Cosmic rays propagation in Galaxy: A fractional approach

Vladimir Uchaikin and Renat Sibatov
Russia, Ulyanovsk State University

vuchaikin@gmail.com
1. Discovery of cosmic rays

Many centuries ago people knew that electrically charged bodies gradually lose charge. In order to find the cause for this phenomena, **Victor Hess** bets his life in several balloon flights (1905). Hess received the Nobel Prize in Physics in 1936 for his discovery that the cause of the discharging is a radiation coming from the outer space!

When cosmic ray particles enter the Earth’s atmosphere and collide with atoms they produce cascades of secondary particles: extensive air showers.
Knowledge about cosmic rays influences the following fields:
Communication, air- and space-flights, climate, weather...
Medicine, biophysics, ecology, physics of atmosphere and geophysics...
Plasma physics, high energy physics, physics of elementary particles...
Solar-physics, astrophysics, space physics, gravitation, cosmology.
Dark matter, dark energy, dark forces…
Cosmic Rays are the tool to find answers to the following questions: “What is our Universe? When did it born? When will it die?”

But first of all we should know where cosmic rays are born and how they come to the Earth.
3. What do people measure?
Energy spectra, mass composition, angular distribution.
4. Propagation of CR’s in the Galaxy

Interstellar medium: dust, gas, plasma, magnetic fields – long lines, clouds, random irregular structures.

Sources: bearing in supernovas bursts and accelerating on theirs remnants.

"Flat halo" model (Ginzburg & Ptuskin 1976)

The problem is to develop such propagation model which could predict all observed characteristics of CR: mass composition, energy spectra, angular distributions.
5. Homogeneous Brownian model

The problem has not been solved yet, because at least three substances interacts with each other: gas, magnetic field, high-energy particles. As a rule, only linearized versions of the complicated process are considered yet. They are based on linear kinetic equation, asymptotically leading to the normal diffusion equation

\[ \frac{\partial f}{\partial t} = D \Delta f(x, t) + \delta(x)\delta(t). \]

This approach was first proposed for CR’s propagation

Ginzburg V.L., Syrovatskii S.I.

6. Non-homogeneous diffusion model

For turbulent diffusion: **L. Richardson (1926)** described this process in frame of ordinary diffusion model with an only assumption: the diffusion coefficient must depend on the space coordinates

\[
\frac{\partial f}{\partial t} = \nabla (D(|\mathbf{x}|\nabla f(\mathbf{x}, t))) + \delta(\mathbf{x})\delta(t), \quad D(r) \propto r^{4/3}.
\]

The width of the diffusion packet did grow more rapidly then in the normal case, \(\sim t^{3/2}\) instead of \(\sim t^{1/2}\) and coincided with what observed in experiments, although the shape of the packet was not investigated in details. Moreover, one couldn't find any physical reason for a distance dependence of the diffusivity.
7. The Levy-motion model

Considering transport in turbulent media, Russian mathematician Monin (1955) obtained the Fourier transform of the following equation with fractional Laplacian

$$\frac{\partial f(k, t)}{\partial t} = D |k|^\alpha f(k, t) + \delta(t), \quad \alpha \in (0, 2)$$

Monin ment hydrodynamic systems, neither cosmic rays nor cosmic plasma.

The introduction of differential equation with fractional Laplacian given by A. S. Monin was rather formal. The more physical way was used by Saichev & Zaslavsky (1997) and independently of them by Uchaikin & Zolotarev (1999). In frame of CTRW formalism, they established that the fractional character of Laplacian introduced by Monin as a formal consequence of Kolmogorov-Obukhov laws of turbulence, is a result of non-exponential but power-law free path distribution in the particles kinetics.
8. The Brown- and Levy-trajectories

\[
\frac{\partial f}{\partial t} = D \Delta f(x, t) + \delta(x) \delta(t).
\]

\[
\frac{\partial f}{\partial t} = -D(-\Delta)^{\alpha/2} f(x, t) + \delta(x) \delta(t), \quad \alpha \in (0, 2)
\]
9. Why $\alpha<2$?


