Sampling type methods for an inverse wave guide problem in the frequency domain and in the time domain

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PART I:

Sampling type methods

for an inverse wave guide problem

in the frequency domain (FD)
Problem
Detect and locate bounded inhomogeneous obstacles in an infinite tubular wave guide, given measurements of pressure waves due to point sources inside the wave guide

Goal
Non-destructive testing of pipes
- A potential application of acoustic techniques for the inspection of underground pipes such as sewers
  *S.N. Chandler-Wilde brought this to our attention!*
- An acoustic source and an array of microphones are lowered into the sewer and then used to collect scattering data from the mouth of the pipe
  *a whole array of transducers may be lowered into the sewer to collect multistatic data!*
State of the art

- Engineers have suggested using acoustic means to inspect sewers (but have not considered multistatic data)
  [F. Podd, M. Ali, K. Horoshenkov, et al.,

- Several authors have considered the use of time reversal methods for small inclusions in wave guides

- Sampling type techniques we will apply:
  - Linear Sampling Method (LSM)
  - Reciprocity Gap Method (RGM)

- LSM for infinite sound-hard acoustic wave guides (i.e. pipes) and sound-soft obstacle
Cross-section of the wave guide: 
\( \Sigma \subset \mathbb{R}^2 \) bounded, smooth and simply connected

Infinite tubular wave guide: 
\( P := \mathbb{R} \times \Sigma \subset \mathbb{R}^3 \)

Penetrable obstacle: 
\( D \subset P \) bounded, with connected complement \( \Omega := P \setminus \overline{D} \) and with boundary \( \partial D \)

We identify each point \( x \in \mathbb{R}^3 \) with \( (x_1, \hat{x}) \in \mathbb{R} \times \mathbb{R}^2 \)
Forward problem: equations

Find $u$ and $v$ such that

$$
\begin{align*}
\triangle u + k^2 u &= 0 \quad \text{in } \Omega \\
\triangle v + k^2 n v &= 0 \quad \text{in } D \\
v - u &= u^i \quad \text{across } \partial D \cap P \\
\partial_{\nu_D} v - \partial_{\nu_D} u &= \partial_{\nu_D} u^i \quad \text{across } \partial D \cap P \\
\partial_{\nu} u &= 0 \quad \text{on } \partial P \setminus \partial D \\
\partial_{\nu} v &= 0 \quad \text{on } \partial P \cap \partial D
\end{align*}
$$

- The total field is $u^t = u^i + u$ in $\Omega$ and $v$ in $D$, being $u^i$ the incident field (known) and $u$ the scattered field (unknown).

- The inhomogeneity is represented by the coefficient $n \in C(\overline{D})$ with $\Re(n) \geq C > 0$ and $\Im(n) \geq 0$ in $D$.

- A further radiation condition on $u$ is required for uniqueness!
Forward problem: radiation condition

Idea
The scattered wave $u$ must propagate away from the inhomogeneity $D$

Formalization: ingredients...

- **cross-sectional modes**
  - let $k_n^2$ be an eigenvalue of the Neumann problem for $-\Delta$ on $\Sigma$, and $\theta_n$ an associated eigenfunction
  - take $k_n \in [0, +\infty)$ sorted so that $k_n \nearrow +\infty$, and $\{\theta_n\}_{n \in \mathbb{N}}$ forming an orthonormal basis of $L^2(\Sigma)$

- **wave guide modes**
  \[ g_n^{\pm}(x_1, \hat{x}) := \theta_n(\hat{x}) e^{\pm i \beta_n x_1} \]
  where we choose $\beta_n := \sqrt{k^2 - k_n^2} \in \mathbb{C}$ with $\Re(\beta_n), \Im(\beta_n) \geq 0$

Remark
- $\Delta g_n^{\pm} + k^2 g_n^{\pm} = 0$ in $P$ and $\partial_{\nu} g_n^{\pm} = 0$ on $\partial P$
- there are a finite number of propagating modes (i.e. $\Im(\beta_n) = 0$), and the remainder are evanescent modes (i.e. $\Im(\beta_n) > 0$)
Forward problem: radiation condition

- For $R > 0$ big enough, i.e. $\overline{D} \subset (-R, R) \times \Sigma$, we truncate the pipe by $P_R := (-R, R) \times \Sigma$, and consider the artificial walls $\Sigma_{\pm R} := \{\pm R\} \times \Sigma$

- The DtN operator $\mathcal{T}_{\pm R} : H^{1/2}(\Sigma_{\pm R}) \to \tilde{H}^{-1/2}(\Sigma_{\pm R})$ has the explicit expression

$$\mathcal{T}_{\pm R} g = \pm i \sum_{n \in \mathbb{N}} \beta_n \int_{\Sigma_{\pm R}} g \bar{\theta}_n dS \theta_n$$

It allows us to rewrite the radiation condition as

$$\mathcal{T}_{\pm R} u = \partial_{\nu_0} u \quad \text{on } \Sigma_{\pm R}$$
We assume that the *incident field* $u^i$ is a smooth solution of

$$\begin{cases}
\Delta u^i + k^2 u^i = 0 \quad &\text{in } P_{R_S} \\
\partial_\nu u^i = 0 \quad &\text{on } \partial P_{R_S} \cap \partial P
\end{cases}$$

for some $R_S \in (0, R)$ with $\bar{D} \subset (-R_S, R_S) \times \Sigma$

and being $P_{R_S} := (-R_S, R_S) \times \Sigma$
Forward problem

We consider $R > 0$ big enough, i.e. $\overline{D} \subset (-R, R) \times \Sigma := P_R$

We look for $u \in H^1(P_R \setminus \overline{D})$ and $v \in H^1(D)$ such that

\[
\begin{align*}
\Delta u + k^2 u &= 0 & \text{in } P_R \setminus \overline{D} \\
\Delta v + k^2 n \cdot v &= 0 & \text{in } D \\
v - u &= u^i & \text{on } \partial D \cap P_R \\
v - u &= u^i & \text{on } \partial D \cap \partial P \\
\partial_{\nu} u &= 0 & \text{on } \Sigma_{\pm R} \\
\partial_{\nu_0} u &= T_{\pm R} u
\end{align*}
\]

Result

Existence and uniqueness of solution... *up to a discrete set of frequencies!* (which we skip in the sequel)
Inverse problem: geometry

