Stability: a motivation to study other inverse problems

Juan Antonio Barceló & Pedro Caro

Numerical Resolution for Inverse Problems
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Outline

The Calderón problem and electrical impedance tomography

Related inverse boundary value problems

Ways to improve the stability/resolution
   Multimodality imaging methods
   Hybrid inverse problems

Final comments and references
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General goal of the Calderón problem

The inverse Calderón problem consists of recovering the electric properties of a medium, namely the conductivity, by non-invasive measurements of many configuration of voltages and currents, more precisely measurements on the surface.
Setting the Calderón problem

- The medium is modelled by a bounded $\Omega \subset \mathbb{R}^n$ with $n \geq 2$.
- The electrical property we are interested in is the conductivity, represented by $\sigma$.
- Consider in $\Omega$, an electric field which has reached a steady state. Assume it to be conservative and determined by the potential $u$. Then, the ohmic current $\sigma \nabla u$, in the absence of sources and sinks, satisfies
  \[ \nabla \cdot (\sigma \nabla u) = 0. \]

Prescribing a voltage $f$ on $\partial \Omega$, the induced potential solves the problem:

\[
\begin{aligned}
\{ & \nabla \cdot (\sigma \nabla u) = 0, \\
& u|_{\partial \Omega} = f.
\end{aligned}
\]
The Calderón problem

In the Calderón problem, the boundary measurements consist of non-invasive measurements of arbitrary configurations of voltages and currents of the boundary. These are nothing but restrictions to $\partial \Omega$ of all possible solutions for the conductivity equation.

- The DN map:

  $\Lambda_\sigma : u|_{\partial \Omega} \longrightarrow \sigma \partial_\nu u|_{\partial \Omega}$.

The inverse Calderón problem consists of reconstructing $\sigma$ from the DN map $\Lambda_\sigma$.

Related questions:

- **Uniqueness**: Does $\Lambda_{\sigma_1} = \Lambda_{\sigma_2}$ imply $\sigma_1 = \sigma_2$?
- **Stability**: Does there exist $\omega$ such that

  $$\|\sigma_1 - \sigma_2\| \leq \omega(\|\Lambda_{\sigma_1} - \Lambda_{\sigma_2}\|_*)?$$
Logarithmic stability under a-priori assumptions

- Sylvester and Uhlmann showed in 1987 that uniqueness holds for $n \geq 3$ under certain smoothness assumptions on $\sigma$ and $\partial \Omega$.
- Nachman provided in 1988 an algorithm to recover $\sigma$ for $n \geq 3$.
- Alessandrini proved in 1988, under certain a-priori assumptions, logarithmic stability for this problem in $n \geq 3$: If $M^{-1} \leq \sigma_j$ and $\|\sigma_j\|_{H^s} \leq M$ for $s > n/2 + 2$, then

$$\|\sigma_1 - \sigma_2\|_{L^\infty} \lesssim_M \omega(\|\Lambda_{\sigma_1} - \Lambda_{\sigma_2}\|_*)$$

with

$$\omega(t) \leq |\log t|^{-\delta}, \quad 0 < t < 1/e$$

for $0 < \delta < 1$.
- Alessandrini also showed that these a-priori assumptions are necessary to prove the previous stability estimate.
- Mandache proved in 2001 that the optimal stability under these a-priori assumptions is logarithmic.
Electrical impedance tomography: high contrast and low resolution

- EIT refers to a non-invasive medical imaging technique in which an image of the conductivity (or permittivity) of part of the body is inferred from surface electrode measurements.
- The strength of EIT basis on its high contrast, which allows to monitor representative changes in the conductivities of tissues.
- EIT is specially promising when monitoring lung functions since lung conductivity fluctuates intensely during the breath cycle.
- The Calderón problem is the mathematical model for EIT and the low resolution of this medical technique is connected with the (optimal) logarithmic stability of the inverse problem.
EIT: high contrast and low resolution

- EIT has also promising applications in breast cancer detection as an alternative or complementary technique to mammography and magnetic resonance imaging since malignant breast tissues present higher conductivities (0.2 siemens) than healthy tissues (0.03 siemens).
- The success of mammography or MRI rests on their high resolution, however, they also present a low specificity, which is result of a relatively high rate of false positive.
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General idea

We are interested in recovering other physical properties of a medium by non-invasive measurements. We look for mathematical models which provide techniques capable of monitoring representative changes of the corresponding physical properties, that is, techniques with high contrast as EIT. Therefore, our guide to find these models is the Calderón problem.
Interested in the electric permittivity

The **electric permittivity** $\varepsilon$ is a measure of how an electric field affects, and is affected by, a dielectric medium. The permittivity of a medium describes how much electric field (more correctly, flux) is 'generated' per unit charge in that medium. More electric flux exists in a medium with a low permittivity (per unit charge) because of polarization effects.

