Completely monotone functions in fractional relaxation processes

Francesco MAINARDI

Department of Physics, University of Bologna
Via Irnerio 46, I-40126 Bologna, Italy
francesco.mainardi@unibo.it
http://www.fracalmo.org

BCAM: 19 November 2015
Acknowledgements

This research has been carried out in the framework of the project *Fractional Calculus Modelling*, see [http://www.fracalmo.org](http://www.fracalmo.org)

This lecture is partly based on the author’s books and papers
Abstract

In this talk the condition of complete monotonicity (CM) is discussed for the response functions characterizing relaxation processes modelled by constitutive equations of fractional order. Our models can concern both mechanical and dielectric relaxation. Indeed, in view of the electro-mechanical analogy, linear viscoelastic and dielectric materials exhibit similar features as far as the time-dependent response functions are concerned.

We point out that CM is an essential property for the physical acceptability and realizability of the models since it ensures, for instance, that in isolated systems the energy decays monotonically as expected from physical considerations. Studying the conditions under which the response function of a system is CM is therefore of fundamental importance.

The purpose of this talk is to summarize the results obtained by the author with some collaborators for dielectric media where the response functions are shown to be of the Mittag-Leffler type.
Contents

1 A survey on completely monotone and Bernstein functions 6
   1.1 Definitions ................................................. 6
   1.2 Some basic criteria and examples ................. 11

2 The Mittag-Leffler functions 20
   2.1 Definitions ................................................. 20
   2.2 The Laplace transform pairs ......................... 21
   2.3 Complete monotonicity ............................ 23

3 The dielectric relaxation phenomenon 40
   3.1 Introduction to dielectric relaxation ............. 40
   3.2 The Cole-Cole relaxation model .................. 44
   3.3 The Davidson–Cole relaxation model .......... 47
   3.4 The Havriliak–Negami relaxation model .......... 50

4 Bibliography 60
5 Appendix A: The spectral distribution of the Prabahakar function 67

6 Appendix B: Formal demonstration of $K_{\alpha,\beta}^1(r) = K_{\alpha,\beta}(r)$ 69

7 Appendix C: Complete monotonicity from Laplace transform 72
1. A survey on completely monotone and Bernstein functions

1.1. Definitions

We recall that a function \( \phi(t) \) defined for \( t \geq 0 \) is said to be completely monotone (CM) if it is a non-negative with infinitely many derivatives that alternate in sign. In other words

\[
(-1)^n \frac{d^n}{dt^n} \phi(t) \geq 0, \quad n = 0, 1, 2, \ldots, \quad t \geq 0. \tag{1.1}
\]

A necessary and sufficient condition is given by Bernstein’s theorem, according to which a function \( \phi(t) \) is CM in \( R^+ \) if and only if it is the restriction to the positive real semi-axis of the Laplace transform of a positive measure.

For sake of simplicity we agree to represent \( \phi(t) \) as

\[
\phi(t) = \int_0^\infty e^{-rt} K_{\phi}(r) \, dr, \quad K_{\phi}(r) \geq 0, \tag{1.2}
\]

where \( K_{\phi}(r) \) is a non-negative (ordinary or generalized) function.
Alternatively, introducing $\tau = 1/r$ we can write Eq. (2) as

$$\phi(t) = \int_0^\infty e^{-t/\tau} H_\phi(\tau) \, d\tau, \quad H_\phi(\tau) \geq 0,$$

where

$$H_\phi(\tau) = K_\phi(1/\tau)/\tau^2,$$

is a non-negative (ordinary or generalized) function alike $K_\phi(r)$. Furthermore, a function $\psi(t)$ defined for $t \geq 0$ is said to be a **Bernstein function** if it is non-negative and its first derivative is a CM function. In other words

$$\psi(t) \geq 0, \quad (-1)^{n-1} \frac{d^n}{dt^n} \psi(t) \geq 0, \quad n = 1, 2, \ldots, \quad t \geq 0.$$ 

Thus the CM and Bernstein functions are special non-negative functions defined on the positive real semi-axis, the former being decreasing and convex, the latter increasing and concave.
Relevant and simplest examples of CM and Bernstein functions are $\phi(t) = e^{-t}$ and $\psi(t) = 1 - e^{-t}$, respectively.

For more details on these classes of functions we refer mainly to [Berg and Forst (1975), Gripenberg et al. (1990), Miller and Samko (2001), Schilling et al. (2012)], where the reader can find rigorous and exhaustive treatments. A more practical approach to this matter can be found in [Feller (1971)].

We agree to represent a Bernstein function $\psi(t)$ as

$$\psi(t) = a + bt + \int_0^\infty (1 - e^{-rt}) K_\psi(r) \, dr, \quad a, b, K_\psi(r) \geq 0,$$

(1.6)

where $K_\psi(r)$ is a non-negative (ordinary or generalized) function, and $a = \psi(0^+)$, $b = \lim_{t \to \infty} \psi(t)/t$. 
Alternatively, introducing \( \tau = 1/r \), we can write Eq. (1.6) as

\[
\psi(t) = a + bt + \int_0^\infty \left(1 - e^{-t/\tau}\right) H_\psi(\tau) \, d\tau, \quad H_\psi(\tau) \geq 0,
\]

(1.7)

where

\[
H_\psi(\tau) = K_\psi(1/\tau)/\tau^2,
\]

(1.8)
is a non-negative (ordinary or generalized) function alike \( K_\psi(r) \).

We note that if for a certain \( \alpha \in (0, 1) \), \( \psi(t) = O(t^\alpha) \) as \( t \to \infty \), then

\[
\int_0^\infty K_\psi(r) \, dr = \int_0^\infty K_\psi(\tau) \, d\tau = \infty.
\]

(1.9)
The functions \( K(r) \) and \( H(\tau) \) are usually referred to as the frequency and time spectral functions, respectively. They are interrelated through the differential form

\[
K(r) \, dr = H(\tau) \, d\tau.
\]

(1.10)
The spectral functions are relevant to characterize the processes of relaxation and creep in linear viscoelastic and dielectric materials.
The spectral functions can be uniquely determined from the derivative of the corresponding functions $\phi(t)$ and $\psi(t)$ by using the following relations akin to Laplace transform pairs. We get

\[
\int_{0}^{\infty} r K_{\phi}(r) e^{-tr} dr = -\frac{d}{dt}\phi(t), \quad (1.11)
\]

\[
\int_{0}^{\infty} r K_{\psi}(r) e^{-tr} dr = \frac{d}{dt}\psi(t) + b, \quad (1.12)
\]

showing that $-r K_{\phi}(r)$ and $r K_{\psi}(r)$ can be viewed as the inverse Laplace transforms of $\dot{\phi}(t)$ and $\dot{\psi}(t) + b$, respectively, where $t$ is now considered the Laplace transform variable instead of the usual $r$ and vice versa. The time spectral functions can be derived from the corresponding frequency spectral functions by Eqs. (1.4), (1.8). Thus, adopting the connective symbol $\div$ for Laplace transform pairs, where in the LHS we put the original function and in the RHS its Laplace transform, we re-write Eqs. (1.11), (1.12) as

\[
-r K_{\phi}(r) \div \dot{\phi}(t), \quad (1.11')
\]

\[
r K_{\psi}(r) = \div \dot{\psi}(t) + b. \quad (1.12')
\]
1.2. Some basic criteria and examples

The following elementary functions are CM

\[ e^{-at}, \quad a \geq 0, \]
\[ \frac{1}{\lambda + \mu t}, \quad \lambda \geq 0, \quad \mu \geq 0, \quad \nu \geq 0, \quad (1.13) \]
\[ \log \left( b + \frac{c}{t} \right), \quad b \geq 1, \quad c > 0. \quad (1.14) \]