On assumption $D(E)=CE^n$

spatial distribution of CR’s produced by a point source is converted into spectrum
10. Why the diffusion is not normal

Traps and flights
11. Time-fractional equations

Let us come back to works of Saichev & Zaslavsky (1997) and Uchaikin & Zolotarev (1999). The equations derived there contained not only fractional Laplacian but fractional time derivative. The physical meaning of the operator depends on the sense of other variable. If it is a space coordinate then the fractional order of time-derivative means existing of traps, braking Markovian character of the process and showing the presence of memory. If the other variable is energy, momentum or type of the particle, then the fractionality means fractal character of the interstellar medium. Both cases have a physical basement to be considered.

\[ 0D_t^\beta f(x, t) = D\Delta f(x, t) + S(x, t) \]

As an example, one can refer to the work of R. Balescu (2000) [Plasma Phys. Control Fusion 42, B1, 2000], where the equation was derived not from CTRW formalism but from kinetic equation for plasma in magnetic field. The case with another sense appeared in problem of multiple scattering of light on gravitation field of galaxies [Uchaikin, Korobko, 1998, 1999].
12. Bifractional model and the “knee”

\[
\frac{\partial N}{\partial t} = -D_\alpha(E) \frac{\partial^{1-\beta}}{\partial t^{1-\beta}} (-\Delta)^{\alpha/2} N(r, t, E) + S(r, t, E).
\]

\[S(r, t, E) = S_0 E^{-\rho} \delta(r) 1+(t)\]

Self-similar solution

\[N(r, t, E) = S_0 E^{-\rho} [D_0 \alpha E^\delta t^\beta]^{-3/\alpha} \Psi_3^{(\alpha, \beta)} \left( r [D_0 \alpha E^\delta t^\beta]^{-1/\alpha} \right)\]

with power asymptotics

\[N(r, t, E) \approx \begin{cases} 
S_0 D_0 \alpha t^{\beta} r^{-3-\alpha} E^{-p+\delta}, & E < E_<; \\
S_0 [D_0 \alpha t^\beta]^{-1} r^{-3+\alpha} E^{-p-\delta}, & E > E_>.
\end{cases}\]

expressed through the fractional stable distribution.
13. Bifractional model and anisotropy

In the case of a point instantaneous isotropic source

\[ J(r, t) = \frac{1}{r^2} \frac{d}{dt} \int_r^\infty N(r, t) r^2 \, dr \]

\[ \int_r^\infty N(r, t) r^2 \, dr = \int_r^\infty \left[ Dt^\beta \right]^{-3/\alpha} \Psi_3^{(\alpha, \beta)} \left( r \left[ Dt^\beta \right]^{-1/\alpha} \right) r^2 \, dr \]

\[ = \int_r^\infty \Psi_3^{(\alpha, \beta)} (\xi) \xi^2 \, d\xi. \]

Finally

\[ J(r, t) = N(r, t) \frac{\beta r}{\alpha t} \]
14. Finite-velocity-diffusion (FVD) model

$T$ and $R$ are uncorrelated

$T$ and $R$ fully correlated

Uchaikin 2010 (JETP Letters)
“About fractional differential model of cosmic rays transport in Galaxy“

$$D_t^\alpha + D(\Delta)^{\alpha/2} \leftrightarrow \langle (D_t + v\Omega \nabla)^\alpha \rangle$$
15. Time-flight correlations effect

Monte Carlo simulation
16. The mean escape time

The mean escape time can be calculated according to the following formula:

$$\tau_{esc} = \varepsilon\tau_{SB} + \varepsilon(1 - \varepsilon)[\tau_{SB} + \tau_{BB}] + \varepsilon(1 - \varepsilon)^2[\tau_{SB} + 2\tau_{BB}] + \ldots = \tau_{SB} + \frac{1 - \varepsilon}{\varepsilon}\tau_{BB},$$

