New numerical results for the Gel’fand-Calderon problem

Matteo Santacesaria

Department of Mathematics and Statistics,
University of Helsinki

Meeting "Numerical Resolution for Inverse Problems"
BCAM, Bilbao
9 January 2015
Gel’fand–Calderón Problem

Schrödinger equation at fixed energy $E \in \mathbb{R}$,

$$(-\Delta + v)\psi = E\psi \quad \text{in } D,$$

where $v \in L^\infty(D)$ is a potential (energy), $D \subset \mathbb{R}^d$ open bounded domain. If 0 is not a Dirichlet eigenvalue of $-\Delta + v - E$ in $D$, we can define:

$\Phi_v(E)f = \frac{\partial u}{\partial \nu}\bigg|_{\partial D}$

Dirichlet-to-Neumann map

- $f \in H^{1/2}(\partial D)$, $\nu$ exterior normal to $\partial D$, $u$ the unique solution $H^1$ of

$$\begin{cases} 
(-\Delta + v)u = Eu & \text{in } D, \\
u|_{\partial D} = f.
\end{cases}$$

- $\Phi_v(E) : H^{1/2}(\partial D) \to H^{-1/2}(\partial D)$

Gel’fand–Calderón inverse problem

Given $\Phi_v(E)$ for a fixed $E \in \mathbb{R}$, determine $v$ on $D$. 
Motivations

Zero energy ($E = 0$): Electrical impedance tomography (Calderón problem)

- $v = \frac{\Delta \sqrt{\sigma}}{\sqrt{\sigma}}$, $\sigma$ scalar conductivity
- $\Phi$ voltage-to-current map

Positive energy ($E > 0$): acoustic tomography

- time-harmonic wave equation (Helmholtz equation) for the acoustic pressure
- reconstruction of the density and velocity of propagation of sound waves from boundary data
Multi-channel case

Multi-channel Schrödinger equation at energy $E \in \mathbb{R}$:

$(-\Delta + v)\psi = E\psi$ in $D$,

- $v$ and $\psi$ are $M_n(\mathbb{C})$-valued functions (the set of $n \times n$ complex matrices).

Motivation: approximation of the 3D equation.

- The Schrödinger equation in a cylindrical domain $D \times L, D \subset \mathbb{R}^2, L = [a, b] \subset \mathbb{R}$, is equivalent to an infinite dimensional system of equations on $D$.

- Principal advantage: the 2D problem is not overdetermined. $\Phi$ and $v$ depend on the same number of variables, while this is not true in higher dimensions.
  (if $D \subset \mathbb{R}^n$, $\Phi$ depends on $2(n-1)$ variables and $v$ on $n$)
The inverse scattering problem

Consider the continuous solutions $\psi^+(x, k)$ of the Schrödinger equation on $\mathbb{R}^2$, where $k$ is a parameter, $k \in \mathbb{R}^2$, $k^2 = E$, such that

$$\psi^+(x, k) = e^{ikx}I - i\pi \sqrt{2\pi}e^{-i\frac{\pi}{4}}f\left(k, \frac{x}{|x|}\right) \frac{e^{i|k||x|}}{\sqrt{|k||x|}} o\left(\frac{1}{\sqrt{|x|}}\right),$$

as $|x| \to \infty$, for some a priori unknown $M_n(\mathbb{C})$-valued function $f$, where $I$ is the identity matrix.

- $f$ is the scattering amplitude for the potential $V$;
- $f$ is defined on $\mathcal{M}_E = \{(k, l) \in \mathbb{R}^2 \times \mathbb{R}^2 : k^2 = l^2 = E\}$.

Inverse scattering problem

Given $f$ on $\mathcal{M}_E$, find $V$ on $\mathbb{R}^2$. 
Questions

- **Uniqueness:**
  - injectivity of $v \mapsto \Phi_v(E)$.

- **Reconstruction**

- **Stability:**
  - there exists $f$ such that
    \[
    \|v_2 - v_1\|_{L^\infty(D)} \leq f(\|\Phi_2(E) - \Phi_1(E)\|),
    \]
    \[
    f(t) \to 0 \text{ as } t \to 0^+.
    \]
Historical remarks for Gel’fand-Calderón

First formulation: Gel’fand (1954).

**First global results (scalar case, full boundary data):**

<table>
<thead>
<tr>
<th>$D \subset \mathbb{R}^d$</th>
<th>$d = 2$</th>
<th>$d \geq 3$</th>
</tr>
</thead>
</table>


Several uniqueness results in the case of partial data on the boundary.
Ill-posedness of the problem

Mandache (2001) and Isaev (2011): for fixed $E$, it is not possible to have a stability estimate like

$$\|v_2 - v_1\|_{L^\infty(D)} \leq c (\log(3 + \|\Phi_2(E) - \Phi_1(E)\|_*^{-1}))^{-\alpha},$$

(1)

for $\alpha > m$, if $v_1, v_2 \in C^m(D, \mathbb{C})$ and $\Phi_1(E), \Phi_2(E)$ the corresponding Dirichlet-to-Neumann operators ($\| \cdot \|_* = \| \cdot \|_{H^{1/2}(\partial D) \to H^{-1/2}(\partial D)}$).

