A semi-analytical method to solve the equilibrium equations of axisymmetric elasticity in a half-space with a hemispherical pit

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Overview

1. Motivation and basic equations
2. Analytical solution in series form
3. Boundary conditions on the hemispherical pit
4. Numerical results and validation
5. Conclusions and perspectives for future work
Motivation and basic equations

Analytical solution in series form

Boundary conditions on the hemispherical pit

Numerical results and validation

Conclusions and perspectives for future work
Main motivation: Stresses in surface excavations

Need for determining gravity stresses in the surrounding rock mass

Estimate potential slope failure or landslide occurring
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Need for determining gravity stresses in the surrounding rock mass
Estimate potential slope failure or landslide occurring
Difficulties associated with this problem

When trying to solve this problem using mathematical and numerical approaches, the following difficulties are encountered:

1. Complex geometry of the excavation and its surrounding area
2. Complex internal structure of the rock mass (inhomogeneous, anisotropic and with discontinuities)
3. The surrounding area is in practice unbounded (to be stored in a computer it needs to be truncated)

Most commercial software to solve this type of problems (usually based on FEM) are able to deal with difficulties 1 and 2, but they fail in resolving adequately difficulty 3 (problem of unboundedness)
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Approach considered in this work

Difficulties associated with complex geometry and internal structure are simplified, focusing the attention on treating appropriately the problem of unboundedness.

We thus assume the following hypotheses:

1. Excavation is an hemispherical pit surrounded by a half-space.
2. Half-space occupied by a homogenous isotropic elastic solid.
3. Gravity force acting downwards everywhere in the half-space.

A simplified mathematical model of our problem is built.

A semi-analytical solution method is considered, which takes advantage of the axisymmetry of the domain.

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Axisymmetric domain

Axisymmetric spherical coordinates \((r, \phi)\)

Unit vectors \(\hat{r}, \hat{\phi}\) and \(\hat{k}\)

Pit radius \(h\), gravity acceleration \(\vec{g}\)

Domain and boundaries:

\[
\Omega = \{(r, \phi) : \ h < r < \infty, \ \pi/2 < \phi < \pi\}
\]
\[
\Gamma_h = \{(r, \phi) : \ r = h, \ \pi/2 < \phi < \pi\}
\]
\[
\Gamma_\infty = \{(r, \phi) : \ r \geq h, \ \phi = \pi/2\}
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\end{align*}
\]
Basic equations

**Displacement field** $\mathbf{u}$, **stress tensor** $\mathbf{\sigma}(\mathbf{u})$

Isotropic Hooke’s law $\mathbf{\sigma}(\mathbf{u}) = \lambda(\nabla \cdot \mathbf{u})\mathbf{I} + \mu(\nabla \mathbf{u} + (\nabla \mathbf{u})^T)$

with $\lambda \geq 0$ and $\mu > 0$ the Lamé constants of the elastic solid

Poisson’s ratio $\nu = \frac{\lambda}{2(\lambda + \mu)}$

Equation of elastostatic equilibrium $\nabla \cdot \mathbf{\sigma}(\mathbf{u}) = -\rho g \hat{\mathbf{k}}$

with $\rho$ the density of the elastic solid

Traction-free boundary conditions (Neumann-type) $\mathbf{\sigma}(\mathbf{u})\hat{\mathbf{n}} = 0$

with $\hat{\mathbf{n}}$ the outward unit normal vector
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Elasticity in axisymmetric spherical coordinates

A generic displacement field \( u \) is given by components as

\[
\mathbf{u}(r, \phi) = u_r(r, \phi) \hat{r} + u_\phi(r, \phi) \hat{\phi}
\]