Figure 4. Escape time versus transparency. The instantaneous point source is situated on the middle plane.
17. PDF of the transverse coordinate
Stochastic acceleration: additive Levy walks in $p$-space

\[ p = p_0 + \Delta p_1 + \Delta p_2 + \Delta p_3 + \ldots \]

\[ 0 D_t^\alpha f(p, t) = \mu A f(p, t) + f_0(p) \delta_\alpha(t) \]

\[ \int [w(\Delta p; p - \Delta p)f(p - \Delta p, t) - w(\Delta p, p)f(p, t)] d\Delta p \]

\[ \int_{|\Delta p| > p} w(\Delta p; p') d\Delta p \propto p^{-\gamma}, \ p \to \infty \]

\[ 0 D_t^\alpha f(p, t) = -K(-\Delta p)^{\nu/2} f(p, t) + f_0(p) \delta_\alpha(t) \]
19. Stochastic acceleration: multiplicative Levy walks

Multiple acceleration of cosmic rays in Supernova remnants Wandel et al. 1987

\[ \Delta \mathbf{p} = p' q, \quad \int_{|\Delta \mathbf{p}| > p} w(\Delta \mathbf{p}; p')d\Delta \mathbf{p} \propto (p/p')^{-\gamma}, \quad p \to \infty \]

\[ \Delta p \sim p' \quad V(q) = \gamma q^{-\gamma - 1}, \quad \gamma > 1 \]

Acceleration of cosmic rays in Supernova remnants distributed fractally Uchaikin 2010

\[ 0 \mathcal{D}_t^\alpha n(E, t) = \mu \left\{ \int_1^\infty \gamma q^{-\gamma - 1} n(E/q, t) dq/q - n(E, t) \right\} + n_0(E) \delta_\alpha(t) \]

Change in spectrum exponent: \( \alpha < 1 \) instead of 1

\[ N_\alpha(E; \tau) = \frac{\mu \tau^\alpha \gamma}{(1 + \mu \tau^\alpha)^2} \left( \frac{E}{E_0} \right)^{-1 - \gamma/(1 + \mu \tau^\alpha)} \frac{1}{E_0} \]
20. Problems and Perspectives

- 20.1. On fractional time-derivatives
- 20.2. Uchaikin-Sibatov equation
- 20.3. On fractional Laplacian
- 20.4. Non-locality
- 20.5. On fractional gradient
- 20.6. Natural definition of frac. gradient.
- 20.7. Natural definition of frac. total derivat.
20.1. On fractional time-derivatives

Carreras et al 2001 Trapping time distribution in plasma Numerical simulation

Cadavid et al 1999 (Anomalous diffusion of solar magnetic elements)—observation data
20.2. Uchaikin-Sibatov equation

The exponential truncation of the time-fractional integral kernel

\[
\frac{\partial^{1-\beta} n(r, t)}{\partial t^{1-\beta}} = \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_0^t (t-\tau)^{-(1-\beta)} n(r, \tau) d\tau
\]

\[
\mapsto \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_0^t (t-\tau)^{-(1-\beta)} e^{-\gamma(t-\tau)} n(r, \tau) d\tau = e^{-\gamma t} \frac{\partial^{1-\beta}}{\partial t^{1-\beta}} \left[ e^{\gamma t} n(r, t) \right]
\]

leads to the following equation (Uchaikin-Sibatov 2010)

\[
\frac{\partial n}{\partial t} = D \Delta e^{-\gamma t} \frac{\partial^{1-\beta}}{\partial t^{1-\beta}} \left[ e^{\gamma t} n(r, t) \right] + \delta(r) \delta(t) =
\]

\[
\Rightarrow \begin{cases} 
\frac{\partial n}{\partial t} = D \Delta n(r, t) + \delta(r) \delta(t), & t \to \infty; \\
\frac{\partial^\beta n}{\partial t^\beta} = D \Delta n(r, t) + \delta(r) \delta_\beta(t), & t \to 0.
\end{cases}
\]
20.3. On fractional Laplacian

A typical spectrum of the Kolmogorov-Obukhov turbulence model.

As should be for a fractional Laplacian.
20.4. Space-craft measurement of magnetic fluctuations

- Form of the spectrum?
  - Inertial range: $k^{-5/3}$
  - Energy range: $k^{-1}$?
  - Dissipation range: $k^{-2.9}$?