A trivial observation is the \( \phi(t) \) is CM, then \( \phi^{(2m)}(t) \) and \(-\phi^{(2m+1)}\) are also CM

**Theorem 1.** If \( \phi_1(t) \) and \( \phi_2(t) \) are CM, then \( a\phi_1(t) + b\phi_2(t) \), where \( a \) and \( b \) are non negative constants and \( \phi_1(t) \phi_2(t) \) are also CM

The following elementary functions are Bernstein

\[ t^\alpha, \quad 0 < \alpha \leq 1, \quad (1.15) \]
\[ \log(1 + at), \quad a > 0, \quad (1.16) \]
\[ \frac{(1 + at)^\alpha - 1}{\alpha}, \quad a > 0, \quad \alpha \leq 1. \quad (1.17) \]
Remark on Eq (1.17): In the extended Jeffrey-Lomnitz creep law the complete range $\alpha \leq 1$ (rather than $0 \leq \alpha \leq 1$) yields a continuous transition from a Hooke elastic solid with no creep ($\alpha \to -\infty$) to a Maxwell fluid with linear creep ($\alpha = 1$) passing through the Lomnitz viscoelastic body with logarithmic creep ($\alpha = 0$), which separates solid-like from fluid-like behaviors. In the expression for the extended Jeffreys-Lomnitz creep law, it is convenient to separately consider four cases:

$$
t \geq 0, \quad \psi(t) = \begin{cases} 
\frac{t}{\tau_0}, & \alpha = 1, \\
\frac{(1 + t/\tau_0)^\alpha - 1}{\alpha}, & 0 < \alpha < 1, \\
\log(1 + t/\tau_0), & \alpha = 0, \\
\frac{1 - (1 + t/\tau_0)^{-|\alpha|}}{|\alpha|}, & \alpha < 0.
\end{cases}
$$
The behaviour of $\psi(t)$ as a function of the dimensionless time $t/\tau_0$ is illustrated in the Figures below, for some values of $\alpha$ in the range $-2 \leq \alpha \leq 1$, adopting a logarithmic time scale and a linear time scale.

**Theorem 2.** Let $\phi(t)$ be CM and let $\psi(t)$ be Bernstein, the $\phi[\psi(t)]$ also is CM function. A noteworthy example is $\phi(t) = \exp[-\psi(t)]$.

**Corollary 1.** Let $f(t)$ and $\phi(t)$ be CM Then

$$f \left( a + b \int_0^t \phi(t') \, dt' \right), \quad a \geq 0, \ b \geq 0, \quad (1.18)$$

also is CM.

**Corollary 2.** Let $f(t)$ be CM and $f(0) < \infty$. Then the functions

$$\frac{1}{[A - f(t)]^\mu}, \quad A \geq f(0), \quad \mu \geq 0, \quad (1.19)$$

and

$$- \log \left[ 1 - \frac{f(t)}{A} \right], \quad A \geq f(0), \quad (1.20)$$

are CM. It also follows that the function

$$\frac{f'(t)}{A - f(t)}, \quad A \geq f(0), \quad (1.21)$$

is c.m since reduces to minus derivative of the function above.
We note some particular cases of CM functions of the above types.

\[ \exp(-at^\alpha), \ a \geq 0, \ 0 \leq \alpha \leq 1, \quad (1.22) \]

\[ \frac{1}{[a + b \log(1 + t)]^\mu}, \ a \geq 0, \ b \geq 0, \ \mu \geq 0, \quad (1.23) \]

\[ \frac{1}{(a - be^{-t})^\mu}, \ a \geq b > 0, \ \mu \geq 0, \quad (1.24) \]

Another version of the statement for composite functions is given in terms of power series with non-negative coefficients by the following theorem.

**Theorem 3.** Let \( y = \phi(t) \) be CM and let the power series

\[ f(y) = \sum_{n=0}^{\infty} a_n y^n \quad (1.25) \]

converge for all \( y \) in the range of the function \( y = \phi(t) \). If \( a_n \geq 0 \) for all \( n = 0, 1, 2, \ldots \). Then \( f[\phi(t)] \) is CM
Corollary 3. If \( \phi(t) \) is CM, then \( \exp[\phi(t)] \) is CM. In particular, the functions

\[
\exp(at^\alpha), \quad a \geq 0, \quad \alpha \leq 0, \quad (1.26)
\]

\[
(1 + t)^{a/t} = \exp[a \log(1 + t)/t], \quad a \geq 0, \quad (1.27)
\]

\[
\left( a + \frac{b}{t} \right)^\mu = \exp \left\{ \mu \log \left( a + \frac{b}{t} \right) \right\}, \quad a \geq 1, \quad b > 0, \quad \mu \geq 0. \quad (1.28)
\]

Remark A natural idea is to pass from series representation with positive coefficients, as in Theorem 3, to integral transforms with non-negative densities. Let

\[
f(t) = \int_c^d K(t, \tau) g(\tau) d\tau, \quad 0 \leq c < d \leq \infty. \quad (1.29)
\]

Obviously, if \( K(t, \tau) \) is CM in \( t \) for all \( \tau \in (0, \infty) \) and \( g(\tau) \) is non-negative, the formal differentiation shows that \( f(t) \) also is CM. In view of this Remark [Miller and Samko (2001)] we may extend the result (1.28) by proving the CM of

\[
\left( a + \frac{b}{t^\alpha} \right)^\mu, \quad a \geq 0, \quad b > 0 \quad \mu \geq 0, \quad 0 \leq \alpha \leq 1. \quad (1.30)
\]
Furthermore we recall a noteworthy property relating CM and Bernstein function in addition to their composition already described in Theorem 2. If \( \psi(t) \) is a Bernstein function \( \phi(t) = \psi(t)/t \) is CM.

By the way we could complement our analysis in noting that there are two relevant subclasses for CM and Bernstein functions: the Stieltjes functions and the Complete Bernstein functions, respectively well discussed in the book [Schilling et al. (2012)] to which we address the interesting readers.

We prefer to conclude this introductory part with two theorems relating CM and Laplace transforms.
There is a sort of inverse of the Bernstein theorem, according to which the inverse Laplace transform of a CM function is non-negative and viceversa, under suitable regularity conditions:

\[ \phi(t) \geq 0, t \geq 0 \iff \tilde{\phi}(s) \text{ CM function}, s > 0, \quad (1.31) \]

where \( \tilde{\phi}(s) \) is the Laplace transform of \( \phi(t) \).

This property may be justified by the previous Remark on integral transforms (1.29) and by the Post-Widder formula for the inversion of the Laplace transform:

\[ \phi(t) = \lim_{n \to \infty} \frac{(-1)^n}{n!} \left( \frac{n}{t} \right)^{n+1} \tilde{\phi}^{(n)} \left( \frac{n}{t} \right). \quad (1.32) \]
Herewith we report a theorem found in [Gripenberg et al. (1990)], see Theorem 2.6, pp. 144-145, that provides necessary and sufficient conditions to ensure the CM of a locally integrable function $f(t)$ in $t \geq 0$ based on its Laplace transform $\tilde{f}(s)$.

**Theorem** The Laplace transform $\tilde{f}(s)$ of a function $f(t)$ that is locally integrable on $\mathbb{R}^+$ and CM has the following properties:

(i) $\tilde{f}(s)$ an analytical extension to the region $\mathbb{C} - \mathbb{R}$;
(ii) $\tilde{f}(s) = \tilde{f}^*(s)$ for $s \in (0, \infty)$;
(iii) $\lim_{s \to \infty} \tilde{f}(s) = 0$;
(iv) $\text{Im}\{\tilde{f}(s)\} < 0$ for $\text{Im}\{s\} > 0$;
(v) $\text{Im}\{s \tilde{f}(s)\} \geq 0$ for $\text{Im}\{s\} > 0$ and $\tilde{f}(x) \geq 0$ for $x \in (0, \infty)$.

