Global Attractors for Semi-Linear PDEs Involving Degenerate Elliptic Operators

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Introduction

Problem

\[
\frac{\partial}{\partial t} u = \Delta_\lambda u + f(u) \quad \Omega \times (0, \infty),
\]

\[
u|_{\partial \Omega} = 0 \quad \partial \Omega \times [0, \infty),
\]

\[
u|_{t=0} = u_0 \quad \Omega \times \{0\},
\]

in a bounded, smooth domain \( \Omega \subset \mathbb{R}^N \), where

\[
\Delta_\lambda := \sum_{i=1}^{N} \partial_{x_i} (\lambda_i^2 \partial_{x_i}), \quad \lambda = (\lambda_1, \ldots, \lambda_N) \quad \text{is sub-elliptic.}
\]

- Local and global well-posedness
- Longtime behavior: Existence and finite fractal dimension of the global attractor, convergence to equilibria
Global Attractors for Semi-Linear PDEs Involving Degenerate Elliptic Operators
∆_λ-Laplacians

∆_λ-Laplacians

- Include, as a particular case, Grushin-type operators
- First introduced and studied in 1983
- Existence and regularity of weak solutions of the semilinear sub-elliptic problem

\[ \Delta_\lambda u = f(u) \]

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Global Attractors for Semi-Linear PDEs Involving Degenerate Elliptic Operators

\[ \Delta_\lambda \text{-Laplacians: } \Delta_\lambda := \sum_{i=1}^{N} \partial x_i (\lambda_i^2 \partial x_i) \]

\( \lambda_1, \ldots, \lambda_N \) continuous, strictly positive and \( C^1 \) outside the coordinate hyperplanes,

- \( \lambda_1(x) \equiv 1, \lambda_i(x) = \lambda_i(x_1, \ldots, x_{i-1}), \ i = 2, \ldots, N. \)
- \( \lambda_i(x) = \lambda_i(x^*), \) where \( x^* = (|x_1|, \ldots, |x_N|). \)
- There exists a group of dilations \( (\delta_r)_{r>0} \)
  \( \delta_r(x_1, \ldots, x_N) = (r^{\varepsilon_1}x_1, \ldots, r^{\varepsilon_N}x_N), \ 1 \leq \varepsilon_1 \leq \varepsilon_2 \leq \cdots \leq \varepsilon_N, \)
  such that \( \lambda_i \) is \( \delta_r \)-homogeneous of degree \( \varepsilon_i - 1, \)
  \[ \lambda_i(\delta_r(x)) = r^{\varepsilon_i-1} \lambda_i(x) \quad \forall x \in \mathbb{R}^N, \ r > 0. \]

\( Q := \varepsilon_1 + \cdots + \varepsilon_N, \) is the homogeneous dimension of \( \mathbb{R}^N \) with respect to \( (\delta_r)_{r>0}. \)
Examples

1. Grushin-type operators:

\[ \Delta_\lambda = \Delta_x + |x|^{2\alpha} \Delta_y, \]

where \((x, y) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}\) and \(\alpha \geq 0\). We find

\[ \delta_r (x, y) = \left( r x, r^{\alpha + 1} y \right), \]

and \(Q = N_1 + N_2 (\alpha + 1)\).
Examples

2. Let $\alpha, \beta, \gamma \geq 0$. For the operator

$$\Delta_\lambda = \Delta_x + |x|^{2\alpha} \Delta_y + |x|^{2\beta} |y|^{2\gamma} \Delta_z,$$

where $(x, y, z) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \times \mathbb{R}^{N_3}$, we find

$$\delta_r (x, y, z) = \left( rx, r^{\alpha+1} y, r^{\beta+(\alpha+1)\gamma+1} z \right),$$

and $Q = N_1 + (\alpha + 1)N_2 + (\beta + (\alpha + 1)\gamma + 1)N_3$. 
Abstract Semilinear Parabolic Problems

We consider
\[ u_t = Au + f(u) \quad t > 0, \]
\[ u|_{t=0} = u_0 \]
where \( A \) is positive, sectorial in the Banach space \( X \).