Two surfaces
- $\Sigma^S := \Sigma_{a_S} := \{a_S\} \times \Sigma$ (with sources)
- $\Sigma^M := \Sigma_{a_M} := \{a_M\} \times \Sigma$ (with receivers)

where
\[ -R < a_S < a_M < R \quad \text{and} \quad D \subset (a_M, R) \times \Sigma \]

Auxiliary sections
- $P^S_{PR} := (a_S, R) \times \Sigma$
- $P^M_{PR} := (a_M, R) \times \Sigma$

so that
\[ D \subset P^M_{PR} \subset P^S_{PR} \subset P_R \]
Inverse problem: point sources

For each source point $\mathbf{x}_0 \in \Sigma^S$, the incident field $u^i_{\mathbf{x}_0}$ is

$$u^i_{\mathbf{x}_0}(\mathbf{x}) := \Phi(\mathbf{x}_0, \mathbf{x}) \quad \text{with} \quad \mathbf{x} \in \mathbb{R} \times \Sigma$$

where $\Phi$ is the fundamental solution for the wave guide

$$\Phi(\mathbf{x}, \mathbf{y}) := \Phi_{\mathbf{x}}(\mathbf{y}) := - \sum_{n \in \mathbb{N}} \frac{e^{i \beta_n |x_1 - y_1|}}{2i \beta_n} \theta_n(\hat{x}) \theta_n(\hat{y})$$

for $\mathbf{x} = (x_1, \hat{x}), \mathbf{y} = (y_1, \hat{y}) \in \mathbb{R} \times \Sigma$ with $\mathbf{x} \neq \mathbf{y}$

For this incident field $u^i_{\mathbf{x}_0}$, the corresponding total field is

$$u^t_{\mathbf{x}_0} = u_{\mathbf{x}_0} + u^i_{\mathbf{x}_0} \quad \text{in} \quad P_R \setminus \overline{D} \quad \text{and} \quad v_{\mathbf{x}_0} \quad \text{in} \quad D$$
Inverse problem

Find the boundary of $D$

given $u_{x_0}^t(x)$ and $\partial_{\nu_0} u_{x_0}^t(x)$ for any

source points $x_0 \in \Sigma^S$

&

measurement points $x \in \Sigma^M$

Remark
In practice we will only have a finite amount of sources and receivers
We consider two qualitative methods:

- **Linear Sampling Method (LSM)**
  - ↑ it only uses measurements of $u_{x_0}$
  - ↓ it requires the fundamental solution of both the pipe and entire domain (i.e. pipe and manhole)

- **Reciprocity Gap Method (RGM)**
  - ↓ it uses the measurements of both $u^t_{x_0}(x)$ and $\partial_{\nu_0} u^t_{x_0}(x)$
  - ↑ it only requires the fundamental solution of the pipe

For the remainder of the talk we focus on the RGM!
Reciprocity Gap operator: function spaces

For any $B \subseteq P_R$, let

$$\mathbb{H}(B) := \left\{ u \in H^1(B) ; \  \triangle u + k^2 u = 0 \text{ in } B, \  \partial_\nu u = 0 \text{ on } \partial B \cap \partial P \right\}$$

and, if $\Sigma_R \subset \partial B$,

$$\mathbb{H}_R(B) := \left\{ u \in \mathbb{H}(B) ; \  \partial_{\nu_0} u = T_R u \text{ on } \Sigma_R \right\}$$

We also define the space of total fields for point sources

$$\mathbb{U} := \left\{ u^{t}_{x_0} = u_{x_0} + u^{i}_{x_0} ; \  x_0 \in \Sigma^S \right\}$$
Reciprocity Gap operator

For \( x_0 \in \Sigma^S \) and \( v \in \mathbb{H}_R(P^S_R) \), with \( P^S_R := (a_S, R) \times \Sigma \),

- the \textit{RG operator} \( R : \mathbb{H}_R(P^S_R) \to L^2(\Sigma^S) \) satisfies

\[
Rv(x_0) := \mathcal{R}(u^{t}_{x_0}, v)
\]

- the \textit{RG bilinear form} is

\[
\mathcal{R}(u^{t}_{x_0}, v) := \int_{\Sigma^M} (u^{t}_{x_0} \partial_{\nu_0} v - \partial_{\nu_0} u^{t}_{x_0} v) \, dS
\]
Single Layer potentials

Idea

Recall the RG operator definition

\[ R\nu(x_0) := R(u^t_{x_0}, \nu) := \int_{\Sigma} \left( u^t_{x_0} \partial_\nu \nu - \partial_\nu u^t_{x_0} \nu \right) dS \]

How do we choose a suitable parametrization for the RG operator argument, \( \nu \in \mathbb{H}_R(P^S_R) \)?

We make use of single layer potentials using densities on yet another open surface

\[ \Sigma^{SL} := \Sigma_{a_{SL}} \quad \text{with} \quad a_{SL} \in (-R, a_S) \]
Single layer potential

\[ S_{\Sigma_{SL}} : \tilde{H}^{-1/2}(\Sigma^{SL}) \rightarrow \mathbb{H}_{\pm R}(P_R \setminus \Sigma^{SL}) \]

such that

\[ (S_{\Sigma_{SL}} g)(x) = \int_{\Sigma^{SL}} \Phi(x, y) g(y) \, dS_y \quad \text{for a.e. } x \in P_R \setminus \Sigma^{SL} \]

Single layer operator

\[ S_{\Sigma_{SL}} : \tilde{H}^{-1/2}(\Sigma^{SL}) \rightarrow H^{1/2}(\Sigma^{SL}) \]

such that

\[ (S_{\Sigma_{SL}} g)(x) = \int_{\Sigma^{SL}} \Phi(x, y) g(y) \, dS_y \quad \text{for a.e. } x \in \Sigma^{SL} \]
We show that

\[ S_{\Sigma SL} : \tilde{H}^{-1/2} (\Sigma^{SL}) \rightarrow H^{1/2} (\Sigma^{SL}) \]

- is linear and continuous
- defines an isomorphism

\[ S_{\Sigma SL} : \tilde{H}^{-1/2} (\Sigma^{SL}) \rightarrow \mathbb{H}_{\pm R} (P_R \setminus \Sigma^{SL}) \]

- provides a dense set of fields in \( \mathbb{H}_R (P_R^S) \) (when we skip mixed eigenvalues for the Laplacian in \( D \))
Reciprocity Gap Method

Location and reconstruction of $D$ by the RGM

Given $z \in P^M_R$

1. we find a regularized solution $f_z \in L^2(\Sigma^{SL})$ of the ill-posed integral equation

$$ (R \circ S_{\Sigma^{SL}}) f_z = R \Phi_z \quad \text{on } \Sigma^S $$

2. we compute the indicator function $1/\|f_z\|_{L^2(\Sigma^{SL})}$

Remark

In practice we fix many sampling points $z \in P^M_R$ and use the RGM
Justification of RGM

Location and reconstruction of $D$ by the RGM

Given $z \in P^M_R$

1. we find a regularized solution $f_z \in L^2(\Sigma^{SL})$ of the ill-posed integral equation

$$(R \circ S_{\Sigma^{SL}})f_z = R\Phi_z \quad \text{on } \Sigma^S$$

is $R \circ S_{\Sigma^{SL}}$ suitable for Tikhonov regularization?