Conversely, every function $\tilde{f}(s)$ that satisfies (i)–(iii) together with (iv) or (v), is the Laplace transform of a function $f(t)$, which is locally integrable on $\mathbb{R}^+$ and CM on $(0, \infty)$. 
2. The Mittag-Leffler functions

2.1. Definitions

The 3-parameter Mittag-Leffler function

\[ E_{\alpha,\beta}^{\gamma}(z) := \sum_{n=0}^{\infty} \frac{(\gamma)^n}{n!\Gamma(\alpha n + \beta)} z^n, \quad (2.1) \]

\[ \text{Re}\{\alpha\} > 0, \text{Re}\{\beta\} > 0, \text{Re}\{\gamma\} > 0, \]

\[ (\gamma)_n = \gamma(\gamma + 1) \ldots (\gamma + n - 1) = \frac{\Gamma(\gamma + n)}{\Gamma(\gamma)}. \]

For \( \gamma = 1 \) we recover the 2-parameter Mittag-Leffler function

\[ E_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad (2.2) \]

and for \( \gamma = \beta = 1 \) we recover the standard Mittag-Leffler function

\[ E_{\alpha}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}. \quad (2.3) \]
2.2. The Laplace transform pairs

Let us now consider the relevant formulas of Laplace transform pairs related to the above three functions, already known in the literature for $\alpha, \beta, \gamma > 0$ when the independent variable is real of type $at$ where $t > 0$ is interpreted as time and $a$ as a certain constant.

Let us start with the most general function. Substituting the series representation of the Prabhakar generalized Mittag-Leffler function in the Laplace transformation yields the identity

$$\int_0^\infty e^{-st} t^\beta - 1 E_{\alpha,\beta}^{\gamma}(at^\alpha) \, dt = s^{-\beta} \sum_{n=0}^\infty \frac{\Gamma(\gamma + n)}{\Gamma(\gamma)} \left(\frac{a}{s}\right)^n. \quad (2.4)$$

On the other hand (binomial series)

$$(1 + z)^{-\gamma} = \sum_{n=0}^\infty \frac{\Gamma(1 - \gamma)}{\Gamma(1 - \gamma - n)n!} z^n = \sum_{n=0}^\infty (-1)^n \frac{\Gamma(\gamma + n)}{\Gamma(\gamma)n!} z^n. \quad (2.5)$$
Comparison of Eq. (2.4) and Eq. (2.5) yields the Laplace transform pair

\[ t^\beta - 1 \ E^{\gamma}_\alpha,\beta (at^\alpha) \div \frac{s^{-\beta}}{(1 - as^{-\alpha})^\gamma} = \frac{s^{\alpha\gamma - \beta}}{(s^{\alpha} - a)^\gamma}. \] (2.6)

Eq. (2.6) holds (by analytic continuation) for \(\text{Re}(\alpha) > 0\), \(\text{Re}(\beta) > 0\).

In particular we get the known Laplace transform pairs, see e.g. [Mainardi (2010), Podlubny (1999)],

\[ t^\beta - 1 \ E^{\alpha,\beta}_\alpha (at^\alpha) \div \frac{s^{\alpha - \beta}}{s^{\alpha} - a} = \frac{s^{-\beta}}{1 - as^{-\alpha}}, \] (2.7)

\[ E_\alpha (at^\alpha) \div \frac{s^{\alpha - 1}}{s^{\alpha} - a} = \frac{s^{-1}}{1 - as^{-\alpha}}. \] (2.8)
2.3. Complete monotonicity

We recall that a function $f(t)$ with $t \geq 0$ is completely monotone (CM) if it is positive and its derivatives are alternating in sign,

$$(-1)^n f^{(n)}(t) > 0, \quad t \geq 0. \quad (2.9)$$

Thus, for the Bernstein theorem that states a necessary and sufficient condition for the CM, the function can be expressed as a real Laplace transform of non-negative (generalized) function,

$$f(t) = \int_0^\infty e^{-rt} K(r) \, dr, \quad K(r) \geq 0, \quad t \geq 0. \quad (2.10)$$

By the way, the determination of such non-negative function $K(r)$ (the Laplace measure) is a standard method to prove the CM of a given function defined in the positive real axis $\mathbb{R}^+$. In physical applications the function $K(r)$ is usually referred to as the spectral distribution function, in that it is related to the fact that the process governed by $f(t)$ can be expressed in terms of a continuous distribution of elementary (exponential) relaxation processes with frequencies $r$ on the whole range $(0, \infty)$. 
In the case of the pure exponential \( f(t) = \exp(-\lambda t) \) with a given relaxation frequency \( \lambda > 0 \) we have \( K(r; \lambda) = \delta(r - \lambda) \).

Since \( \tilde{f}(s) \) turns to be the iterated Laplace transform of \( K(r) \) we recognize that \( \tilde{f}(s) \) is the Stieltjes transform of \( K(r) \) and therefore the spectral distribution can be determined as the inverse Stieltjes transform of \( \tilde{f}(s) \) via the Titchmarsh inversion formula, see e.g. \([\text{Titchmarsh (1937), Widder (1946)}]\),

\[
\tilde{f}(s) = \int_{0}^{\infty} \frac{K(r)}{s + r} \, dr, \quad K(r) = \mp \frac{1}{\pi} \text{Im} \left[ \tilde{f}(s) \bigg|_{s = re^{\pm i\pi}} \right]. \tag{2.11}
\]

As a consequence, a method for proving CM suitable for our Mittag-Leffler functions (2.6) (2.7) (2.8) is that to arrive at the non-negative spectral distribution \( K(r) \) by inverting the corresponding Laplace transform by using the Bromwich contour integral. Indeed this was the method followed by \([\text{Gorenflo and Mainardi (1997)}]\) to prove the CM of \( E_{\alpha}(-t^{-\alpha}) \) for \( 0 < \alpha < 1 \) and to determine the corresponding spectral function.
We recall the result

\[ E_\alpha(-t^\alpha) = \int_0^\infty e^{-rt} K_\alpha(r) \, dr, \quad K_\alpha(r) = -\frac{1}{\pi} \text{Im} \left[ \frac{s^{\alpha-1}}{s^\alpha + 1} \right]_{s=re^{i\pi}}, \quad (2.12) \]

\[ K_\alpha(r) = \frac{1}{\pi r} \frac{\sin(\alpha\pi)}{r^{\alpha} + 2 \cos(\alpha\pi) + r^{-\alpha}} = \frac{1}{\pi} \frac{r^{\alpha-1} \sin(\alpha\pi)}{r^{2\alpha} + 2r^\alpha \cos(\alpha\pi) + 1}, \quad (2.13) \]

As a matter of fact the function \( K_\alpha(r) \) was derived as an exercise in complex analysis by evaluating the contribution on the branch cut (the negative real axis) of the Bromwich integral (bending the Bromwich path into the equivalent Hankel loop) and turns out to be provided by the so-called Titchmarsh formula. We easily recognize

\[ K_\alpha(r) \geq 0 \quad \text{if} \quad 0 < \alpha \leq 1, \quad (2.14) \]

including the limiting case \( \alpha = 1 \) where our Mittag-leffler function reduces to the exponential \( \exp(-t) \) and \( K_1(r) = \delta(r - 1) \). In fact, the denominator is non negative being greater or equal to \((r^\alpha - 1)^2\) and the numerator is non-negative as soon as the \(\sin\) function is non-negative.
In Fig. 2.1 we show some plots of $\Psi(t) = E_\alpha(-t^\alpha)$ for some values of the parameter $\alpha$. It is worth to note the different rates of decay of $E_\alpha(-t^\alpha)$ for small and large times. In fact the decay is very fast as $t \to 0^+$ and very slow as $t \to +\infty$.