- Leukemia increases $\varepsilon$ and decreases $\sigma$ in bone marrow (a factor of two).
Inverse boundary value problem in electrodynamics

- The relevant electrical properties now are: the conductivity \( \sigma \) and the permittivity \( \varepsilon \).
- Consider in \( \Omega \), a time-harmonic electric field with frequency \( \omega \). Assume it to be conservative and determined by the potential \( u \):

\[
E(t, x) = e^{i\omega t} \nabla u(x)
\]

Then, the ohmic current \( \sigma e^{i\omega t} \nabla u \) plus the derivative of electric displacement \( \varepsilon e^{i\omega t} \nabla u \), in the absent of sources and sinks, satisfies

\[
\nabla \cdot (\gamma \nabla u) = 0, \quad \gamma = \sigma + i\omega \varepsilon.
\]

Prescribing a voltage \( f \) on \( \partial \Omega \), the induced potential solves the problem:

\[
\nabla \cdot (\gamma \nabla u) = 0, \quad u|_{\partial \Omega} = f.
\]

The DN map:

\[
\Lambda_\gamma : u|_{\partial \Omega} \rightarrow \gamma \partial_\nu u|_{\partial \Omega}.
\]

The IBVP consists of reconstructing \( \gamma \) from the DN map \( \Lambda_\gamma \).
Inverse boundary value problems in electromagnetism

We are now interested in recovering the electromagnetic properties of a medium in $\mathbb{R}^3$, that is, the electric conductivity and permittivity and the magnetic permeability (it is denoted by $\mu$ and it describes the degree of magnetization that a material obtains in response to an applied magnetic field).

- Assume the electric and magnetic fields to be time-harmonic

$$E(t, x) = e^{-i\omega t}E(x), \quad H(t, x) = e^{-i\omega t}H(x).$$

- $E, H$ satisfy the time-harmonic Maxwell equations:

$$\begin{cases} 
\nabla \times H + i\omega \varepsilon E = \sigma E, \\
\nabla \times E - i\omega \mu H = 0.
\end{cases}$$

- The boundary measurements are described by the Cauchy data set:

$$C = \{(\nu \times E|_{\partial\Omega}, \nu \times H|_{\partial\Omega})\}.$$

- The IBVP consists of recovering the conductivity, the permittivity and the permeability from Cauchy data set.
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How to improve the resolution of high contrast techniques

High contrast techniques allow to monitor representative changes in the medium, but the mathematical problems to model such methods are severely ill-posed, which translate in applications in low resolution. There are several mathematical proposal to get higher resolution for these methods:

▶ One can introduce a-priori information of different character by carefully taking into account the applied context from which the problem arises. These a-priori data is obtained from a different imaging modality for example MRI and EIT.

▶ One can use two different physical effects to couple one modality with high contrast (EIT for example) with another with high resolution (MRI for example). The mathematical analysis of such couplings forms the class of hybrid inverse problems.

▶ In certain frameworks, as in the conductivity equation with complex coefficient and the time-harmonic Maxwell equations, one can assume extra knowledge for a range of frequencies $\omega$ and prove an effect of increasing stability as $\omega$ increases. (Attend Santacesaria’s talk)
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Alessandrini and Vessella’s approach

- Keeping in mind the medical imaging motivation EIT Alessandrini and Vessella proposed different a-priori assumptions to improve the stability. In particular, they assume the conductivity $\sigma$ to be piecewise constant:

$$\sigma(x) = \sum_{j=1}^{N} \sigma_j \mathbf{1}_{D_j}(x)$$

where $D_1, \ldots, D_N$ are known subdomains of $\Omega$ and $\sigma_1, \ldots, \sigma_N$ are unknown constants.

- This assumption is motivated by the idea that $D_1, \ldots, D_N$ may represent the areas occupied by different tissues and whose geometrical configuration may be detected in advance by other imaging devices, such as MRI. Then, one can use the high contrast of EIT to detect the conductivities $\sigma_1, \ldots, \sigma_N$ of the different tissues.

- Alessandrini and Vessella proved in 2005 that

$$\|\sigma - \bar{\sigma}\|_{L^\infty} \lesssim_N \omega(\|\Lambda_{\sigma} - \Lambda_{\bar{\sigma}}\|_*)$$

with

$$\omega(t) \leq |t|^\delta, \quad 0 < t < 1$$

for $0 < \delta \leq 1$. 
Resolution limit for EIT

- Let $C$ denote the implicit constant in Alessandrini and Vessella’s inequality. Then, Rondi showed in 2006 that

$$ C \geq Ae^{BN^{1/(2n-1)}} $$

where $A$ and $B$ are absolute constants.