The stress tensor \( \sigma \) has the following four components:

\[
\sigma_r(u) = (\lambda + 2\mu) \frac{\partial u_r}{\partial r} + \frac{\lambda}{r} \left[ 2u_r + \frac{\partial u_\phi}{\partial \phi} + \cot \phi u_\phi \right]
\]

\[
\sigma_\phi(u) = \lambda \frac{\partial u_r}{\partial r} + \frac{1}{r} \left[ 2(\lambda + \mu)u_r + (\lambda + 2\mu) \frac{\partial u_\phi}{\partial \phi} + \lambda \cot \phi u_\phi \right]
\]

\[
\sigma_\theta(u) = \lambda \frac{\partial u_r}{\partial r} + \frac{1}{r} \left[ 2(\lambda + \mu)u_r + \lambda \frac{\partial u_\phi}{\partial \phi} + (\lambda + 2\mu) \cot \phi u_\phi \right]
\]

\[
\sigma_{r\phi}(u) = \mu \left[ \frac{1}{r} \frac{\partial u_r}{\partial \phi} + \frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r} \right]
\]
Elasticity in axisymmetric spherical coordinates

A generic displacement field $u$ is given by components as

$$u(r, \phi) = u_r(r, \phi) \hat{r} + u_\phi(r, \phi) \hat{\phi}$$

The stress tensor $\sigma$ has the following four components:

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Boundary-value problem (BVP) in $u$

In absence of pit the half-space undergoes lithostatic displacement

$$u_g(x, y, z) = -\frac{\rho g z^2}{2(\lambda + 2\mu)} \hat{k}$$

In presence of pit the displacement field $u$ satisfies the BVP

$$\nabla \cdot \sigma(u) = -\rho g \hat{k} \quad \text{in } \Omega$$

$$\sigma(u)\hat{k} = 0 \quad \text{on } \Gamma_\infty$$

$$\sigma(u)\hat{r} = 0 \quad \text{on } \Gamma_h$$

$$\sigma(u)\hat{\phi} \cdot \hat{r} = u \cdot \hat{\phi} = 0 \quad \text{on } \Gamma_s$$

$$|u - u_g| = O\left(\frac{1}{r}\right) \quad \text{as } r \to \infty$$
Motivation
Analytical solution
Boundary conditions on the pit
Numerical results
Conclusions

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Boundary-value problem (BVP) in $\nu$

We use the decomposition $u = u_g + \nu$, with $\nu$ satisfying the BVP

\[ \nabla \cdot \sigma(\nu) = 0 \quad \text{in } \Omega \]
\[ \sigma(\nu) \hat{k} = 0 \quad \text{on } \Gamma_\infty \]
\[ \sigma(\nu) \hat{r} = f \quad \text{on } \Gamma_h \]
\[ \sigma(\nu) \hat{\phi} \cdot \hat{r} = \nu \cdot \hat{\phi} = 0 \quad \text{on } \Gamma_s \]
\[ |\nu| = O\left(\frac{1}{r}\right) \quad \text{as } r \to \infty \]

with $f(\phi) = \frac{\rho gh \cos \phi}{1 - \nu} \left((\nu + (1 - 2\nu) \cos^2 \phi) \hat{r} - (1 - 2\nu) \cos \phi \sin \phi \hat{\phi}\right)$

In what follows, this BVP in $\nu$ is solved
Boundary-value problem (BVP) in $v$

We use the decomposition $u = u_g + v$, with $v$ satisfying the BVP

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$$\sigma(v)\hat{k} = 0 \quad \text{on } \Gamma_\infty$$

$$\sigma(v)\hat{r} = f \quad \text{on } \Gamma_h$$

$$\sigma(v)\hat{\phi} \cdot \hat{r} = v \cdot \hat{\phi} = 0 \quad \text{on } \Gamma_s$$

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Displacement in terms of Boussinesq potentials

The displacement $v$ is sought under the form

$$2\mu v = \nabla(\Phi + z\Psi) - 4(1 - \nu)\Psi \hat{k}$$

where $\Phi$ and $\Psi$ are scalar functions (Boussinesq potentials).

This $v$ satisfies the equation of elastostatic equilibrium provided that $\Phi$ and $\Psi$ are harmonic functions in $\Omega$.