**Fig. 2.1** Plots of the Mittag-Leffler function $E_\alpha(-t^\alpha)$ for $\alpha = 0.25, 0.50, 0.75, 1.$; Left: in linear-linear scales; Right in log-log scales.
In Fig. 2.2 we show $K_\alpha(r)$ for some values of the parameter $\alpha$. Of course for $\alpha = 1$ the Mittag-Leffler function reduces to the exponential function $\exp(-t)$ and the corresponding spectral distribution is the Dirac delta generalized function centred at $r = 1$, namely $\delta(r - 1)$.

\[ \text{Fig. 2.2 Plots of the spectral function } K_\alpha(r) \text{ for } \alpha = 0.25, 0.50, 0.75, 0.90 \text{ in the frequency range } 0 \leq r \leq 2. \]
As a matter of fact, $K_\alpha(r)$ provides an interesting spectral representation of $e_\alpha(t)$ in frequencies. With the change of variable $\tau = 1/r$ we get the corresponding spectral representation in relaxation times, namely

$$E_\alpha(t) = \int_0^\infty e^{-t/\tau} H_\alpha(\tau) \, d\tau, \quad H_\alpha(\tau) = \tau^{-2} K_\alpha(1/\tau), \quad (2.15)$$

that can be better interpreted as a continuous distribution of elementary (i.e. exponential) relaxation processes. As a consequence we get the identity between the two spectral distributions, that is $K_\alpha(r) = H_\alpha(\tau)$, because we get

$$H_\alpha(\tau) = \frac{1}{\pi} \frac{\tau^{\alpha-1} \sin(\alpha \pi)}{\tau^{2\alpha} + 2\tau^\alpha \cos(\alpha \pi) + 1}, \quad (2.13')$$

a surprising fact pointed out in Linear Viscoelasticity by the author in his book [Mainardi (2010)]. This kind of universal/scaling property seems a peculiar one for our Mittag-Leffler function $E_\alpha(-t^\alpha)$. 
Before deriving the conditions of CM and the corresponding spectral function for the Mittag-Leffler function in three parameters in Eq.(2.6) let us revisit the conditions of CM for the function in two parameters in Eq.(2.7) following the approach by Gorenflo and Mainardi [Gorenflo and Mainardi (1997)]. Since the argument of our function $at^\alpha$ must be negative we assume $a = -1$ (without loss of generality) so the corresponding Laplace transform pair reads from Eq.(2.7),

$$t^{\beta-1} E_{\alpha,\beta}(-t^\alpha) \div \frac{s^{-\beta}}{1 + s^{-\alpha}} = \frac{s^{\alpha-\beta}}{s^\alpha + 1}. \quad (2.16)$$

We prove the existence of the corresponding spectral distribution using the complex Bromwich formula to invert the Laplace transform.

Taking $0 < \alpha < 1$ the denominator does not exhibit any zero so, bending the Bromwich path into the equivalent Hankel path (the well known loop around the negative real semi-axis), we get
\[
t^\beta-1 E_{\alpha,\beta}(-t^\alpha) = \int_0^\infty e^{-rt} K_{\alpha,\beta}(r) \, dr,
\]

\[
K_{\alpha,\beta}(r) = -\frac{1}{\pi} \text{Im} \left[ \frac{s^{\alpha-\beta}}{s^\alpha + 1} \right]_{s=r e^{i\pi}}
\]  
(2.17)

\[
K_{\alpha,\beta}(r) = \frac{r^{\alpha-\beta}}{\pi} \frac{\sin \left[ (\beta - \alpha)\pi \right]}{r^{2\alpha} + 2r^\alpha \cos(\alpha\pi) + 1}.
\]  
(2.18)

We easily recognize

\[
K_{\alpha,\beta}(r) \geq 0 \quad \text{if} \quad 0 < \alpha \leq \beta \leq 1,
\]  
(2.19)

In fact, the denominator in Eq.(2.18) is non negative being greater or equal to \((r^\alpha - 1)^2\) and the numerator is non negative as soon as the two sin functions are both non-negative.

The particular cases \(\beta = 1\) with \(0 < \alpha \leq 1\) are of course recovered.
We recall that for the Mittag-Leffler functions in one and two-order parameters $E_\alpha(z)$, $E_{\alpha,\beta}$ the conditions to be CM were proved respectively by Pollard [Pollard (1948)] in 1948 and by Schneider [Schneider (1996)] in 1996, see also [Miller and Samko (1997), Miller and Samko (2001)] for further details. These conditions require the independent variable to be real and negative (we write $z = -x$ with $x \geq 0$) and the order-parameters such that $0 < \alpha \leq 1$ for $E_\alpha(-x)$, and $0 < \alpha \leq 1$ and $\beta \geq \alpha$ for $E_{\alpha,\beta}(-x)$.

As a consequence, the CM property of $E_\alpha(-t^\alpha)$ can also be seen as a consequence of the result by Pollard because the transformation $x = t^\alpha$ is just a Bernstein function for $\alpha \in (0, 1]$. Furthermore, we note that the conditions (2.19) on the parameters $\alpha$ and $\beta$ can also be justified by noting that in this case the function $t^{\beta-1}E_{\alpha,\beta}(-t^\alpha)$ is CM as a product of two CM functions. In fact $t^{\beta-1}$ is CM if $\beta \leq 1$ whereas $E_{\alpha,\beta}(-t^\alpha)$ is CM if $0 < \alpha \leq 1$ and $\beta \geq \alpha$. 
Fig. 2.3 $K_{\alpha,\beta}(r)$ calculated for $\alpha = 0.9$. 

$$K_{\alpha,\beta}(r) = \frac{r^{\alpha-\beta}}{\pi} \frac{\sin[(\beta-\alpha)\pi]}{r^{2\alpha} + 2r^\alpha \cos(\alpha\pi) + 1}$$
Fig. 2.4 $K_{\alpha,\beta}(r)$ calculated for $\alpha = 0.75$. 

$$K_{\alpha,\beta}(r) = \frac{r^{\alpha-\beta}}{\pi} \frac{\sin[(\beta - \alpha)\pi] + r^{\alpha} \sin(\beta \pi)}{r^{2\alpha} + 2r^{\alpha} \cos(\alpha \pi) + 1}$$
Fig. 2.5 $K_{\alpha,\beta}(r)$ calculated for $\alpha = 0.5$. 

$$K_{\alpha,\beta}(r) = \frac{r^{\alpha-\beta}}{\pi} \frac{\sin[(\beta - \alpha)\pi] + r^\alpha \sin(\beta \pi)}{r^{2\alpha} + 2r^\alpha \cos(\alpha \pi) + 1}$$
Fig. 2.6 $K_{\alpha,\beta}(r)$ calculated for $\alpha = 0.25$. 

$$K_{\alpha,\beta}(r) = \frac{r^{\alpha-\beta}}{\pi} \frac{\sin[(\beta - \alpha)\pi] + r^\alpha \sin(\beta \pi)}{r^{2\alpha} + 2r^\alpha \cos(\alpha \pi) + 1}$$
We finally devote our attention to the more general three-parameters function

\[ \xi_G(t) := t^\beta - 1 E_{\alpha,\beta}^\gamma (-t^\alpha), \quad (2.20) \]

with Laplace transform (as derived from (2.7) with \( a = -1 \))

\[ \tilde{\xi}_G(s) = \frac{s^{\alpha \gamma - \beta}}{s^\alpha + 1} \gamma, \quad (2.21) \]

where the notation \( \xi_G(t) \) been introduced for future convenience in dielectric models.