- Assume $\varepsilon$ to be the error on the measured DN map and say we can tolerate an error up to $C_0\varepsilon$ on the reconstructed conductivity. The error amplification tolerance $C_0$ provides an upper bound on the number of subdomains $D_1, \ldots, D_N$:

$$ N \leq \left( \frac{1}{B} \log \left( \frac{C_0}{A} \right) \right)^{2n-1} \frac{1}{2n-1} . $$

- Assuming that $|D_j| \sim r^n$ for some $r$, we have that $r \sim N^{-1/n}$. The number $r$ can be interpreted as a resolution parameter and resolution limit is

$$ r \geq \left( \frac{1}{B} \log \left( \frac{C_0}{A} \right) \right)^{-(2n-1)/n} . $$

- For fix $C_0$, no detail smaller than the resolution limit can be detected.
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Coupled-physics inverse problems

- Hybrid inverse problems concern the combination of a high contrast modality with a high resolution modality. By combination, we mean the existence of a physical mechanism that couples these two modalities.

- In the multimodality imaging methods the different physical frameworks are not coupled in one modality, they produce data independently.

- Hybrid inverse problems typically involve two steps. In a first step, a well-posed problem involving the high-resolution low-contrast modality is solved from knowledge of boundary measurements. In a second step, a quantitative reconstruction of the parameters of interest is performed from knowledge of the pointwise, internal, functionals of the parameters reconstructed during the first step.

- The physical coupling between the two modalities may happen in several ways but we are not get into this.
## Classification of potential couplings

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Magnetic Resonance Electrical Impedance Tomography

- MREIT aims at reconstructing the conductivity in a medium using MRI.
- The idea of MREIT basis on the facts that an injected current in an electrically conducting medium produces electric and magnetic fields, and that the magnetic field inside the medium can be measured by a non-contact method (using an MRI scanner). Thus, we can obtain internal information (the magnetic field) from non-invasive measurements, to reconstruct the conductivity of the medium.

Mathematically, the problem is as follows:

- An injected current on $\partial \Omega$ produces a magnetic field $B$ in $\Omega$:
  \[
  \begin{align*}
  \nabla \cdot (\sigma \nabla u) &= 0, \\
  \sigma \partial_{\nu} u|_{\partial \Omega} &= g
  \end{align*}
  \right\} \leadsto B
  \]

- Assuming the magnetic permeability $\mu_0$ to be constant and known, the ohmic current $J = \sigma \nabla u$ satisfies, by the Ampère law, that
  \[
  J = \frac{1}{\mu_0} \nabla \times B.
  \]
The two steps for this HIP

- In the first step, one reconstructs $B$ in $\Omega$ from non-invasive measurements of the Neumann data and the generated magnetic field on $\partial \Omega$.

- In the second step, one uses the internal data $\sigma \nabla u$ in $\Omega$ obtained in the previous step, by the Ampère law, to recover the conductivity $\sigma$ in $\Omega$.

- This coupled modality is Lipschitz stable since every step is Lipschitz stable.

However, this procedure is ideal since in practical applications one can not recover the whole $B$, only one of its component. Indeed, the MRI scanner measures only one component of $B$ that is parallel to the direction of the main magnetic field of the scanner. Measuring all components require rotations, which seems to be impractical.

- **Open problem**: Reconstruct $\sigma$ from internal data of only one of the components of $B$. 
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- High contrast techniques allow to monitoring significant changes in the properties of a medium.
- High contrast techniques usually provide low resolution recoveries, which is unsatisfactory in some real applications.
- The low resolution is described mathematically by the weak stability of the inverse problem associated, which are at most of logarithmic type.
- The way to improve the resolution is to modify the mathematical model to improve the stability, to Lipschitz type.
- Currently there are several ways to do so:
  - Multimodality imaging methods.
  - Hybrid inverse problems.
References


- For general information about EIT, see *Wikipedia* article Electrical impedance tomography. (Some pictures for this presentation were borrowed from Wikipedia)

- For general facts about the stability of the Calderón problem and more particularly, Alessandrini and Vessella’s approach, see Alessandrini, Open issues of stability for the inverse conductivity problem *Journal of Inverse and Ill-posed Problems*, 15, (2007).

- For general facts about hybrid inverse problems, see Bal, *Inside Out II* MSRI Publications Volume 60, 2012 (editor Uhlmann).

- Some pictures were borrowed from the blog *Scattering ideas* by Montalalto Cruz.