As $v$ has to fulfill the decaying condition $|v| = O\left(\frac{1}{r}\right)$ as $r \to \infty$, Boussinesq potentials $\Psi$ and $\Phi$ must satisfy

$$\begin{cases}
\Delta \Psi = 0 \quad \text{in } \Omega \\
\Psi = O\left(\frac{1}{r}\right) \quad \text{as } r \to \infty
\end{cases}$$

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Expressions for the Boussinesq potentials

Laplace equation in spherical coordinates \((r, \phi)\) for \(F = \Psi, \Phi\)

\[
\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial F(r, \phi)}{\partial r} \right) + \frac{1}{r^2 \sin \phi} \frac{\partial}{\partial \phi} \left( \sin \phi \frac{\partial F(r, \phi)}{\partial \phi} \right) = 0
\]

Standard separation of variables in \(\Omega\) yields \(F_n(r, \phi) = \frac{P_n(\cos \phi)}{r^{n+1}}\)
where \(P_n(\cdot)\) is the Legendre polynomial of order \(n \geq 0\)

Solutions \(\Psi\):

Decaying condition is \(\Psi = O\left(\frac{1}{r}\right)\) as \(r \to \infty\)

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Solutions \(\Phi\):

Decaying condition is \(|\nabla \Phi| = O\left(\frac{1}{r}\right)\) as \(r \to \infty\)

\[
\Phi_n(r, \phi) = \begin{cases} 
\ln(r - r \cos \phi) & n = -1 \\
\frac{P_n(\cos \phi)}{r^{n+1}} & n \geq 0
\end{cases}
\]
Obtaining the series

Solutions in **series form** are built by **substituting** the following two combinations of **Boussinesq potentials**

1. \( \Phi = \Phi_n \quad \Psi = 0 \quad n \geq -1 \)
2. \( \Psi = (2n + 1)\Psi_n \quad \Phi = -(n - 4 + 4\nu)\Phi_{n-1} \quad n \geq 0 \)

in the expression

\[
2\mu v = \nabla (\Phi + z\Psi) - 4(1 - \nu)\Psi \hat{k}
\]

These **displacements** are denoted respectively by \( v_n^{(1)} \) and \( v_n^{(2)} \).

The associated **stress tensors**, denoted by \( \sigma_n^{(1)} \) and \( \sigma_n^{(2)} \) respectively, are obtained explicitly with the aid of the Hooke's law in spherical coordinates.
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Obtaining the series

The basic displacements and stresses can be reexpressed as

\[
\begin{align*}
    v_n^{(1)}(r, \phi) &= \frac{w_n^{(1)}(\phi)}{r^{n+2}}, & \sigma_n^{(1)}(r, \phi) &= \frac{\tau_n^{(1)}(\phi)}{r^{n+3}}, & n \geq -1 \\
    v_n^{(2)}(r, \phi) &= \frac{w_n^{(2)}(\phi)}{r^{n+1}}, & \sigma_n^{(2)}(r, \phi) &= \frac{\tau_n^{(2)}(\phi)}{r^{n+2}}, & n \geq 0
\end{align*}
\]

where for \( j = 1, 2, w_n^{(j)}, \tau_n^{(j)} \) are functions depending on \( \phi \)

The general solution \((v, \sigma)\) is then expressed as

\[
\begin{align*}
    v(r, \phi) &= \sum_{n=-1}^{\infty} \frac{1}{r^{n+2}} \left( a_n^{(1)} w_n^{(1)}(\phi) + a_{n+1}^{(2)} w_{n+1}^{(2)}(\phi) \right) \\
    \sigma(r, \phi) &= \sum_{n=-1}^{\infty} \frac{1}{r^{n+3}} \left( a_n^{(1)} \tau_n^{(1)}(\phi) + a_{n+1}^{(2)} \tau_{n+1}^{(2)}(\phi) \right)
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Traction-free boundary conditions on $\Gamma_\infty$

Let us recall the **boundary conditions** imposed on the **plane surface**

$$\sigma(u)\hat{k} = 0 \quad \text{on } \Gamma_\infty$$

For $z = 0$, $\hat{k} = -\hat{\phi} \implies \sigma(v)\hat{k} = -\sigma_r\phi(v)\hat{r} - \sigma_\phi(v)\hat{\phi}$

Imposing $\sigma_\phi(v) = \sigma_r\phi(v) = 0$ to the general solution yields the following relations for the coefficients $a_n^{(1)}$ and $a_n^{(2)}$

$$a_{-1}^{(1)} = (3 - 2\nu)a_0^{(2)}$$
$$ (2n + 1)a_{2n}^{(1)} = \alpha_{2n} a_{2n+1}^{(2)} \quad n \geq 0 $$
$$ (2n + 2)a_{2n+1}^{(1)} = \alpha_{2n+2} a_{2n+2}^{(2)} \quad n \geq 0 $$