2. we compute the indicator function $1/\|f_z\|_{L^2(\Sigma^{SL})}$

can this be used as an indicator function?
Justification of RGM: the ITP

Interior Transmission Problem (ITP)

Find $v_1 \in H^1(D)$, $v_2 \in H^1(D)$ such that

\[
\begin{cases}
\triangle v_1 + k^2 v_1 = 0 & \text{in } D \\
\triangle v_2 + k^2 n v_2 = 0 & \text{in } D \\
v_1 - v_2 = G & \text{across } \partial D \cap P_R \\
\partial_{\nu_D} v_1 - \partial_{\nu_D} v_2 = g & \text{across } \partial D \cap P_R \\
\partial_{\nu} v_1 = \partial_{\nu} v_2 = 0 & \text{on } \partial D \cap \partial P
\end{cases}
\]

for suitable data $g$ and $G$ on $\partial D \cap P_R$

Remark

- The ITP is involved in the RGM analysis
- This is a generalized ITP due to the mixed boundary conditions
We rewrite the ITP as a *mixed 4th order boundary valued problem* for
\[ w = v_1 - v_2 \]

Then\(^1\) \( w \) satisfies an additional natural boundary condition on the boundary \( \partial D \cap \partial P \), and we can show:

**Lemma**

- *If* \( \Re(n) > C > 1 \) *in* \( D \), *the ITP is well-posed for any* \( k \) *except for, at most, a discrete set of* \( k \) *values (called transmission eigenvalues)*
- *If* \( \Im(n) > 0 \) *there are no transmission eigenvalues*

---

\(^1\) Personal communication of F. Cakoni
First properties of the RG operator

We use this lemma and follow

[F.Cakoni, M.Cayoeren and D.Colton, Inv. Prob., 24 (2008), 065016]

to show:

**Lemma**

*If $k$ is not a transmission eigenvalue and $D \neq \emptyset$, the RG operator*

$$R : \mathbb{H}_R(P_S^R) \rightarrow L^2(\Sigma^S)$$

*is injective*
Justification of RGM: regularization

**Hypothesis**
We skip values of $k$ which are transmission eigenvalue or eigenvalues for the mixed Laplacian

**Lemma**
For any $-R < a_{SL} \leq a_S \leq a_M < R$, the operator

$$R \circ S_{\Sigma SL} : L^2(\Sigma^{SL}) \rightarrow L^2(\Sigma^S)$$

- is linear continuous
- is compact and injective
- has a dense range

**Consequence**
The operator $R \circ S_{\Sigma SL}$ is suitable for Tikhonov regularization!
Justification of RGM: the indicator function

**Hypothesis**
We skip values of $k$ which are transmission eigenvalue or eigenvalues for the mixed Laplacian.

**Theorem**

*For any $\varepsilon > 0$, there exists $f_{\varepsilon}^{z} \in L^2(\Sigma^{SL})$ with*

$$\left\| (R \circ S_{\Sigma^{SL}} f_{\varepsilon}^{z} - R\Phi_{z}(x_0)) \right\|_{L^2(\Sigma^S)} < \varepsilon$$

*and in addition*

(i) if $z \in D$:

$$\{ S_{\Sigma^{SL}} f_{\varepsilon}^{z}; \varepsilon > 0 \} \text{ converges in } L^2(\Sigma^{SL}) \text{ as } \varepsilon \to 0$$

(ii) if $z \in P_{R}^{M} \setminus D$:

$$\left\| f_{z}^{\varepsilon} \right\|_{L^2(\Sigma^{SL})} \to \infty \text{ for } \varepsilon \to 0$$

**Consequence**

$1/\left\| f_{z}^{\varepsilon} \right\|_{L^2(\Sigma^{SL})}$ can be used as an indicator function!
Forward synthetic data

- Pipe: a cylinder with radius 0.075m and the Ox₁ as axis of rotation
- Model equation: \textit{slightly more general!} 
  \[
  \text{div} \left( \frac{1}{\rho(x)} \nabla u \right) + \frac{\omega^2}{\rho(x)c(x)^2} u = 0
  \]
  with piecewise constant coefficients:
  - sound speed in the pipe: \( c = 343 \text{ m/s} \)
  - density of air: \( \rho = 1.2 \text{ kg/m}^3 \)
  - frequencies: \( f = 2\text{KHz}, 6\text{KHz}, 10\text{KHz} \) (then \( \omega = f/(2\pi) \))
- Approx. solution of the forward problem: the Ultra Weak Variational Formulation ([Cessenat and Després]) closed with the DtN map, over a fine tetrahedral grid

<table>
<thead>
<tr>
<th>( f )</th>
<th>( k = \omega/c )</th>
<th>( \lambda = 2\pi/k )</th>
<th># prop. modes</th>
<th># total modes</th>
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<tr>
<td>2</td>
<td>3.6637</td>
<td>1.71</td>
<td>3</td>
<td>38</td>
</tr>
<tr>
<td>6</td>
<td>10.9910</td>
<td>0.57</td>
<td>21</td>
<td>38</td>
</tr>
<tr>
<td>10</td>
<td>18.1530</td>
<td>0.2</td>
<td>53</td>
<td>77</td>
</tr>
</tbody>
</table>
Forward synthetic data

- Point sources: on \( x_1 = a_S := -4.1 \text{m} \), positioned at \( M_q \) Gauss-Jacobi points radially and \( N_q \) in angle

<table>
<thead>
<tr>
<th>wave number ( k )</th>
<th>( M_q )</th>
<th>( N_q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.6637 &amp; 10.9910</td>
<td>4</td>
<td>12</td>
</tr>
<tr>
<td>18.1530</td>
<td>6</td>
<td>20</td>
</tr>
</tbody>
</table>