- How to model the spectrum?
  $$G(k) \propto \frac{k^q}{(1 + k^2)^{(s+q)/2}}$$

- Time dependence
  - MHD plasma waves
  - Dynamic decorrelation

- Parallel/perpendicular directions?
  - Goldreich-Sridhar
20.5. Non-locality

1\textsuperscript{st} consequence

\[ |k|^{\alpha} = (k_x^2 + k_y^2 + k_z^2)^{\alpha/2} \neq |k_x|^{\alpha} + |k_y|^{\alpha} + |k_z|^{\alpha} \]

The variable separation method is inapplicable

2\textsuperscript{nd} consequence

The operator becomes regional, including boundary conditions

\[-(-\Delta)^{\alpha/2}_G f(x) \equiv \lim_{\varepsilon \downarrow 0} C(\alpha) \int_{G, \ |x-y| > \varepsilon} \frac{f(x) - f(y)}{|y - x|^{d+\alpha}} dy \]

(Rafeiro, Samko 2005; Guan, Ma 2005; Krepysheva et al 2006; Zoia et al 2007; ...)

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20.6. On fractional gradient

\[ \nabla^\alpha f(x, y) = f_x^{\alpha}(x, y)e_x + f_y^{\alpha}(x, y)e_y. \]

\[
\frac{1}{\Gamma(1 - \alpha)} \frac{\partial}{\partial y} \int_y^y \frac{f(x, y')dy'}{(y - y')^\alpha}
\]
20.7. Fractional gradient depends on a large part of the curve.
20.8. Natural definition of fractional gradient

\[
\frac{1}{\Gamma(1 - \alpha)} \frac{\partial}{\partial y} \int_0^y \frac{f(x'(t'), y'(t')) dy'(t')}{(y - y'(t'))^\alpha}
\]

Here the curve is given in a parametric form and \( t \) is a parameter.
20.9. Natural definition of the total derivative

In a similar way, the definition

\[
\left( \frac{\partial}{\partial t} + \mathbf{v} \nabla \right)^\alpha f(\mathbf{r}, t) = \frac{1}{\Gamma(1 - \alpha)} \left( \frac{\partial}{\partial t} + \mathbf{v} \nabla \right) \int_{-\infty}^{t} \frac{f(\mathbf{r} - (t - \tau)\mathbf{v}, \tau)}{(t - \tau)^\alpha} d\tau
\]

can be naturalized as

\[
\left( \frac{\partial}{\partial t} + \mathbf{v} \nabla \right)^\alpha f(\mathbf{r}, t) = \frac{1}{\Gamma(1 - \alpha)} \left( \frac{\partial}{\partial t} + \mathbf{v} \nabla \right) \int_{-\infty}^{t} \frac{f(\mathbf{R}(\tau; \mathbf{r}, t), \tau)}{(t - \tau)^\alpha} d\tau,
\]

where \( \mathbf{R}(\tau; \mathbf{r}, t) \) is determined from a real (non-linear) trajectory of a particle.
Fractional model for CR diffusion along random magnetic lines

Vladimir Uchaikin and Renat Sibatov
Ulyanovsk State University, Russian Federation

vuchaikin@gmail.com
1. Different types of magnetic field turbulence

Xu and Yan, 2013

- Isotropic (super-Alfvénic) turbulence (large scales)
- Anisotropic (sub-Alfvénic) turbulence (small scales)

Diagram showing magnetic field lines and CR trajectories.
1. Transport of Cosmic Rays in the Solar System

- - - - - charged particles trajectories
- - - - - the magnetic field of the Sun
----- the turbulent fields of the solar wind
2. Compound model

Now the CR particles are assumed to scatter back and forth along the field lines, and the perpendicular transport of particles across the mean field is due to random walk of the field lines (see also Getmantsev 1963, Kota and Jokipii 2000). The mean magnetic field is directed along the z-axis, and \( d_\xi \) represents arc length along the field.