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Rearranging appropriately the general solution, it is reexpressed as

\[
\begin{align*}
\mathbf{v}(r, \phi) &= \sum_{n=0}^{\infty} A_n \left( \frac{h}{r} \right)^{2n+2} \mathbf{w}_n^{(A)}(\phi) + \sum_{n=-1}^{\infty} B_n \left( \frac{h}{r} \right)^{2n+3} \mathbf{w}_n^{(B)}(\phi) \\
\mathbf{\sigma}(r, \phi) &= \frac{1}{h} \left[ \sum_{n=0}^{\infty} A_n \left( \frac{h}{r} \right)^{2n+3} \mathbf{\tau}_n^{(A)}(\phi) + \sum_{n=-1}^{\infty} B_n \left( \frac{h}{r} \right)^{2n+4} \mathbf{\tau}_n^{(B)}(\phi) \right]
\end{align*}
\]

where functions \( \mathbf{w}_n^{(A)}, \mathbf{w}_n^{(B)}, \mathbf{\tau}_n^{(A)} \) and \( \mathbf{\tau}_n^{(B)} \) are defined in terms of \( \mathbf{w}_n^{(1)}, \mathbf{w}_n^{(2)}, \mathbf{\tau}_n^{(1)} \) and \( \mathbf{\tau}_n^{(2)} \), and \( A_n, B_n \) are generic real coefficients.

It is easily verified that this general solution also satisfies the axisymmetric boundary conditions on \( \Gamma_s: v_\phi = \sigma_{r\phi}(\mathbf{v}) = 0 \)
Analytical solution expressed in series form

Rearranging appropriately the general solution, it is reexpressed as

\[
v(r, \phi) = \sum_{n=0}^{\infty} A_n \left( \frac{h}{r} \right)^{2n+2} w^{(A)}_n(\phi) + \sum_{n=-1}^{\infty} B_n \left( \frac{h}{r} \right)^{2n+3} w^{(B)}_n(\phi)
\]

\[
\sigma(r, \phi) = \frac{1}{h} \left[ \sum_{n=0}^{\infty} A_n \left( \frac{h}{r} \right)^{2n+3} \tau^{(A)}_n(\phi) + \sum_{n=-1}^{\infty} B_n \left( \frac{h}{r} \right)^{2n+4} \tau^{(B)}_n(\phi) \right]
\]

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Truncation of the series

Up to now we have obtained a solution that is fully analytical.

It satisfies the equation of elastostatic equilibrium, boundary conditions on $\Gamma_\infty$ and $\Gamma_s$, and a decaying condition at infinity.

The boundary condition on $\Gamma_h$ cannot be imposed analytically, so it is enforced numerically, giving rise to a semi-analytical solution.

First of all the infinite series are truncated at a finite order $N$:

$$v(r, \phi) = \sum_{n=0}^{N} A_n \left( \frac{h}{r} \right)^{2n+2} w_n^{(A)}(\phi) + \sum_{n=-1}^{N} B_n \left( \frac{h}{r} \right)^{2n+3} w_n^{(B)}(\phi)$$

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Motivation Analytical solution Boundary conditions on the pit Numerical results Conclusions

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Let us recall the **boundary condition** on the **hemispherical surface**

\[ \sigma(v) \hat{r} = f \quad \text{on } \Gamma_h \]

The numerical enforcement of this boundary condition yields approximate values of a finite number of coefficients \( A_n \) and \( B_n \). For this, we define the following **quadratic functional of energy**

\[
J(v) = -\frac{1}{2h^2} \int_{\Gamma_h} \sigma \hat{r} \cdot v \, ds + \frac{1}{h^2} \int_{\Gamma_h} f \cdot v \, ds
\]

The numerical minimisation of this functional leads to a **linear system of equations** for the coefficients \( A_n \) and \( B_n \), whose particular structure allows us to solve it in a very efficient form.
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Matrix form of the quadratic functional