- Measurement points: on \( x_1 = a_M := -4.0 \text{m} \)

we assume that the Fourier coefficients of the scattered field can be measured on \( \Sigma^M \)
Pipe with penetrable ball

Obstacle:
- a penetrable ball of radius 0.03m inside the pipe
- sound speed $c = 600\text{m/s}$ and density $\rho = 20\text{kg/m}^3$

An isosurface of the indicator function $1/\|f_z\|_{L^2(\Sigma^S)}$:

- Exact object
- Reconstruction at 10KHz
- Reconstruction at 6KHz
- Reconstruction at 2KHz
Pipe with penetrable ball: 6KHz

Contour plots of $1/||f_z||_{L^2(\Sigma SL)}$ on several cross-sections:

$z_1, z_3$ plane at $z_2 = 0$

$z_2, z_3$ plane at $z_1 = 0.2$

$z_2, z_3$ plane at $z_1 = 0$

$z_2, z_3$ plane at $z_1 = -0.2$
Pipe with penetrable ball: 10KHz

Contour plots of $\frac{1}{\|f_z\|_{L^2(\Sigma^SL)}}$ on several cross-sections:

$z_1, z_3$ plane at $z_2 = 0$

$z_2, z_3$ plane at $z_1 = 0.2$

$z_2, z_3$ plane at $z_1 = 0$

$z_2, z_3$ plane at $z_1 = -0.2$
Pipe with hard obstruction

**Hard obstruction:**
\[ \partial_n u = 0 \text{ on the surface of the obstruction} \]

An isosurface of the indicator function \( \frac{1}{\| f_z \|_{L^2(\Sigma^{SL})}} \):

- Exact object
- Reconstruction at 10KHz
- Reconstruction at 6KHz
Pipe with hard obstruction: 6KHz

Contour plots of $1/\|f_z\|_{L^2(\Sigma^SL)}$ on several cross-sections:

$z_1, z_3$ plane at $z_2 = 0$

$z_2, z_3$ plane at $z_1 = 0.2$

$z_2, z_3$ plane at $z_1 = 0$

$z_2, z_3$ plane at $z_1 = -0.2$
Pipe with hard obstruction: 10KHz

Contour plots of $1/||f_z||_{L^2(\Sigma_{SL})}$ on several cross-sections:

- $z_1, z_3$ plane at $z_2 = 0$
- $z_2, z_3$ plane at $z_1 = 0.2$
- $z_2, z_3$ plane at $z_1 = -0.2$
Pipe with penetrable obstruction

Obstacle:
- same parameters as for the penetrable ball

An isosurface of the indicator function $1/\| f_z \|_{L^2(\Sigma^{SL})}$:

- Exact object
- Reconstruction at 10KHz
- Reconstruction at 6KHz
Pipe with penetrable obstruction: 6KHz

Contour plots of $1/||f_z||_{L^2(\Sigma^SL)}$ on several cross-sections:

$z_1, z_3$ plane at $z_2 = 0$

$z_2, z_3$ plane at $z_1 = 0.2$

$z_2, z_3$ plane at $z_1 = 0$

$z_2, z_3$ plane at $z_1 = -0.2$
**Blocked pipe:**

- a semi-infinite sound hard domain \((-\infty, 0) \times \Sigma\)
- *not covered by our theory*

The method fails to detect the end of the pipe:

```
Exact object  Reconstruction at 6KHz
```
Remark

The failure detecting the end of the pipe is not caused by the wrong choice of isosurface!

Contour plots of $1/||f_z||_{L^2(\Sigma SL)}$ on several cross-sections:

- $z_1, z_3$ plane at $z_2 = 0$
- $z_2, z_3$ plane at $z_1 = 0.2$
- $z_2, z_3$ plane at $z_1 = 0$
- $z_2, z_3$ plane at $z_1 = -0.2$
Remark

*Evanescent modes help detecting the end of the pipe!*  

We move the measurement surface to $x_1 = -0.05 m$ (near field):

*Now the end of the pipe is detected!*
Remark

*Evanescent modes help detecting the end of the pipe!*

The end of the pipe is clearly indicated:

- $z_1, z_3$ plane at $z_2 = 0$
- $z_2, z_3$ plane at $z_1 = 0.192$
- $z_2, z_3$ plane at $z_1 = -0.192$
Comments and conclusions about the RGM in the FD

- The RGM is rapid and easy to implement
- The RGM can detect some scatterers in the pipe even laying the walls
- We need to understand the scatterers that the RGM fails to find

\(^2\)A modal solution of the blocked pipe problem confirms our observations concerning the blocked pipe (personal communication of L.Bourgeois)
PART II:

Sampling type methods

for an inverse wave guide problem

in the time domain (TD)
Problem
Detect and locate bounded obstacles in an infinite tubular wave guide, given measurements of causal waves in the time domain (TD) due to point sources inside the wave guide.

Goal
- A potential application of acoustic techniques for the inspection of underground pipes such as sewers. *thanks to S.N. Chandler-Wilde for bringing this to our attention!*
- An acoustic source and an array of microphones is lowered into the sewer and then used to collect scattering data from the mouth of the pipe. *a whole array of transducers may be lowered into the sewer to collect multistatic data... even in the TD!*
State of the art

- Previous work in the frequency domain:
  - LSM for an infinite sound hard acoustic wave guide and a sound soft obstacle at a fixed frequency
    [L. Bourgeois, E. Lunéville (2008)]
  - LSM and RGM for an infinite sound hard acoustic wave guide and a penetrable obstacle at a fixed frequency
    [P. Monk, V. Selgas (2012)] or PART I of this talk

- Now we want to work with the data in the TD, and not to transform them into the FD
Cross-section of the wave guide: \( \Sigma \subset \mathbb{R}^2 \) bounded, smooth and simply connected

Infinite tubular wave guide: 
\[ P := \mathbb{R} \times \Sigma \subset \mathbb{R}^3 \]

Impenetrable obstacle:
\[ D \subset P \text{ bounded, with connected complement } \Omega := P \setminus \overline{D} \]
and with Lipschitz boundary \( \partial D \)

We identify each point \( x \in \mathbb{R}^3 \) with \( (x_1, \hat{x}) \in \mathbb{R} \times \mathbb{R}^2 \)
Incident field