Assuming that \( P(x|z) \) for magnetic FRW does not depend on time, Webb et al used the Chapman-Kolmogorov equation

\[
P_\perp(x, t|x_0, t_0) = \int_{-\infty}^{\infty} dz P_{\text{FRW}}(x|z) P_{p\parallel}(z, t|z_0, t_0)
\]
3. Parallel transport model

\[ v = \text{const} \]

Integral equations of the process

\[
G_{\|}(z, t) = \int_{t}^{0} \left[ \gamma_1 f(z - v\tau, t - \tau) + \gamma_2 f(z + v\tau, t - \tau) \right] P(v\tau) d\tau
\]

\[
f(z, t) = \int_{0}^{t} \left[ \gamma_1 f(z - v\tau, t - \tau) + \gamma_2 f(z + v\tau, t - \tau) \right] p(v\tau) d\tau + \delta(z)\delta(t)
\]

\[
\tilde{G}_{\|}(k, \lambda) = \frac{\lambda/v - ik\beta - \gamma_1(\lambda/v - ik)\tilde{p}(\lambda/v + ik) - \gamma_2(\lambda/v + ik)\tilde{p}(\lambda/v - ik)}{v[k^2 + (\lambda/v)^2][1 - \gamma_2\tilde{p}(\lambda/v - ik) - \gamma_1\tilde{p}(\lambda/v + ik)]}
\]
4. Asymptotical propagators for $0 < \alpha < 1$

\[
\left[ \gamma_2 \left( \frac{\partial}{\partial t} + v \frac{\partial}{\partial x} \right)^\alpha + \gamma_1 \left( \frac{\partial}{\partial t} - v \frac{\partial}{\partial x} \right)^\alpha \right] G_{\parallel}(x, t) = \frac{t^{-\alpha}}{\Gamma(1 - \alpha)} \left[ \gamma_2 \delta(x + vt) + \gamma_1 \delta(x - vt) \right].
\]

\[
G_{\parallel}(x, t) = \frac{2 \sin \pi \alpha}{\pi vt} \frac{\gamma_1 \gamma_2 (1 - x^2/v^2t^2)^{\alpha - 1}}{\gamma_1^2 (1 - x/\nu t)^{2\alpha} + \gamma_2^2 (1 + x/\nu t)^{2\alpha} + 2 \gamma_1 \gamma_2 (1 - x^2/v^2t^2)^\alpha \cos \pi \alpha}
\]
5. Tempered diffusion

Ballistic restriction of packet spreading when $\alpha < 1$
Dots show our results of Monte Carlo simulation

Comparison of propagators for restricted and non-restricted diffusion processes
6. Parallel propagators for different $\alpha$ and various time
7. Diffusion model for FRW

Field line Random Walk

(Webb et al, 2006)
8. But MF lines can not be Brownian


the Brownian model produces an ensemble of nowhere differentiable lines each element of which has an infinite length.
9. Smoothed Brownian lines

Moving average method?

\[ X(z) \mapsto X(z, \delta) = \frac{1}{\delta} \int_{z-\delta/2}^{z+\delta/2} X(z') \, dz'. \]

Fractional Integral?

\[ X^H(t) = \frac{1}{\Gamma(H + 1/2)} \int_0^t (t - s)^{H-1/2} \, dB(s) \]

Fractional Brownian motion!

\[
\frac{1}{\Gamma(H + 1/2)} \left\{ \int_{-\infty}^0 \left[ (t - s)^{H-1/2} - (-s)^{H-1/2} \right] \, dB(s) + \int_0^t (t - s)^{H-1/2} \, dB(s) \right\}
\]
10. Fractional Brownian field lines

- $n=1000, H=0.1$
- $n=1000, H=0.3$
- $n=1000, H=0.5$
- $n=1000, H=0.7$
- $n=1000, H=0.8$
- $n=1000, H=0.9$
11. Ensembles of fB field lines
12. Equations for field line process

From plasma turbulence theory [Shalchi & Kourakis 2007]

\[ \left\langle (\Delta x)^2 \right\rangle \sim \left[ \frac{9\pi}{2B_0^2} \sqrt{\frac{\pi}{2}} g^2 D(0) \right]^{2/3} \left| z \right|^{4/3} \]

Assumption on Gaussian

\[ f(x(z)) = \frac{1}{2\pi \left\langle (\Delta x)^2 \right\rangle} \exp \left[ -\frac{x^2}{2 \left\langle (\Delta x)^2 \right\rangle} \right] \]

leads to an fBm equation

\[ \frac{\partial f(x, z)}{\partial z} = 2HK \cdot z^{2H-1} \frac{\partial^2 f(x, z)}{\partial x^2} \]

not to fractional one.