\( A \) generates an analytic semigroup \( e^{-At}, t \geq 0 \), in \( X \), \( X^\gamma, \gamma \in [0, 1] \), associated fractional power spaces,
\[ \mathcal{D}(A) = X^1 \leftrightarrow X^\gamma \leftrightarrow X^0 = X. \]
Abstract Semilinear Parabolic Problems

Theorem
If $f : X^\gamma \rightarrow X$ is Lipschitz on bounded subsets of $X^\gamma$, then $\forall u_0 \in X^\gamma$ there exists a unique solution, defined on the max. interval of existence $[0, T[$,

$$u \in C([0, T[; X^\gamma) \cap C^1((0, T); X^\beta) \quad \forall \beta \in [0, 1],$$

either $T = \infty$ or, if $T < \infty$, then $\limsup_{t \rightarrow T} \|u(t)\|_{X^\gamma} = \infty$, and $u$ satisfies the variation of constants formula

$$u(t) = e^{-At} u_0 + \int_0^t e^{-A(t-s)} f(u(s)) ds \quad t \in [0, T[.$$
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Functional Setting and Embedding Properties

Functional Setting

Let $\mathring{W}^{1,2}_\lambda(\Omega)$ be the closure of $C^1_0(\Omega)$ with respect to

$$
\|u\|_{\mathring{W}^{1,2}_\lambda(\Omega)} := \left( \int_{\Omega} |\nabla_\lambda u(x)|^2 \, dx \right)^{\frac{1}{2}},
$$

where $\nabla_\lambda u = (\lambda_1 \partial_{x_1} u, \ldots, \lambda_1 \partial_{x_N} u)$, $|\nabla_\lambda u|^2 := \sum_{i=1}^N |\lambda_i \partial_{x_i} u|^2$.

Lemma (Poincaré-type inequality)

There exists $C > 0$ such that

$$
\|u\|_{L^2(\Omega)} \leq C \|u\|_{\mathring{W}^{1,2}_\lambda(\Omega)} \quad \forall u \in C^1_0(\Omega).
$$
Furthermore, $-\Delta_\lambda$ is selfadjoint and densely defined in $L^2(\Omega)$
\[ \implies A := -\Delta_\lambda \text{ positive, sectorial}, \]
\[ D(A) = X^1 \hookrightarrow \dot{W}^{1,2}_\lambda(\Omega) = X^1_2 \hookrightarrow L^2(\Omega) = X^0. \]

$A$ can be extended and restricted to a positive sectorial operator in $X^\alpha$ with domain $X^{\alpha+1}$, $\alpha \geq -1$.

The analytic semigroups in $X^\alpha$ and $X^\beta$ are obtained from each other by natural extension and restriction,
\[ \| e^{-At} \|_{L(X^\alpha;X^\beta)} \leq \frac{C_{\alpha,\beta}}{t^{\alpha-\beta}} \quad -1 \leq \beta < \alpha < \infty, \ t > 0. \]
Embedding Properties\(^1\)

\[
\frac{\partial}{\partial t} u = \Delta_\lambda u + f(u),
\]

\[
u|_{\partial \Omega} = 0,
\]

\[
u|_{t=0} = u_0,
\]

\[
u_0 \in \dot{W}^{1,2}_\lambda(\Omega) = X^{\frac{1}{2}}.
\]

**Theorem (Sobolev-type embedding)**

Let \(2^*_\lambda := \frac{2Q}{Q-2}\). Then, the embedding

\[
\dot{W}^{1,2}_\lambda(\Omega) \hookrightarrow L^p(\Omega)
\]

is continuous for \(p \in [1, 2^*_\lambda]\) and compact for every \(p \in [1, 2^*_\lambda[\).

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Local Well-Posedness

We assume $f : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz and

$$|f(u) - f(v)| \leq c|u - v|(1 + |u|^{\rho} + |v|^{\rho}), \quad 0 < \rho < \frac{4}{Q-2}.$$  

The Sobolev-type embedding theorem implies

- **Case 1:** $0 < \rho \leq \frac{2}{Q-2}$
  
  $f : X^{\frac{1}{2}} \to X^{0}$ is Lipschitz on bounded subsets of $X^{\frac{1}{2}}$.

- **Case 2:** $\frac{2}{Q-2} < \rho < \frac{4}{Q-2}$
  
  There exists $\alpha \in (0, \frac{1}{2})$ such that $f : X^{\frac{1}{2}} \to X^{-\alpha}$ is Lipschitz on bounded subsets of $X^{\frac{1}{2}}$. 
Local Well-Posedness

**Theorem**

For every \( u_0 \in X^{1/2} = \dot{W}^{1,2}_\lambda(\Omega) \) there exists a unique solution, defined on the maximal interval of existence \([0, T[\),

\[
u \in C([0, T[; X^{1/2}) \cap C^1((0, T); X^{1/2}),
\]

either \( T = \infty \) or, if \( T < \infty \), then \( \limsup_{t \to T} \| u(t) \|_{X^{1/2}} = \infty \), and \( u \) satisfies the variation of constants formula

\[
u(t) = e^{-At} u_0 + \int_0^t e^{-A(t-s)} f(u(s)) ds, \quad t \in [0, T[.
\]
Global Existence of Solutions