In analogy with the previous computing, we get

\[ \xi_G(t) = \int_0^\infty e^{-rt} K_{\alpha,\beta}^\gamma (r) \, dr, \]

where the spectral distribution is to be computed by applying the Titchmarsh formula to the Laplace transform (2.21).

We get:
\[ K_{\alpha,\beta}^{\gamma}(r) = \frac{r^{-\beta}}{\pi} \text{Im} \left\{ \frac{e^{i\beta\pi}}{r^\alpha + 2 \cos(\alpha\pi) + r^{-\alpha}} \right\}^{\gamma} \]

\[ = -\frac{r^{\alpha\gamma-\beta}}{\pi} \text{Im} \left\{ \frac{e^{i(\alpha\gamma-\beta)\pi}}{r^\alpha e^{i\alpha\pi} + 1} \right\}^{\gamma} \]

from which, after standard manipulations in complex analysis, we get

\[ K_{\alpha,\beta}^{\gamma}(r) = \frac{r^{\alpha\gamma-\beta}}{\pi} \frac{\sin \left[ \gamma \theta_\alpha(r) + (\beta - \alpha\gamma)\pi \right]}{(r^{2\alpha} + 2r^\alpha \cos(\alpha\pi) + 1)^{\gamma/2}}, \quad (2.23) \]

where

\[ \theta_\alpha(r) := \arctan \left[ \frac{r^\alpha \sin(\pi \alpha)}{r^\alpha \cos(\pi \alpha) + 1} \right] \in [0, \pi]. \quad (2.24) \]

For details we refer to the Appendix A where a warning on the correct branch of the function \( \arctan \) is also outlined.
We easily recognize that $\theta_\alpha(r)$ is a non-negative and increasing function of $r$ limited by $\alpha \pi \leq \pi$, as shown in the Fig. 2.7, where the dotted lines indicate the limit values $\alpha \pi$. In fact for $r \gg 1$ it is direct to check its asymptotic behaviour

$$\frac{r^\alpha \sin(\pi \alpha)}{r^\alpha \cos(\pi \alpha) + 1} = \frac{\sin(\pi \alpha)}{\cos(\pi \alpha) + 1/r^\alpha} \leq \frac{\sin(\pi \alpha)}{\cos(\pi \alpha)} = \tan(\pi \alpha).$$

**Fig. 2.7** The function $\theta_\alpha(r)$ for $\alpha = 0.25$, $\alpha = 0.50$ and $\alpha = 0.75$. 
Then we recognize that in order the spectral distribution to be non-negative the argument of the sin function in the numerator must be included in the closed interval $[0, \pi]$ and henceforth we find the conditions

$$0 < \alpha \leq 1, \quad 0 < \alpha \gamma \leq \beta \leq 1. \quad (2.25)$$

These conditions were formerly stated by the author in collaboration with the Brazilian colleagues Capelas and Vaz in [Capelas, Mainardi and Vaz (2011)] by using a different method. Based on the requirements on the Laplace transform stated in the treatise [Gripenberg et al. (1990)], see Theorem 2.6, pp. 144-145. Indeed these requirements provide necessary and sufficient conditions to ensure the CM of a locally integrable function from its Laplace transform.

For this approach we refer the reader to Appendix B.

It is not straightforward to derive the spectral function (2.18) for $\gamma = 1$ from the the spectral function (2.23)-(2.24) with a generic $\gamma \neq 1$. For reader’s convenience the derivation is reported in Appendix C.
3. The dielectric relaxation phenomenon

3.1. Introduction to dielectric relaxation

We briefly outline the theory of dielectric polarization in order to introduce the mathematical models for anomalous relaxation present in complex materials based on the Mittag-Leffler functions.

Under the influence of the electric field, the matter becomes polarized. For a perfect isotropic dielectric the interdependence between the electric field $E$ and the polarization $P$ is described by the law:

$$ P = \varepsilon_0[(\varepsilon_s - \varepsilon_\infty)(\hat{\varepsilon}(i\omega) - 1)]E = \varepsilon_0[(\varepsilon_s - \varepsilon_\infty)\hat{\chi}(i\omega)]E, $$  \hspace{1cm} (3.1)

where $\varepsilon_s$ and $\varepsilon_\infty$ are the static and infinite dielectric constants, and $\hat{\varepsilon}(i\omega)$ and $\hat{\chi}(i\omega)$ are respectively the normalized permittivity and susceptibility depending on the frequency $\omega$ of the external field. They are specific characteristics of the medium, determined by matching experimental data into a theoretical model.
The classical Debye model has given an expression for the normalized permittivity as

$$\hat{\varepsilon}(i\omega) = \frac{1}{1 + i\omega\tau} = \varepsilon'(\omega) - i\varepsilon''(\omega), \quad (3.2)$$

where $\tau$ is the only relaxation time expected. As a consequence the response function $\xi(t)$ (=$\text{inverse Laplace transform of the dielectric permittivity}$) is purely exponential.

The Debye model was the first relaxation law based on the Statistical Mechanics and formulated in terms of the Brownian motion. However, it is useful in a little variety of cases. Indeed, in complex materials, the Debye phenomenology of relaxation processes breaks down and a power law decay is generally found. Furthermore it does not fit the observed distributions of relaxation times.

Models for the non-Debye (= anomalous) relaxation are thus necessary especially in complex materials like the biological ones.
The Laplace transform pairs for the Mittag-Leffler functions in one, two and three parameters with $t \geq 0$ and $s = i\omega$ discussed in the previous Section can be used as mathematical modes for the response function $\xi(t)$ and for the complex permittivity $\hat{\varepsilon}(\omega)$ to take into account the anomalous relaxation relaxation in dielectrics.

The corresponding spectral functions can be used to characterize the models by means of suitable distributions of exponential relaxation processes.

Furthermore, it is known that the dielectric models are also distinguishable by inspection of their (so-called) Cole-Cole plots that exhibit the imaginary part versus real part of the complex permittivity.
We now show how the three classical models for anomalous relaxation referred to Cole–Cole (C-C), Davidson–Cole (D-C) and Havriliak–Negami (H-N) are contained in our general model when $\alpha \gamma = \beta$ (taking for simplicity $\tau = 1$)

$$\frac{s^{\alpha \gamma - \beta}}{(s^\alpha + 1)^\gamma} \xrightarrow{\alpha \gamma = \beta} \frac{1}{(s^\alpha + 1)^\gamma}$$

so that

$$\alpha \gamma = \beta \begin{cases} 
0 < \alpha < 1, \gamma = 1 & \text{C-C } \{\alpha\}, \\
\alpha = 1, 0 < \gamma < 1 & \text{D-C } \{\gamma\}, \\
0 < \alpha < 1, 0 < \gamma < 1 & \text{H-N } \{\alpha, \gamma\}. 
\end{cases} \tag{3.3}$$

However we can generalize the H-N model extending its original range of $\gamma$ to

$$1 < \gamma < 1/\alpha,$$

keeping the complete monotonicity of the corresponding response function, see [Capelas, Mainardi and Vaz (2011)], [Mainardi and Garrappa (2015)].
3.2. The Cole-Cole relaxation model