Substituting in $J$ the expressions for $\mathbf{v}$ and $\mathbf{\sigma}$ as truncated series evaluated at $r = h$, and rearranging appropriately yields

$$J(\mathbf{v}) = \frac{1}{2} \sum_{n=0}^{N} \sum_{k=0}^{N} Q_{nk}^{(AA)} A_n A_k + \frac{1}{2} \sum_{n=0}^{N} \sum_{k=-1}^{N} Q_{nk}^{(AB)} A_n B_k$$

$$+ \frac{1}{2} \sum_{n=-1}^{N} \sum_{k=0}^{N} Q_{nk}^{(BA)} B_n A_k + \frac{1}{2} \sum_{n=-1}^{N} \sum_{k=-1}^{N} Q_{nk}^{(BB)} B_n B_k$$

$$- \sum_{n=0}^{N} c_n^{(A)} A_n - \sum_{n=-1}^{N} c_n^{(B)} B_n$$

for matrices $Q^{(AA)}$, $Q^{(AB)}$, $Q^{(BA)}$, $Q^{(BB)}$ and vectors $c^{(A)}$, $c^{(B)}$.
Matrix form of the quadratic functional

Substituting in $J$ the expressions for $\nu$ and $\sigma$ as truncated series evaluated at $r = h$, and rearranging appropriately yields

$$J(\nu) = \frac{1}{2} \sum_{n=0}^{N} \sum_{k=0}^{N} Q_{nk}^{(AA)} A_n A_k + \frac{1}{2} \sum_{n=0}^{N} \sum_{k=-1}^{N} Q_{nk}^{(AB)} A_n B_k$$

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Matrix form of the quadratic functional

For $i, j = A, B$ the entries of matrices $Q^{(ij)}$ and vectors $c^{(i)}$ are defined as

\[
Q^{(ij)}_{nk} = - \int_{\pi/2}^{\pi} \left( \left[ w_n^{(i)} \right] r(\phi) \left[ \tau_k^{(j)} \right] r(\phi) + \left[ w_n^{(i)} \right] \phi(\phi) \left[ \tau_k^{(j)} \right] r(\phi) \right) \sin \phi \, d\phi
\]

\[
c_n^{(i)} = - h \int_{\pi/2}^{\pi} \left( \left[ w_n^{(i)} \right] r(\phi) f_r(\phi) + \left[ w_n^{(i)} \right] \phi(\phi) f_\phi(\phi) \right) \sin \phi \, d\phi
\]

All these entries are calculated explicitly with the aid of integral formulae for Legendre polynomials and their derivatives, e.g.

\[
\int_{\pi/2}^{\pi} P_{2n}(\cos \phi) P_{2k+1}(\cos \phi) \sin \phi \, d\phi = - \frac{(2k + 1) P_{2n}(0) P_{2k}(0)}{(2k + 1 - 2n)(2k + 2 + 2n)}
\]
Matrix form of the quadratic functional

For $i, j = A, B$ the entries of matrices $Q^{(ij)}$ and vectors $c^{(i)}$ are defined as

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Structure of matrices

$$Q^{(AA)}$$, $$Q^{(AB)}$$, and $$Q^{(BB)}$$ are symmetric and positive definite matrices.

$$Q^{(BA)} = [Q^{(AB)}]^T$$

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\[ Q^{(AA)} \]

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Structure of matrices

\[ Q^{(AA)} \quad Q^{(AB)} \quad Q^{(BB)} \]

\[ \begin{align*}
Q^{(AA)} \text{ and } Q^{(BB)} \text{ are symmetric and positive definite matrices} \\
Q^{(BA)} &= [Q^{(AB)}]^T
\end{align*} \]
Minimisation of the functional and linear system

Defining the vectors \( x^{(A)} = (A_0 \ A_1 \ldots \ A_N)^T \in \mathbb{R}^{N+1} \) and 
\( x^{(B)} = (B_{-1} \ B_0 \ldots \ B_N)^T \in \mathbb{R}^{N+2} \)

and the matrices and vectors by blocks

\[
Q = \begin{bmatrix}
Q^{(AA)} & Q^{(AB)} \\
[Q^{(AB)}]^T & Q^{(BB)}
\end{bmatrix}
\quad
x = \begin{bmatrix}
x^{(A)} \\
x^{(B)}
\end{bmatrix}
\quad
c = \begin{bmatrix}
c^{(A)} \\
c^{(B)}
\end{bmatrix}
\]

the quadratic functional is reexpressed as 
\( J(x) = \frac{1}{2} x^T Q x - x^T c \)