$u^i_y \equiv u^i_y(t, x)$ is a (regularized) causal incident field due to a source at $y \in P$ if

$$\begin{cases}
\partial_{tt}^2 u^i_y - \Delta u^i_y = \chi(t) \delta_y & \text{in } P, \text{ for } t > 0 \\
\partial_{\nu} u^i_y = 0 & \text{on } \partial P, \text{ for } t > 0 \\
u^i_y = \partial_t u^i_y = 0 & \text{in } P, \text{ for } t = 0
\end{cases}$$

being

- $\chi \in C_\infty^\infty([0, \infty))$ a given smooth function
- $\delta_y$ the Dirac delta function centered at $y$
Forward problem: equations

**Scattered field**

$u_y \equiv u_y(t, x)$ is the scattered field due to the causal incident field $u^i_y$ if

$$
\begin{cases}
\partial_{tt}^2 u_y - \Delta u_y = 0 & \text{in } \Omega, \text{ for } t > 0 \\
\partial_\nu u_y = 0 & \text{on } \partial P, \text{ for } t > 0 \\
u_y = -u^i_y & \text{on } \partial D, \text{ for } t > 0 \\
u_y = \partial_t u_y = 0 & \text{in } P, \text{ for } t \leq 0
\end{cases}
$$

**Remark**

- Boundary conditions:
  a sound-hard wave guide and a sound-soft impenetrable obstacle as in [L. Bourgeois, E. Lunéville (2008)] for the FD!
- No need of a radiation condition on $u$: $u^i$ is causal and the wave speed is finite
- Analysis of the forward problem:
  by means of *the* Fourier-Laplace transform
Fourier-Laplace transform

For any Hilbert space $X$ and $\sigma, p \in \mathbb{R}$, we take

- $\mathcal{D}'(\mathbb{R}, X)$ the space of $X$-valued distributions
- $\mathcal{S}'(\mathbb{R}, X)$ the space of $X$-valued tempered distributions
- $\mathcal{L}'_{\sigma}(\mathbb{R}, X) := \{ f \in \mathcal{D}'(\mathbb{R}, X); \; e^{-\sigma t} f(t) \in \mathcal{S}'(\mathbb{R}, X) \}$

Fourier-Laplace transform

If $f \in \mathcal{L}'_{\sigma}(\mathbb{R}, X)$ with $e^{-\sigma t} f(t) \in L^1(\mathbb{R}, X)$,

$$\mathcal{L}[f](s) := \int_{-\infty}^{\infty} e^{ist} f(t) \, dt \quad \text{for a.e. } s \in \mathbb{R} + i\sigma$$

Related Hilbert spaces

$$\mathcal{H}^p_{\sigma}(\mathbb{R}, X) := \left\{ f \in \mathcal{L}'_{\sigma}(\mathbb{R}, X); \int_{-\infty+i\sigma}^{\infty+i\sigma} |s|^{2p} \| \mathcal{L}[f](s) \|_X^2 \, ds < \infty \right\}$$

endowed with norm $\| f \|_{\mathcal{H}^p_{\sigma}(\mathbb{R}, X)} := \left( \int_{-\infty+i\sigma}^{\infty+i\sigma} |s|^{2p} \| \mathcal{L}[f](s) \|_X^2 \, ds \right)^{1/2}$
Forward problem in the Fourier-Laplace domain

**Strategy**
We take formally the Fourier-Laplace transform in time for each $s \in \mathbb{C}_\sigma$, being $\mathbb{C}_\sigma = \{ s \in \mathbb{C}; \Im(s) > \sigma \}$ and $\sigma > 0$ fixed.

**Incident field**

\[ U^{i}_{s,y} := \mathcal{L}[u^{i}_{y}](s) = \mathcal{L}[\chi](s) \Phi_{s,y} \]

**Remark**

$\Phi_{s,y} \equiv \Phi_{s}(x,y)$ is the fundamental solution in the Fourier-Laplace domain, i.e., it solves

\[
\begin{cases}
\Delta \Phi_{s,y} + s^2 \Phi_{s,y} = -\delta_{y} & \text{in } P \\
\partial_{\nu} \Phi_{s,y} = 0 & \text{on } \partial P \\
\text{a suitable radiation condition}
\end{cases}
\]

and *it can be written explicitly in terms of modes* in the wave guide.
Problem
The forward problem in the Fourier-Laplace domain is written in terms of $U_{s,y}(x) = \mathcal{L}[u_y](s,x)$ and for $F = -U_{s,y}^i|_{\partial D}$ as

$$
\begin{cases}
\Delta U_{s,y} + s^2 U_{s,y} = 0 & \text{in } \Omega \\
\partial_\nu U_{s,y} = 0 & \text{on } \partial P \\
U_{s,y} = F & \text{on } \partial D
\end{cases}
$$

Remark
We impose a suitable radiation condition as for the FD, i.e.

1st Truncate the pipe:
- fix $R > 0$ such that $D \subset P_R := (-R, R) \times \Sigma$
- denote $\Omega_R := P_R \setminus \overline{D}$ and $\Sigma_{\pm R} := \{\pm R\} \times \Sigma$
**Forward problem: equations**

**Problem**
The forward problem in the Fourier-Laplace domain is written in terms of $U_s,y(x) = \mathcal{L}[u_y](s,x)$ and for $F = -U^i_{s,y}|_{\partial D}$ as

$$
\begin{aligned}
\Delta U_{s,y} + s^2 U_{s,y} &= 0 \quad \text{in } \Omega \\
\partial_\nu U_{s,y} &= 0 \quad \text{on } \partial P \\
U_{s,y} &= F \quad \text{on } \partial D
\end{aligned}
$$

a suitable radiation condition

**Remark**
We impose a suitable radiation condition as for the FD, i.e.

1st Truncate the pipe

2nd Define $\mathcal{T}^s_{\pm R} : H^{1/2}(\Sigma_{\pm R}) \to \tilde{H}^{-1/2}(\Sigma_{\pm R})$ the DtN map on $\Sigma_{\pm R}$

As in the FD, we get an explicit expression of $\mathcal{T}^s_{\pm R}$ using a modal expansion in $(-\infty, -R) \times \Sigma$ and in $(R, \infty) \times \Sigma$