The C-C relaxation model is a non-Debye relaxation model depending on one parameter, say $\alpha$ ($0 < \alpha < 1$), see [Cole and Cole (1941), Cole and Cole (1942)], that for $\alpha = 1$ reduces to the standard Debye model. We have for $0 < \alpha < 1$:

$$\tilde{\xi}_{\text{C-C}}(s) = \frac{1}{1 + s^\alpha} \div \tilde{\xi}_{\text{C-C}}(t) = t^{\alpha - 1} E_{\alpha,\alpha}^1(-t^\alpha) = -\frac{d}{dt}E_{\alpha}(-t^\alpha).$$

(3.4)

The spectral distribution is thus obtained from that of the two-parameter Mittag-Leffler function (2.18) for $\alpha = \beta$ and also from that of $E_{\alpha}(-t^\alpha)$ by multiplying by $r$:

$$K_{\text{C-C}}(r) := K_{1,\alpha}^1(r) = K_{\alpha,\alpha}(r) = \frac{1}{\pi} \frac{r^\alpha \sin(\alpha \pi)}{r^{2\alpha} + 2r^\alpha \cos(\alpha \pi) + 1}.$$

(3.5)

We note its power law decay for $r \to \infty$: $K_{\text{C-C}}(r) \sim 1/r^\alpha$, and its maximum value in $r = 1$

$$K_{\text{C-C}}(r = 1) = \frac{1}{2\pi} \frac{\sin \alpha \pi}{\cos \alpha \pi + 1} = \frac{1}{2\pi} \tan \frac{\alpha \pi}{2}.$$

(3.6)
Fig. 3.1 The spectral distribution for the Cole-Cole model

\[ K_{C-C}(r) = \frac{1}{\pi} \frac{r^\alpha \sin(\alpha \pi)}{r^{2\alpha} + 2r^\alpha \cos(\alpha \pi) + 1} \]

calculated for \( \alpha = 0.9, \alpha = 0.75, \alpha = 0.5, \alpha = 0.25 \).
The Cole-Cole plot for this model is shown in Fig. 3.2 where the apexes of the arcs correspond to the mean relaxation frequency.

**Fig 3.2** Cole-Cole plot for Cole-Cole and Debye models
3.3. The Davidson–Cole relaxation model

The D-C relaxation model is a non-Debye relaxation model depending on one parameter, say $\gamma$ ($0 < \gamma < 1$), see [Davidson and Cole], that for $\gamma = 1$ reduces to the standard Debye model. The corresponding complex susceptibility ($s = i\omega$) and response function read for $0 < \gamma < 1$:

$$\tilde{\xi}_{D-C}(s) = \frac{1}{(1 + s)^\gamma} \div \xi_{D-C}(t) = t^{\gamma-1} E_{1,\gamma}(-t) = \frac{t^{\gamma-1}}{\Gamma(\gamma)} e^{-t}, \quad . \quad \quad (3.7)$$

The spectral distribution for the Cole-Davidson model is easily obtained

$$K_{D-C}(r) := K_{1,\gamma}^\gamma(r) = \begin{cases} 0, & 0 < r < 1, \\ (r - 1)^{-\gamma} \frac{\sin(\gamma \pi)}{\pi}, & r > 1, \end{cases} \quad \quad (3.8)$$

where we have used the identity $\Gamma(\gamma)\Gamma(1 - \gamma) = \frac{\pi}{\sin(\gamma \pi)}$. 

Fig. 3.3 The spectral distribution for the Cole-Davidson model, $K_{D-C}(r) := K^\gamma_{1,\gamma}(r)$ calculated for $\gamma = 0.9$, $\gamma = 0.75$, $\gamma = 0.5$, $\gamma = 0.25$. 

$$K_{D-C}(r) = \begin{cases} 0, & 0 < r < 1 \\ (r - 1)^{\gamma} \frac{\sin(\gamma \pi)}{\pi}, & r > 1 \end{cases}$$
In Fig. 3.4 the Cole-Cole plots are shown for this model. We note skewed arcs where the maximum deviates from $\omega = 1$.

3.4 Cole-Cole plot for Davidson-Cole model
3.4. The Havriliak–Negami relaxation model

The classical H-N relaxation model is a non-Debye relaxation model depending on two parameters, say $\alpha$ ($0 < \alpha < 1$) and $\gamma$ ($0 < \gamma < 1$), see [Havriliak and Negami (1966), Havriliak and Negami (1967)], that for $\alpha = \gamma = 1$ reduces to the standard Debye model.

It is worth to note that we have been able to extend the validity of the classical Havriliak-Negami model for $1 < \gamma < 1/\alpha$ for any $\alpha \in (0, 1)$ because the CM of the response function is still valid.

The corresponding complex permittivity and response function read

$$\tilde{\xi}_{H-N}(s) = \frac{1}{(1 + s^\alpha)\gamma} \div \xi_{H-N}(t) = t^{\alpha\gamma - 1} E_{\alpha,\alpha\gamma}^{\gamma}(-t^\alpha). \quad (3.9)$$
We recognize that the H-N relaxation model for $\gamma = 1$ and $0 < \alpha < 1$ reduces to the C-C model, while for $\alpha = 1$ and $\gamma < 1$ to the D-C model.

We also note that whereas for the C-C and H-N models (with $0 < \alpha < 1$) the corresponding response functions decay like a certain negative power of time, for the D-C model, being $\alpha = 1$ the response function exhibits an exponential decay.

In the following slides we show the spectral functions and the Cole-Cole plots for the H-N model with $\alpha = 0.50, 0.75$ both in the range $0 < \gamma < 1$ and $1 \leq \gamma < 1/\alpha$. 
Fig. 3.5 The spectral distribution for the Havriliak-Negami model $K_{H-N}(r)$ for $\alpha = 0.5$ and $0 < \gamma < 1$. 
Fig. 3.6 The spectral distribution for the Havriliak-Negami model $K_{H-N}(r)$, for $\alpha = 0.5$, and $1 \leq \gamma < 2 = 1/\alpha$. 
Fig. 3.7 The spectral distribution for the Havriliak-Negami model, $K_{H-N}(r)$, for $\alpha = 0.75$ and $0 < \gamma < 1$. 
Fig. 3.8 The spectral distribution for the Havriliak-Negami model $K_{H-N}(r)$, for $\alpha = 0.75$ and $1 \leq \gamma < \frac{4}{3} = \frac{1}{\alpha}$. 

\[ \gamma = 1.00 \quad \gamma = 1.10 \quad \gamma = 1.20 \quad \gamma = 1.30 \]
Fig. 3.9 The Cole-Cole plot for the Havriliak-Negami model for $\alpha = 0.50$ and $0 < \gamma < 1$. 
Fig. 3.10 The Cole-Cole plot for the Havriliak-Negami model for $\alpha = 0.50$ and $1 \leq \gamma < 1/\alpha$. 
Fig. 3.11 The Cole-Cole plot for the Havriliak-Negami model for $\alpha = 0.75$ and $0 < \gamma < 1$. 
**Fig. 3.12** The Cole-Cole plot for the Havriliak-Negami model for $\alpha = 0.75$ and $1 \leq \gamma < 1/\alpha$. 
4. Bibliography

References


[Havriliak and Negami (1967)] S. Havriliak and S. Negami, A complex plane representation of dielectric and mechanical relaxation processes in some polymers, Polymer 8 No 4, 161–210 (1967).


5. Appendix A: The spectral distribution of the Prabahakar function

For ease of presentation we have collected in this Appendix some details regarding the derivation of the spectral distribution $K_{\alpha,\beta}^{\gamma}(r)$.

