regge action $I_R$, representing the total curvature derived from edge lengths and angles (the latter computed in terms of lengths), see item A.

• (The action functional, according to the Lagrangian formulation in classical field theory, gives the field equations upon applying Hamilton’s variational principle)

• Recall now the last prescription:

C. The curvature turns out to be ‘concentrated’ into sub-simplices of dimension $(D-2)$, the hinges or bones
The Regge action in 2D

BUILDING BLOCKS: triangles (Euclidean 2-simplices)

BONES: vertices (0-simplices) : \( V \)

\[
\mathbf{I}_R (v) = \varepsilon (v) = 2 \pi - \sum_{\text{tri}} \alpha_{\text{tri}}
\]

\[
\mathbf{I}_R (2D\text{-lattice}) = \sum_v \varepsilon (v)
\]

\( \varepsilon (v) \): deficit angle of the vertex \( v \); \( \alpha \) is the angle between the two edges of the triangle ‘tri’ that share vertex \( v \)
Ex: Regge action for the (surface of the) regular tetrahedron

In each $v$ three equilateral triangles

$$I_R (v) = \varepsilon (v) = 2\pi - \sum_{\text{tri}} a_{\text{tri}}$$

$$= 2\pi - 3 \left(\pi/3\right) = \pi$$

$$I_R (\text{tetrahedron}) = \sum_v \varepsilon (v) = 4\pi$$

(the total Gaussian curvature of the sphere)
Still Regge action in 2D

\[ I_R (v) = \varepsilon (v) = 2 \pi - \sum_{\text{tri}} \alpha_{\text{tri}} \]

\( \varepsilon (v) \), the deficit angle at \( v \), can be positive, as before, or null, or negative

At the inner vertex: the sum over the angles is \( > 2 \pi \) and then \( \varepsilon (v) < 0 \)

Regular triangulation of the plane: \( \varepsilon (v) = 0 \)
Regge action in 3D

BUILDING BLOCKS: solid tetrahedra (Euclidean 3-simplices)

BONES: edges (1-simplices)

\[ \epsilon (l) = 2 \pi - \sum_{\text{tetra}} \beta_{\text{tetra}} \]

\[ I_R (3D\text{-lattice}) = \sum_l \epsilon (l) |l| \]

3D Regge action is a weighted sum of deficit angles, the weights being the edge lengths of the bones

|l| is the length of the edge l

\( \epsilon (l) \) is the deficit angle of the edge l; \( \beta \) is the dihedral angle between the two triangles in the tetrahedron ‘tetra’ that share edge l.
Ex) Solid regular tetrahedra do not fill Euclidean 3D space: there are gaps

4 + 1 tetrahedra joined at a common edge; the fifth one (grey edge) does not close up the configuration; at this edge a deficit angle $\epsilon > 0$ is detected
A brief history of Regge Calculus

- After his 1961 paper Tullio Regge did not work anymore on Regge calculus, with two exceptions
  - Giorgio Ponzano e T Regge, Semiclassical limit of Racah coefficients (*Racah memorial volume*, 1968) (see below)
  - T Regge e Ruth Williams *Discrete structures in gravity*, arXiv:gr-qc/0012035 (review)
- S Hawking 1978 *Space-time foam*: idea of using Regge calculus to deal with ‘quantum’ spacetimes
- Since 1981, up to now: ‘Simplicial’ quantum gravity and many other discretized models; extensions & connections with statistical mechanics, geometric topology, etc.
Regge 3D geometry out of the quantum theory of angular momentum

Roger Penrose:
Angular momentum: an approach to Combinatorial Spacetimes (1971)
Spin networks as quantum substrata for emergent spacetime geometry

Independently:
Ponzano and Regge (1968), by exploiting the tetrahedral symmetry of the Racah-Wigner 6j symbol, proved that Regge geometry emerges in the semiclassical limit
The Ponzano-Regge formula (1968)

\[
\begin{bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{bmatrix} \sim \frac{1}{\sqrt{24\pi V}} \cos \left( \sum_{i=1}^{6} J_i \theta_i + \frac{\pi}{4} \right)
\]

- $V$ is the Euclidean volume of the tetrahedron
- $J_i = j_i + \frac{1}{2} (\hbar = 1)$
- $\theta_i$ is the angle between the outer normals to the faces which share the edge $J_i$
‘Semiclassical’ or asymptotic limit:
all entries of the $6j$ symbol become $>> 1$

Angular momentum variables ↔ edge lengths of a solid tetrahedron, or Euclidean 3-simplex
Interpretations

• The asymp. \{6j\} is a semiclassical, WKB-type wavefunction and as such it includes
  a slowly varying amplitude
  a phase (the argument of cos)

  But

  the phase is the Regge action for the simplest 3D lattice, the tetrahedron itself. Then

• The asymp. \{6j\}, written in terms of \(\exp\) of the phase, represents a semiclassical ‘partition function’ of a discretized simplicial 3D geometry which ‘emerges’ at the classical level, when all Js are large.
Conclusions

• The second interpretation (mentioned as an aside remark in Ponzano & Regge’s paper) is the origin of longstanding interest for the machinery of the quantum theory of angular momentum in gravitational physics.

• Out of atomic and molecular physics, other applications (and interpretations) involving the $6j$ symbol and the Ponzano-Regge result range from geometric topology and special function theory up to quantum computing, cf. M Carfora, A Marzuoli, M Rasetti, *Quantum Tetrahedra* J. Phys. Chem. A 113 (2009) 15376 (V Aquilanti *Festschrift*).
Il quadrisimplesso

Politopo regolare in 4 dimensioni, analogo del tetraedro in 3D
← ← Dal suo grafo si riconoscono:
5 vertici (0-simplessi)
10 lati (1-simplessi)
10 facce triangolari (2-simplessi)
5 tetraedri (3-simplessi)

Realizzazione geometrica:
prendere un tetraedro, aggiungere un vertice ‘nella quarta dimensione’, congiungere il quinto vertice con quelli originari generando così gli altri 4 tetraedri
Azione di Regge per reticoli 4D

BLOCCHI: 4-simplessi euclidei

CARDINI: triangoli (2-simplessi) : $T$

$$\varepsilon \left( T \right) = 2 \pi - \sum_{\text{4sim}} \theta_{\text{4sim}}$$

$$\mathbf{I}_R \left( \text{reticolo 4D} \right) = \sum_T \varepsilon \left( T \right) \mid T \mid$$

In 4D l’ azione di Regge è la somma degli angoli di deficit, attribuiti ai triangoli $T$, ‘pesati’ con l’ area del rispettivo triangolo, $\mid T \mid$ (esprimibile in termini dei quadrati dei suoi lati, cf. la formula di Erone)

$\varepsilon \left( T \right)$ è l’ angolo di deficit del triangolo $T$; $\theta$ è l’ angolo (iperdiedrale) formato dai due tetraedri del 4-simplesso ‘4sim’ che si incontrano nel triangolo $T$