Problem
The forward problem in the Fourier-Laplace domain is written in terms of $U_{s,y}(x) = \mathcal{L}[u_y](s, x)$ and for $F = -U^i_{s,y}|_{\partial D}$ as

$$\begin{cases}
\Delta U_{s,y} + s^2 U_{s,y} = 0 & \text{in } \Omega_R \\
\partial_\nu U_{s,y} = 0 & \text{on } \partial P \cap \overline{\Omega_R} \\
U_{s,y} = F & \text{on } \partial D \\
\partial_{\nu_0} U_{s,y} = \mathcal{T}^s_{\pm R}(U_{s,y}|_{\Sigma_{\pm R}}) & \text{on } \Sigma_{\pm R}
\end{cases}$$
Objective
Analysis of the forward problem in the TD

Strategy
Step 1: Study the forward problem in the Fourier-Laplace domain. In particular, provide stability bounds of $U_{s,y}$ which make clear its dependence w.r.t. $s \in \mathbb{C}_\sigma$

Step 2: Deduce information about the forward problem in the TD
Step 1a
Find stability bounds for $T_{s}^{\pm R} : H^{1/2}(\Sigma_{\pm R}) \to \tilde{H}^{-1/2}(\Sigma_{\pm R})$ which make clear its dependence w.r.t. $s \in \mathbb{C}_{\sigma}$

Lemma

There exists a positive constant $C > 0$ such that

$$\|T_{s}^{\pm R}g\|_{\tilde{H}^{-1/2}(\Sigma_{\pm R})} \leq C \frac{|s|}{\min\{1, \Im(s)\}} \|g\|_{H^{1/2}(\Sigma_{\pm R})}$$

for any $s \in \mathbb{C}_{\sigma}$ and $g \in H^{1/2}(\Sigma_{\pm R})$

Proof
Using the explicit expression of $T_{s}^{\pm R}$ in terms of modes, and an equivalence of norms in $H^{1/2}(\Sigma_{\pm R})$ based on modes expansions
Step 1b
Find stability bounds for $\mathcal{T}_{\pm R}^s : H^{1/2}(\Sigma_{\pm R}) \to \tilde{H}^{-1/2}(\Sigma_{\pm R})$ in terms of \textit{weighted norms}

**Frequency dependent norms in** $H^1(\Omega_R)$

$$\|v\|_{1,s,\Omega_R} = \left( \int_{\Omega_R} (|s \mathbf{v}(\mathbf{x})|^2 + |\nabla \mathbf{v}(\mathbf{x})|^2) \, d\mathbf{x} \right)^{1/2}$$

**Frequency dependent norms in** $H^m(\Sigma_{\pm R})$ ($|m| < 1$)

$$\| \cdot \|_{m,s,\Sigma_{\pm R}} \cdots$$
Forward problem in the Fourier-Laplace domain

**Step 1b**
Find **stability bounds** for $\mathcal{T}^s_{\pm R} : H^{1/2}(\Sigma_{\pm R}) \to \tilde{H}^{-1/2}(\Sigma_{\pm R})$ in terms of weighted norms.

**Lemma**

*There exists a positive constant $C > 0$ such that*

$$\left| \langle \mathcal{T}^s_{\pm R}v_1|_{\Sigma_{\pm R}}, v_2|_{\Sigma_{\pm R}} \rangle_{\Sigma_{\pm R}} \right| \leq C \|v_1\|_{1,s,\Omega_R} \|v_2\|_{1,s,\Omega_R}$$

*for any $s \in \mathbb{C}_\sigma$ and $v_1, v_2 \in H^1(\Omega_R)$*

**Proof**

First, show suitable equivalences of weighted norms on $H^{1/2}(\Sigma_{\pm R})$ using modes expansions. Then, use here again the explicit expression of $\mathcal{T}^s_{\pm R}$ in terms of modes.
Step 1c
Find stability bounds for $U_s \in H^1(\Omega_R)$ in terms of weighted norms

**Proposition**
For any $s \in \mathbb{C}_\sigma$ and $F \in \tilde{H}^{1/2}(\partial D)$, the forward problem in the Fourier-Laplace domain is well-posed, and its unique solution $U_s \in H^1(\Omega_R)$ satisfies

$$\|U_s\|_{1,s,\Omega_R} \leq |s| \frac{C}{\sigma} \|F\|_{1/2,s,\partial D}$$

where $C$ is independent of $s \in \mathbb{C}_\sigma$

**Proof**
It follows by Lax-Milgram’s Lemma, thanks to the stability bounds of $\mathcal{T}_s^{\pm} : H^{1/2}(\Sigma_{\pm R}) \rightarrow \tilde{H}^{-1/2}(\Sigma_{\pm R})$ and $\cdot|_{\partial D} : H^1(\Omega_R) \rightarrow H^{1/2}(\partial D)$ in weighted norms
Forward problem in the time domain

Step 2
Deduce information about the forward problem in the TD out of that in the Fourier-Laplace domain

Weighted Sobolev spaces
For any $p, q, \sigma \in \mathbb{R}$ with $|q| < 1$, we denote

\[
\begin{align*}
H_{\sigma, \Omega_R}^{p,1} &:= H_{\sigma}^{p}(\mathbb{R}, H^{1}(\Omega_R)) \\
H_{\sigma, \partial D}^{p,q} &:= H_{\sigma}^{p}(\mathbb{R}, H^{q}(\partial D)) \\
H_{\sigma, \Sigma_{\pm R}}^{p,1/2} &:= H_{\sigma}^{p}(\mathbb{R}, H^{1/2}(\Sigma_{\pm R})) \\
\tilde{H}_{\sigma, \Sigma_{\pm R}}^{p,-1/2} &:= H_{\sigma}^{p}(\mathbb{R}, \tilde{H}^{-1/2}(\Sigma_{\pm R}))
\end{align*}
\]
Forward problem in the time domain

Step 2a
Formalize the DtN map in the TD

Lemma

There exists a linear continuous operator

$$\mathcal{T}_{\pm R} : H_{\sigma, \Sigma_{\pm R}}^{p+1, 1/2} \rightarrow \tilde{H}_{\sigma, \Sigma_{\pm R}}^{p, -1/2}$$

whose Fourier-Laplace transform is $\mathcal{T}_{\pm R}^s$ for a.e. $s \in \mathbb{C}_\sigma$

Proof
By Parseval's formula, since $\mathcal{T}_{\pm R}^s : H^{1/2}(\Sigma_{\pm R}) \rightarrow \tilde{H}^{-1/2}(\Sigma_{\pm R})$