We additionally assume the dissipativity condition:

$$\limsup_{|u| \to \infty} \frac{f(u)}{u} < \mu,$$

where $\mu > 0$ denotes the first eigenvalue of $-\Delta_\lambda$ on $\Omega$,

and consider the Lyapunov functional $\Phi : X^{\frac{1}{2}} \to \mathbb{R}$,

$$\Phi(u) := \int_\Omega \left( \frac{1}{2} |\nabla_\lambda u|^2 - F(u) \right), \quad F(u) := \int_0^u f(s)ds.$$
Global Existence of Solutions

If $u$ is a solution of (1), then

$$
\frac{d}{dt} \Phi(u(t)) = -\|u_t(t)\|_{L^2(\Omega)}^2 \quad t > 0,
$$

and moreover, for some constants $c_*, c^* \geq 0$,

$$
c_* (1 + \|u(t)\|_{X_1^2}^2) \leq \Phi(u(t)) \leq \Phi(u_0) \leq c^* (1 + \|u_0\|_{X_1^2}^2 + \|u_0\|_{L^{\rho+2}(\Omega)}^{\rho+2}).
$$

Consequently, solutions exist globally.
Main Result \(^1\)

Let \( S(t), t \geq 0, \) be the semigroup in \( \dot{W}_\lambda^{1,2}(\Omega) \) generated by Problem (1),
\[
S(t)u_0 = u(t; u_0) \quad t \geq 0.
\]

Theorem

*The semigroup \( S(t), t \geq 0, \) possesses a global attractor \( A \) in \( \dot{W}_\lambda^{1,2}(\Omega) \), which is connected and of finite fractal dimension. Furthermore,
\[
A = \mathcal{W}^u(\mathcal{E}),
\]
the omega-limit set \( \omega(u_0) \subset \mathcal{E} = \{u \mid \Delta_\lambda u + f(u) = 0\} \) and
\[
\lim_{t \to \infty} \text{dist}_H(S(t)u_0, \mathcal{E}) = 0, \quad \forall u_0 \in \dot{W}_\lambda^{1,2}(\Omega).
\]

Generalizations

Equations involving $X$-elliptic operators

$$\mathcal{L} = \sum_{i,j=1}^{N} \partial_i (a_{ij} \partial_j u), \quad a_{ij} = a_{ji},$$

$X := \{ X_1, \ldots, X_m \}$ vector fields in $\mathbb{R}^N$, $X_j = \sum_{k=1}^{N} \alpha_{jk} \partial x_k$.

$$\frac{1}{C} \sum_{j=1}^{m} \langle X_j(x), \xi \rangle^2 \leq \sum_{i,j=1}^{N} a_{ij}(x) \xi_i \xi_j \leq C \sum_{j=1}^{m} \langle X_j(x), \xi \rangle^2 \quad \forall x, \xi \in \mathbb{R}^N,$$

$$\langle X_j(x), \xi \rangle = \sum_{k=1}^{N} \alpha_{jk}(x) \xi_k, \quad j = 1, \ldots, m.$$
Examples of Admissible $X$-elliptic Operators

- $\Delta_\lambda$-Laplacians,
  e.g., Grushin-type operators
  \[
  \Delta_x + |x|^{2\alpha} \Delta_y
  \]

- Sub-Laplacians on Carnot groups,
  e.g., Kohn Laplacian on the Heisenberg group
  \[
  \Delta_{\mathbb{H}^N} = \sum_{j=1}^{N} (X_j^2 + Y_j^2),
  \]
  where the vector fields
  \[
  X_j = \partial_{x_j} + 2y_j \partial_z, \quad Y_j = \partial_{y_j} - 2x_j \partial_z, \quad (x, y, z) \in \mathbb{R}^{2N+1}.
  \]
Generalizations \(^1\)

Global well-posedness, existence and finite fractal dimension of global attractors for

- semi-linear degenerate parabolic problems

\[ u_t = \mathcal{L}u + f(u), \]

- semi-linear degenerate damped hyperbolic problems

\[ u_{tt} + \beta u_t = \mathcal{L}u + f(u), \]

where \(\mathcal{L}\) is \(X\)-elliptic, \(f\) is dissipative and satisfies appropriate growth restrictions determined by the vector fields \(\{X_1, \ldots, X_m\}\).

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\(^1\) A.E. Kogoj, S. Sonner, *Attractors met \(X\)-elliptic operators*, submitted.
Some references