After applying the Titchmarsh inversion formula, from (2.\textsuperscript{?}) we have

$$K_{\alpha,\beta}^{\gamma}(r) = \frac{r^{-\beta}}{\pi} \Im \left\{ e^{i\beta \pi} \left( \frac{r^\alpha + e^{-i\alpha \pi}}{r^\alpha + 2 \cos(\alpha \pi) + r^{-\alpha}} \right)^\gamma \right\}$$

$$= -\frac{r^{\alpha \gamma - \beta}}{\pi} \Im \left\{ \frac{e^{i(\alpha \gamma - \beta) \pi}}{(r^\alpha e^{i\alpha \pi} + 1)^\gamma} \right\}$$

$$= -\frac{r^{\alpha \gamma - \beta}}{\pi} \Im \left\{ \frac{e^{i(\alpha \gamma - \beta) \pi}}{(r^\alpha e^{i\alpha \pi} + 1)^\gamma (r^\alpha e^{-i\alpha \pi} + 1)^\gamma} \right\}$$

(A.1)

It is now easy to check that the denominator is real and non-negative, so we set

$$\xi := (r^\alpha e^{i\alpha \pi} + 1)(r^\alpha e^{-i\alpha \pi} + 1) = r^{2\alpha} + 2 r^\alpha \cos(\alpha \pi) + 1,$$  \hspace{1cm} (A.2)

with $\xi \geq (r^\alpha - 1)^2 \geq 0$ and consequently $\xi^{\gamma/2} \geq 0$. For the numerator we set

$$z := (r^\alpha e^{-i\alpha \pi} + 1) = [r^\alpha \cos(\alpha \pi) + 1] - i r^\alpha \sin(\alpha \pi) = \rho e^{-i\theta},$$  \hspace{1cm} (A.3)

where $0 \leq \theta \leq \pi$ (being $0 < \alpha \leq 1$). Then

$$\rho = |z| = \sqrt{[\text{Re}(z)]^2 + [\text{Im}(z)]^2} = \xi^{\frac{1}{2}}$$  \hspace{1cm} (A.4)
and
\[
\tan \theta = -\frac{\text{Im}(z)}{\text{Re}(z)} = \frac{r^\alpha \sin(\pi \alpha)}{r^\alpha \cos(\pi \alpha) + 1}. \tag{A.5}
\]

Using the de Moivre’s formula \((\cos \psi + i \sin \psi)^n = \cos(n\psi) + i \sin(n\psi)\) we get
\[
K_{\alpha,\beta}^{\gamma}(r) = -\frac{r^{\alpha-\beta}}{\pi} \frac{\text{Im} \{[\cos(\alpha \gamma - \beta)\pi + i \sin(\alpha \gamma - \beta)\pi] \left[\cos(\gamma \theta) - i \sin(\gamma \theta)\right]\}}{\xi^{\gamma/2}}
\]
\[
= -\frac{r^{\alpha-\beta}}{\pi} \frac{[-\cos(\alpha \gamma - \beta)\pi \sin(\gamma \theta) + \sin(\alpha \gamma - \beta)\pi \cos(\gamma \theta)]}{\xi^{\gamma/2}}
\]
\[
= -\frac{r^{\alpha-\beta}}{\pi} \frac{\sin[(\alpha \gamma - \beta)\pi - \gamma \theta]}{\xi^{\gamma/2}} = \frac{r^{\alpha-\beta}}{\pi} \frac{\sin[\gamma \theta + (\beta - \alpha \gamma)\pi]}{\xi^{\gamma/2}} \tag{A.6}
\]

where
\[
\theta = \theta_\alpha(r) := \arctan \left[\frac{r^\alpha \sin(\pi \alpha)}{r^\alpha \cos(\pi \alpha) + 1}\right] \in [0, \pi]. \tag{A.7}
\]

As noted by Zorn for the Havriliak-Negami model [Zorn (1999)], see our analysis for \(\alpha \gamma - \beta = 0\), we need to chose the arctangent’s value in the interval \([0, \pi]\), which is possible if one considers \(\arctan(x)\) to be a multivalued function. In this sense our proposed formula is always valid if only correctly interpreted. Staying with the usual definition of \(\arctan(x)\) as a function into \([-\pi/2, \pi/2]\), one has to add \(\pi\) to avoid the negative values instead of the changing of sign.
6. Appendix B: Formal demonstration of $K_{\alpha,\beta}^1(r) = K_{\alpha,\beta}(r)$

In this appendix we show how, for $\gamma = 1$, the spectral density of 3-parameters Mittag-Leffler function $K_{\alpha,\beta}^\gamma(r)$ reduces to the 2-parameters spectral density $K_{\alpha,\beta}(r)$. Recalling Eqs. (2.23)-(2.24), for $\gamma = 1$ we have:

$$K_{\alpha,\beta}^1(r) = \frac{1}{\pi} \frac{r^{\alpha-\beta}}{(r^{2\alpha} + 2r^\alpha \cos(\alpha\pi) + 1)^{\gamma/2}} \sin \left[ \arctan \left( \frac{r^\alpha \sin(\pi\alpha)}{r^\alpha \cos(\pi\alpha) + 1} \right) + (\beta - \alpha)\pi \right]$$

$$= \frac{1}{\pi} \frac{r^{\alpha-\beta}}{\xi^{1/2}} \sin[\theta + (\beta - \alpha)\pi],$$

where we call $\xi = r^{2\alpha} + 2r^\alpha \cos(\alpha\pi) + 1$, and $\theta = \arctan \left( \frac{r^\alpha \sin(\pi\alpha)}{r^\alpha \cos(\pi\alpha) + 1} \right)$ as well as in Appendix A, just for brevity.

Recalling the trigonometric formula

$$\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta),$$
we have:

\[ K_{\alpha,\beta}^1(r) = \frac{1}{\pi} \frac{r^{\alpha-\beta}}{\xi^{1/2}} \left[ \sin \theta \cos(\beta - \alpha)\pi + \cos \theta \sin(\beta - \alpha)\pi \right] \]

\[ = \frac{1}{\pi} \frac{r^{\alpha-\beta}}{\xi^{1/2}} \cos \left[ \frac{\tan \theta \cos(\beta - \alpha)\pi + \sin(\beta - \alpha)\pi}{\pi} \right] \]

\[ = \frac{1}{\pi} \frac{r^{\alpha-\beta}}{\xi^{1/2}} \cos \left[ \frac{r^{\alpha} \sin(\pi \alpha)}{r^{\alpha} \cos(\pi \alpha) + 1} \cos(\beta - \alpha)\pi + \sin(\beta - \alpha)\pi \right] . \]

Noting that

\[ \tan^2 \theta = \left( \frac{r^{\alpha} \sin(\pi \alpha)}{r^{\alpha} \cos(\pi \alpha) + 1} \right)^2 = \frac{\sin^2 \theta}{\cos^2 \theta} = \frac{1 - \cos^2 \theta}{\cos^2 \theta} = \frac{1}{\cos^2 \theta} - 1 \]

we can write:

\[ \cos^2 \theta = \frac{1}{\tan^2 \theta + 1} = \frac{(r^{\alpha} \cos(\pi \alpha) + 1)^2}{r^{2\alpha} \left( \sin^2(\pi \alpha) + \cos^2(\pi \alpha) \right) + 2r^{\alpha} \cos(\pi \alpha) + 1} = \frac{(r^{\alpha} \cos(\pi \alpha) + 1)^2}{\xi} . \]
Extracting the square root we find:

\[
K_{\alpha,\beta}^1(r) = \frac{1}{\pi} \frac{r^{\alpha-\beta}}{\xi^{\frac{1}{2}}} \left( \frac{r^\alpha \cos(\pi \alpha) + 1}{\xi^{\frac{1}{2}}} \right) \left[ \frac{r^\alpha \sin(\pi \alpha)}{r^\alpha \cos(\pi \alpha) + 1} \cos(\beta - \alpha)\pi + \sin(\beta - \alpha)\pi \right] \\
= \frac{1}{\pi} \frac{r^{\alpha-\beta}}{\xi} \left[ r^\alpha \sin(\pi \alpha) \cos(\beta - \alpha)\pi + r^\alpha \cos(\pi \alpha) \sin(\beta - \alpha)\pi + \sin(\beta - \alpha)\pi \right] \\
= \frac{1}{\pi} \frac{r^{\alpha-\beta}}{\xi} \left[ r^\alpha \sin(\beta \pi) + \sin(\beta - \alpha)\pi \right] \\
= \frac{r^{\alpha-\beta}}{\pi} \frac{\sin[(\beta - \alpha)\pi] + r^\alpha \sin(\beta \pi)}{r^{2\alpha} + 2r^\alpha \cos(\alpha \pi) + 1} = K_{\alpha,\beta}(r)
\]