- is analytic w.r.t. $s \in \mathbb{C}_\sigma$
- satisfies suitable stability bounds in terms of $s \in \mathbb{C}_\sigma$

Remark
By Paley-Wiener Theorem, we also know that $\mathcal{T}_{\pm R}$ preserves causality
Step 2b
Rewrite the problem in the TD in the truncated wave guide

Problem
Given any function $f : \mathbb{R} \times \partial D \to \mathbb{C}$, find $u : \mathbb{R} \times \Omega_R \to \mathbb{C}$ such that

$$
\begin{cases}
-\Delta u + \partial_{tt}^2 u = 0 & \text{in } \Omega_R, \text{ for } t > 0 \\
\partial_\nu u = 0 & \text{on } \partial P \cap \Omega_R, \text{ for } t > 0 \\
u = f & \text{on } \partial D, \text{ for } t > 0 \\
\partial_{\nu_0} u = T_{\pm R}(u|_{\Sigma_{\pm R}}) & \text{on } \Sigma_{\pm R}, \text{ for } t > 0 \\
u = \partial_t u = 0 & \text{in } \Omega_R, \text{ for } t \leq 0
\end{cases}
$$
Forward problem in the time domain

Step 2c
Analyze the problem in the TD in the truncated wave guide

**Proposition**

Let us fix a positive real number $\sigma_0 > 0$

For any $p, \sigma \in \mathbb{R}$ with $\sigma \geq \sigma_0$, and $f \in H_{\sigma_0, \partial D}^{p+1, 1/2}$, the forward problem in the TD for the truncated wave guide is well-posed, and its unique solution $u \in H_{\sigma, \Omega_R}^{p, 1}$ satisfies

$$\|u\|_{H_{\sigma, \Omega_R}^{p, 1}} \leq C \|f\|_{H_{\sigma_0, \partial D}^{p+1, 1/2}}$$

where $C > 0$ is some constant independent of $p, \sigma$ and $f$
Let us fix $\sigma_0 > 0$, and consider $p, \sigma \in \mathbb{R}$ with $\sigma \geq \sigma_0$

**Space of solutions in $\Omega_R$**

$$X^p_{\sigma,\Omega_R} = \left\{ v \in H^{p,1}_{\sigma,\Omega_R} \text{ such that for a.e. } t > 0 \right.$$  

$$\partial_{tt}^2 v - \Delta v = 0 \text{ in } \Omega_R, \quad \partial_{\nu} v = 0 \text{ on } \partial P \cap \Omega_R, \quad \partial_{\nu_0} v = T_{\pm R} v \text{ on } \Sigma_{\pm R} \right\}$$

**Solution operator**

$$S_{\Omega_R} : H^{p+1,1/2}_{\sigma_0,\partial D} \to X^p_{\sigma,\Omega_R}$$

We have seen that $S_{\Omega_R} : H^{p+1,1/2}_{\sigma_0,\partial D} \to X^p_{\sigma,\Omega_R}$

- is well-defined and linear continuous
- preserves causality (by Paley-Wiener Theorem)
Objective
Aproximate solutions by means of regularized retarded (TD) single layer potentials, defined on $\Sigma^{SL} := \{a_{SL}\} \times \Sigma$ with $a_{SL} \in (-R,R)$

Strategy
Deduce properties of the retarded single layer potential and operator from their behaviour in the Fourier-Laplace domain $\rightsquigarrow \ldots$
Lemma

For any $p \in \mathbb{R}$ and $\sigma > 0$, there exist well-defined and linear continuous operators

$$\text{SL}_{\Sigma_{SL}} : \tilde{H}_{\sigma, \Sigma_{SL}}^{p+1, -1/2} \rightarrow H_{\sigma, P_R}^{p, 1}$$

$$\text{V}_{\Sigma_{SL}} : \tilde{H}_{\sigma, \Sigma_{SL}}^{p+1, -1/2} \rightarrow H_{\sigma, \Sigma_{SL}}^{p, 1/2}$$

whose respective Fourier-Laplace transforms at $s \in \mathbb{C}_\sigma$ are just $\text{SL}^s_{\Sigma_{SL}}$ and $\text{V}^s_{\Sigma_{SL}}$ (the single layer operator and potential at frequency $s$)

Moreover, the images of $\text{SL}_{\Sigma_{SL}}$ are weak solutions of the forward problem in the TD posed in $P_R \setminus \Sigma_{SL}$

Remark

The above $\text{SL}_{\Sigma_{SL}} : \tilde{H}_{\sigma, \Sigma_{SL}}^{p+1, -1/2} \rightarrow H_{\sigma, P_R}^{p, 1}$ and $\text{V}_{\Sigma_{SL}} : \tilde{H}_{\sigma, \Sigma_{SL}}^{p+1, -1/2} \rightarrow H_{\sigma, \Sigma_{SL}}^{p, 1/2}$ are the so called retarded single layer operator and potential
Retarded single layers: usage for the TD-LSM

Objective (reminder...)

1. Approximate solutions of the forward problem in the TD by means of regularized retarded single layer potentials
2. Further, study their behaviour on $\partial D$ for their usage in the TD-LSM

Let $p \in \mathbb{R}$ and $\sigma > 0$

1. Approximation of a solution of the forward problem in the TD:
   For any $v \in X_{\sigma, \Omega_R}^p$, there exists $\{\psi_n\}_{n \in \mathbb{N}} \subset \tilde{H}_{\sigma, \Sigma_{SL}}^{p+1,-1/2}$ such that
   $$SL_{\Sigma_{SL}} \psi_n \rightarrow v \quad \text{in} \quad H_{\sigma, \Omega_R}^{p-1,1}$$

2. Trace of single layer potentials on $\partial D$:
   $$g \in \tilde{H}_{\sigma, \Sigma_{SL}}^{p+1,-1/2} \mapsto (SL_{\Sigma_{SL}} g)_{\mathbb{R} \times \partial D} \in H_{\sigma, \partial D}^{p,1/2}$$
   It is one-to-one and has dense image
Inverse problem

Objective
Deal with measurements on a surface \( \Sigma^M := \{a_M\} \times \Sigma \), being \( a_M \in [a_{SL}, R) \) such that \( D \subset (a_M, R) \times \Sigma =: P^M_R \)

\[
\begin{align*}
\Sigma_R & \quad \Sigma^SL & \quad \Sigma^M & \quad \Sigma_R \\
\end{align*}
\]

Near-field operator

\textit{Traces of solutions on } \( \Sigma^M \)