As the matrix \( Q \) is symmetric and positive definite, \( J \) has a global minimum, which is reached when \( \nabla J(x) = 0 \)

This is equivalent to \( Qx = c \), which corresponds to a linear system of equations for the coefficients \( A_n \) and \( B_n \), stored in the vector \( x \)
Minimisation of the functional and linear system

Defining the vectors \( x^{(A)} = (A_0 \ A_1 \ \ldots \ A_N)^T \in \mathbb{R}^{N+1} \) and
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and the matrices and vectors by blocks

\[
Q = \begin{bmatrix}
    Q^{(AA)} & Q^{(AB)} \\
    Q^{(AB)^T} & Q^{(BB)}
\end{bmatrix}
\]

\[
x = \begin{bmatrix}
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    x^{(B)}
\end{bmatrix}
\]

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INGMAT

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Solution of the linear system

To solve

\[
\begin{bmatrix}
Q^{(AA)} & Q^{(AB)} \\
[Q^{(AB)}]^T & Q^{(BB)}
\end{bmatrix}
\begin{bmatrix}
x^{(A)} \\
x^{(B)}
\end{bmatrix}
=
\begin{bmatrix}
c^{(A)} \\
c^{(B)}
\end{bmatrix}
\]

we use the Schur-Banachiewicz blockwise inversion formula:

\[
x^{(A)} = ([Q^{(AA)}]^{-1} + [Q^{(AA)}]^{-1} Q^{(AB)} [\tilde{Q}^{(BB)}]^{-1} [Q^{(AB)}]^T [Q^{(AA)}]^{-1}) c^{(A)}
- [Q^{(AA)}]^{-1} Q^{(AB)} [\tilde{Q}^{(BB)}]^{-1} c^{(B)}
\]

\[
x^{(B)} = -[\tilde{Q}^{(BB)}]^{-1} [Q^{(AB)}]^T [Q^{(AA)}]^{-1} c^{(A)} + [\tilde{Q}^{(BB)}]^{-1} c^{(B)}
\]

where

\[
\tilde{Q}^{(BB)} = Q^{(BB)} - [Q^{(AB)}]^T [Q^{(AA)}]^{-1} Q^{(AB)}
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is the Schur complement of \(Q^{(BB)}\) in \(Q\).
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To evaluate the **Schur-Banachiewicz blockwise inversion formula** it is required to invert two matrices:

- The symmetric tridiagonal matrix $Q^{(AA)}$: It is efficiently inverted using the Thomas algorithm for tridiagonal systems.
- The positive definite symmetric full matrix $\tilde{Q}^{(BB)}$: It is efficiently inverted with the aid of its Cholesky factorisation.

The evaluation of the formula yields approximate values of coefficients $A_0, A_1, \ldots, A_N$ and $B_{-1}, B_0, B_1, \ldots, B_N$.

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A semi-analytical method to solve the equilibrium equations of axisymmetric elasticity in a half-space with a hemispherical pit.
The semi-analytical solution was numerically evaluated using the following parameter values:

<table>
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<tr>
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<th>$\nu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Value</td>
<td>600 m</td>
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<td>0.3</td>
</tr>
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The Lamé’s constants are obtained through the formulae

$$\lambda = \frac{\nu E}{(1 + \nu)(1 - 2\nu)}, \quad \mu = \frac{E}{2(1 + \nu)}$$

A truncation order of $N = 50$ was considered.

Recall that the physical displacement is obtained as $u = u_g + \nu$
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To validate the semi-analytical procedure of solution, the same axisymmetric boundary-value problem was numerically solved using the commercial FEM software *COMSOL Multiphysics*.

Dirichlet boundary conditions on the right and bottom boundaries.

Different values of the square length $L$ were considered.

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Thanks for your attention!

Any question?