*quod erat demonstrandum.*
7. Appendix C: Complete monotonicity from Laplace transform

Herewith we report a theorem found in [Gripenberg et al. (1990)], see Theorem 2.6, pp. 144-145, that provides necessary and sufficient conditions to ensure the CM of a locally integrable function \( f(t) \) in \( t \geq 0 \) based on its Laplace transform \( \tilde{f}(s) \).

This theorem was used by [Capelas, Mainardi and Vaz (2011)] in order to provide a proof for the inequalities ensuring the CM of the function in the LHS of the Laplace transform pair

\[
\xi_G(t) := t^{\beta - 1} E_{\alpha,\beta}^\gamma (-t^\alpha) \div \tilde{\xi}_G(s) = \frac{s^{\alpha\gamma-\beta}}{(s^\alpha + 1)^\gamma}. \tag{2.6}
\]

that is the conditions ensuring the non-negativity of the corresponding spectral function \( K_{\alpha,\beta}^\gamma(r) \) for \( r \geq 0 \).

We recall these conditions consisting in the following inequalities that we like to write in two equivalent forms,

\[
0 < \alpha \leq 1, \ 0 < \beta \leq 1, \ 0 < \gamma \leq \frac{\beta}{\alpha} \iff 0 < \alpha \leq 1, \ 0 < \alpha \gamma \leq \beta \leq 1, \quad \text{(C.1)}
\]
Hereafter, we recall this theorem from Section 1.

**Theorem** The Laplace transform $\tilde{f}(s)$ of a function $f(t)$ that is locally integrable on $\mathbb{R}^+$ and CM has the following properties:

(i) $\tilde{f}(s)$ an analytical extension to the region $\mathbb{C} - \mathbb{R}^-$;

(ii) $\tilde{f}(s) = \tilde{f}^*(s)$ for $s \in (0, \infty)$;

(iii) $\lim_{s \to \infty} \tilde{f}(s) = 0$;

(iv) $\text{Im}\{\tilde{f}(s)\} < 0$ for $\text{Im}\{s\} > 0$;

(v) $\text{Im}\{s \tilde{f}(s)\} \geq 0$ for $\text{Im}\{s\} > 0$ and $\tilde{f}(x) \geq 0$ for $x \in (0, \infty)$.

Conversely, every function $\tilde{f}(s)$ that satisfies (i)-(iii) together with (iv) or (v), is the Laplace transform of a function $f(t)$, which is locally integrable on $\mathbb{R}^+$ and CM on $(0, \infty)$. 
We recognize that the requirements (i)–(iii) for $\tilde{\xi}_G(s)$ are surely satisfied with the first two conditions in Eq.(C.1), that is $0 < \alpha < 1$, $0 < \beta < 1$ but for any $\gamma > 0$. So for us it suffices to determine which additional condition is implied from the requirement (iv).

We will prove that this relevant condition is just $0 < \alpha \gamma - \beta \leq 0$, namely $0 < \gamma \leq \beta/\alpha$, as stated in Eq.(C.1).

Recalling the expression of the Laplace transform in Eq.(2.6), the requirement (iv) reads:

$$\Lambda(s) := \text{Im} \left[ \frac{s^{\alpha \gamma - \beta}}{(s^\alpha + 1)^\gamma} \right] < 0 \quad \text{where} \quad \text{Im}\{s\} > 0. \tag{C.2}$$

Setting $s = re^{i\phi}$ in the complex upper half-plane ($\text{Im}\{s\} > 0$) we consider

$$\Lambda(r, \phi) := \text{Im} \left[ \frac{(re^{i\phi})^{\alpha \gamma - \beta}(1 + r^{\alpha}e^{-i\alpha \phi})^{\gamma}}{|1 + r^{\alpha}e^{i\alpha \phi}|^{2\gamma}} \right] \quad \text{with} \quad r > 0, \quad 0 < \phi < \pi. \tag{C.3}$$

To prove that $\Lambda(r, \phi)$ is negative it is sufficient to consider the numerator because the denominator is always non-negative. Setting

$$z = (re^{i\phi})^{\alpha \gamma - \beta}(1 + r^{\alpha}e^{-i\alpha \phi})^{\gamma} = \rho e^{i\Psi}, \tag{C.4}$$

we must verify that the conditions on $\{\alpha, \beta, \gamma\}$ stated in Eq.(C.1) ensure that $z$ has negative imaginary part so it is located in the lower half plane with

$$-\pi < \Psi < 0. \tag{C.5}$$
Let
\[ z_1 = r^{\alpha \gamma - \beta} e^{i(\alpha \gamma - \beta)\phi} = \rho_1 e^{i\Psi_1}, \quad \rho_1 = r^{\alpha \gamma - \beta}, \quad \Psi_1 = (\alpha \gamma - \beta)\phi, \quad (C.6) \]
\[ z_2 = r^{\alpha} e^{-i\alpha \phi} = \rho_2 e^{i\Psi_2}, \quad \rho_2 = r^{\alpha}, \quad \Psi_2 = -\alpha \phi, \quad (C.7) \]
and
\[ z_3 = (1 + z_2)^\gamma = \rho_3 e^{i\Psi_3}, \quad \rho_3 = |1 + r^{\alpha} e^{-i\alpha \phi}|\gamma, \quad -\alpha \gamma \phi < \Psi_3 < 0, \quad (C.8) \]
so we can write the complex number in Eq. (C.4) as
\[ z = z_1 \cdot z_3 = \rho_1 e^{i\Psi_1} \rho_3 e^{i\Psi_3} = \rho e^{i\Psi} \quad \text{with} \quad \rho = \rho_1 \rho_3, \quad \Psi = \Psi_1 + \Psi_3. \quad (C.9) \]
Now assuming \( 0 < \phi < \pi \) a we find for \( \alpha \gamma - \beta < 0 \):
\[ - (\beta - \alpha \gamma)\pi < \Psi_1 < 0, \quad (C.10) \]
\[ -\alpha \gamma \pi < \Psi_3 < 0. \quad (C.11) \]
For \( \alpha \gamma - \beta = 0 \) we find \( \Psi_1 = 0 \) and \( -\alpha \gamma \pi = -\beta \pi < \Psi_3 < 0. \)
As a consequence for \( \alpha \gamma - \beta \leq 0 \), by summing \( (\Psi = \Psi_1 + \Psi_3) \), we finally get
\[ -\pi < -\beta \pi < \Psi < 0, \quad (C.12) \]
so the inequality (C.5) is proved since \( 0 < \beta < 1 \). The limiting case \( \beta = 1 \) can be seen to be included.