Given data on \( \partial D \), 1st lift to \( \Omega_R \) and 2nd restrict to \( \Sigma^M \) \( \leadsto \ldots \)

\( S_{\Sigma^M} : g \in H^{p+1,1/2}_{\sigma_0, \partial D} \hookrightarrow (S_{\Omega_R}g)|_{\mathbb{R} \times \Sigma^M} \in H^{p,1/2}_{\sigma, \Sigma^M} \)

which is linear continuous, one-to-one and with dense image

\textit{Near-field operator}

Feed the operator above with retarded single layer potentials \( \leadsto \ldots \)

\[
N := S_{\Sigma^M} \circ |_{\mathbb{R} \times \partial D} \circ SL_{\Sigma^SL}
\]
Inverse problem

Objective
Deal with measurements on a surface $\Sigma^M := \{a_M\} \times \Sigma$, being $a_M \in [a_{SL}, R)$ such that $D \subseteq (a_M, R) \times \Sigma =: P_R^M$

Near-field operator

$$N := S_{\Sigma^M} \circ \cdot |_{\mathbb{R} \times \partial D} \circ SL_{\Sigma^M}$$

Proposition

For any $p \in \mathbb{R}$ and $\sigma \geq \sigma_0 > 0$,

$$N : \tilde{H}^{p+1,-1/2}_{\sigma_0,\Sigma^M} \rightarrow H^{p-1,1/2}_{\sigma,\Sigma^M}$$

is well-defined, linear continuous, one-to-one and its image is dense
Inverse problem: TD-LSM

Location and reconstruction of $D$ by the TD-LSM

Given $z \in P_R$

1. Find a regularized solution of the near-field equation

$$Ng_{\varepsilon,p,z} = u_z^i$$

2. Plot the indicator function

$$1/\|g_{\varepsilon,p,z}\|_{\tilde{H}^{p+1,-1/2}_{\sigma,\Sigma SL}}$$

which is expected to be large in $D$ and small in $\Omega_R$, and to blow up on the boundary $\partial D$ ...
For any $z \in D$ and $\varepsilon > 0$, there exists $g_{\varepsilon, p, z} \in \tilde{H}^{p+1, -1/2}_{\sigma, \Sigma_{SL}}$ such that

$$\| (S L \Sigma_{SL} g_{\varepsilon, p, z}) |_{\mathbb{R} \times \partial D} - u_z^i |_{\mathbb{R} \times \partial D} \|_{H^{p, 1/2}_{\sigma, \partial D}} < \varepsilon$$

$$\| N g_{\varepsilon, p, z} - u_z^i \|_{H^{p-1, 1/2}_{\sigma, \Sigma^M}} < \varepsilon$$

Moreover, when $z \in D$ approaches $\partial D$,

$$\| g_{\varepsilon, p, z} \|_{\tilde{H}^{p+1, -1/2}_{\sigma, \Sigma_{SL}}} \rightarrow +\infty$$
Theorem (2)

For any \( z \in \Omega_R \) and \( \{g_{\varepsilon,p,z}\}_{\varepsilon>0} \subset \tilde{H}^{p+1,-1/2}_{\sigma,\Sigma_S} \) satisfying that

\[
\|Ng_{\varepsilon,p,z} - u^i_z\|_{H^{p-1,1/2}_{\sigma,\Sigma_M}} < \varepsilon
\]

it holds

\[
\lim_{\varepsilon \to 0} \|(SL\Sigma_ag_{\varepsilon,p,z})|_{\mathbb{R} \times \partial D}\|_{H^{p,1/2}_{\sigma,\partial D}} = \infty
\]
Numerical examples
For a bounded scatterer $D$ in a 2d wave guide with synthetic data

- **Synthetic scattering data**
  By a time domain boundary integral equation (TD-BIE) on the surface of a given scatterer in the pipe

  Thanks to M. Hassell and F.J. Sayas for making their TD-BIE code available to us!

- **Experiments**
  TD counterpart of the back-scattering experiments in
  [L. Bourgeois, E. Lunéville (2008)]
Numerics: numerical examples

Geometry

- Pipe: $P = \mathbb{R} \times (0, 1)$
- Obstacle: $D$ a circle at $(0, 0.6)$ with radius $r$
- Sources and receivers: on $x_1 = a < 0$

Auxiliary functions

- Regularization: $\chi$ with “central frequency” $\omega$
- Fundamental solution: $\Phi_s$ approx. by 100 images or 20 modes
Numerics: numerical examples

Geometry and auxiliary functions

- Wave guide: \( P = \mathbb{R} \times (0, 1) \)
- Obstacle: \( D \) a circle at \((0, 0.6)\) with radius \( r \)
- Sources and receivers: on \( x_1 = a < 0 \)
- Regularization: \( \chi \) with “central frequency” \( \omega = 10 \) or 15
- Fundamental solution: \( \Phi_s \) approx. by 100 images or 20 modes

\[
\chi(t) = \left( -3.2 \sin(\omega t) t + \omega \cos(\omega t) + 9.6 \sin(\omega t) \right) \exp(-1.6(t - 3)^2)
\]
Obstacle with radius $r = 0.2$ and transducers at $x_1 = -2$

$\chi$ with $\omega = 10$

$\chi$ with $\omega = 15$
Obstacle with radius $r = 0.2$ and transducers at $x_1 = -2$

$\chi$ with $\omega = 10$

$\chi$ with $\omega = 15$
Obstacle with radius $r = 0.2$ and transducers at $x_1 = -5$

$\chi$ with $\omega = 10$

$\chi$ with $\omega = 15$
Numerics: reconstructions by TD-LSM

Obstacle with radius $r = 0.2$ and transducers at $x_1 = -5$

$\chi$ with $\omega = 10$  \hspace{1cm}  $\chi$ with $\omega = 15$
Obstacle with radius $r = 0.1$ and transducers at $x_1 = -2$
Target with radius $r = 0.1$ and transducers at $x_1 = -2$
Comments and conclusions in the TD

Forward problem in the TD

- The TD forward problem is well-posed
- The TD-BIE performs well once the fundamental solution is well approximated (by means of images or modes)

Inverse problem in the TD: numerical solution by the TD-LSM

- The TD-LSM is justified as usual
- It is rapid and easy to implement
- It detects some scatterers in the pipe better than its frequency counterpart

Future work

- Penetrable obstacles ⇝ . . .
- Obstacles touching the pipe walls