The perpendicular walk of lines (with respect to z) is A SUPERDIFFUSION WITH A FINITE VARIANCE
13. Particle walk on random frozen field line
14. Perpendicular diffusion
15. Perpendicular diffusion
16. Ion superdiffusion at the solar wind termination shock


The power-law fits better than the exponential for all energy channels observed by Voyager 2, with \( \gamma = 0.68-0.71 \) corresponding to superdiffusion:

\[
\left\langle \Delta x(t)^2 \right\rangle \propto t^{1.29-1.32}
\]

Fit Parameters for the Ion Time Profiles at the Termination Shock

<table>
<thead>
<tr>
<th>Energy (keV)</th>
<th>( \gamma )</th>
<th>( \alpha )</th>
<th>( \chi^2_{pl} )</th>
<th>( \chi^2_e )</th>
</tr>
</thead>
<tbody>
<tr>
<td>540–990 keV</td>
<td>0.70 ± 0.07</td>
<td>1.30</td>
<td>0.22</td>
<td>0.40</td>
</tr>
<tr>
<td>990–2140 keV</td>
<td>0.71 ± 0.08</td>
<td>1.29</td>
<td>0.18</td>
<td>0.25</td>
</tr>
<tr>
<td>2140–3500 keV</td>
<td>0.68 ± 0.15</td>
<td>1.32</td>
<td>0.05</td>
<td>0.07</td>
</tr>
</tbody>
</table>
17. Data set from Ulysses spacecraft

Perri & Zimbardo, 2007, ULYSSE

Proton fluxes in log-log axes for the shock crossing observed by Ulysses at 5AU. Solid lines show tour calculation results.

Electron fluxes in log-log axes for the shock crossing observed by Ulysses at 5AU. Solid lines show tour calculation results.
18. Isotropic and anisotropic diffusion of galactic cosmic ray protons

Effenberger et al, 2012
16. Conclusions

We considered a few problems from cosmic rays astrophysics and each time we should seek and represent some special ground for introducing fractional derivatives as tools for their solutions. At the same time, most of theorists solve cosmic ray problems without using fractional calculus.

The question arises: is the use of fractional calculus in our works the result of our personal liking? Or there exist some objective reasons for this? Is there a common reason for appearance of fractional derivatives in calculation? And in what calculations?

We think the answer is "yes".

Therefore, one can advance the hypothesis: fractional derivatives can result from hiding variables.

What is another subsystem which variables are hidden when considering CR problems in a linear approximation? With no doubt, it is the interstellar magnetic field. Both subsystems, CR and IMF interact with each other. Motion of the total system is a Markovian process described by equations with natural derivatives. We obtain fractional equation for CR only after elimination of IMF variables.

Of course, this is only a hypothesis yet, because nobody performs this elimination in a general case. Perhaps, it should added by self-similarity condition, coming from turbulent nature of the process. Nevertheless, we may check: is this hypothesis self-concordant one? That is, if we eliminate CR variables, do we arrive at the factional equations for ISM? The question is positively answered in a row of works using fractional calculus for description of ISM with no taking account the CR influence.
My new book

Fractional Derivatives for Physicists and Engineers

Volume I  Background and Theory

Volume II  Background and Theory
Acknowledgements

Fractional phenomenology of cosmic ray anomalous diffusion

V.V. Uchaikin
Ul’yanovsk State University,
ul. L. Tolstogo 42, 432970 Ul’yanovsk, Russian Federation
E-mail: vuchaikin@gmail.com

Evolution of the cosmic ray diffusion concept from the ordinary (Einsteinian) ray propagation in the Galaxy appeared in the last decade is reviewed. Many perspectives are discussed.

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