Question:

- What is the behaviour of estimate (1) with respect to $E$?
Exact reconstruction algorithms

- Calderón problem:

  $\Lambda_\sigma \rightarrow t(\lambda) \rightarrow \sigma(x)$. 

  - $\Lambda_\sigma \rightarrow t(\lambda)$ logarithmic stable;
  - $t(\lambda) \rightarrow \sigma(x)$ Lipschitz stable.

- Positive energy:

  $\Phi_V(E) \rightarrow (h_{\pm}(\lambda, \lambda'), r(\lambda)) \rightarrow V(x)$. 

  - $\Phi_V(E) \rightarrow h_{\pm}(\lambda, \lambda')$ Lipschitz stable;
  - $\Phi_V(E) \rightarrow r(\lambda)$ logarithmic stable.
Approximate reconstruction algorithm

Fact:
- for $V \in W^{m,1}(\mathbb{R}^2, M_n(\mathbb{C}))$, $m \geq 3$ we have
  $$|r(\lambda)| \leq O(E^{-(m-2)/2}).$$

New algorithm at positive (sufficiently large) energy:

$$\Phi_V(E) \rightarrow h_{\pm}(\lambda, \lambda') \rightarrow V_{appr}(x).$$

- $\Phi_V(E) \rightarrow h_{\pm}(\lambda, \lambda')$ Lipschitz stable;
- $h_{\pm}(\lambda, \lambda') \rightarrow V_{appr}(x)$ Lipschitz stable.

$$\|V_{appr}(\cdot, E) - V(\cdot)\|_{L^\infty(D)} \leq O(E^{-(m-2)/2}).$$
Approximate inverse scattering

**Uniqueness fails** for inverse scattering in two-dimensions at fixed energy (transparent potentials).

Approximate reconstruction (for $V \in \mathcal{W}_{\varepsilon}^{m,1}(\mathbb{R}^2, M_n(\mathbb{C})), \varepsilon > 0$):

$$f \rightarrow h_{\pm}(\lambda, \lambda') \rightarrow V_{\text{appr}}(x).$$

- $f \rightarrow h_{\pm}(\lambda, \lambda')$ Lipschitz stable;
- $h_{\pm}(\lambda, \lambda') \rightarrow V_{\text{appr}}(x)$ Lipschitz stable.

Same rate of convergence:

$$\|V_{\text{appr}}(\cdot, E) - V(\cdot)\|_{L^\infty(D)} \leq O(E^{-(m-2)/2}).$$
Faddeev functions

- \( V \in \mathcal{W}^{m,1}(\mathbb{R}^2, M_n(\mathbb{C})), m \geq 3 \) with \( \text{supp} \, V \subset D \),
- or \( V \in \mathcal{W}^{m,1}_\varepsilon(\mathbb{R}^2, M_n(\mathbb{C})) \).

We consider the Faddeev functions \( \psi(x, k) \):

- \( (-\Delta + V)\psi(x, k) = E\psi(x, k) \),
- \( \psi(x, k) \to e^{ikx}I \) as \( |x| \to +\infty \),

where \( x = (x_1, x_2), k = (k_1, k_2) \in \mathbb{C}^2 \setminus \mathbb{R}^2 \),
\( k^2 = k_1^2 + k_2^2 = E > 0 \), \( I \) is the identity matrix.

They are constructed using the Faddeev Green function
\( G(x, k) = e^{ikx}g(x, k) \), where

\[
g(x, k) = -\left( \frac{1}{2\pi} \right)^2 \int_{\mathbb{R}^2} \frac{e^{i\xi \cdot x}}{\xi^2 + 2k\xi} \, d\xi.
\]
Faddeev functions

\[ \psi(x, l) = e^{ikx} \mu(x, l), \]
\[ \mu(x, k) = I + \int_{\mathbb{R}^2} g(x - y, k)V(y)\mu(y, k)dy, \]
\[ h(k, l) = \left( \frac{1}{2\pi} \right)^2 \int_{\mathbb{R}^2} e^{-ilx}V(x)\psi(x, k)dx, \]

where \( x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2, k = (k_1, k_2) \in \mathbb{C}^2 \setminus \mathbb{R}^2, \)
\( l = (l_1, l_2) \in \mathbb{C}^2, \text{Im}k = \text{Im}l \neq 0 \) and \( I \) is the identity matrix.

\( h \) is a generalised scattering amplitude in the complex domain for the potential \( V. \)

For \( k, l \in \mathbb{R}^2 \) these formulas are not well defined, but certain limits are.
Faddeev functions

For $\gamma \in S^1 = \{\gamma \in \mathbb{R}^2 : |\gamma| = 1\}$, we consider

$$G_\gamma(x, k) = G(x, k + i0\gamma),$$
$$\psi_\gamma(x, k) = e^{ikx} \mu_\gamma(x, k), \quad \mu_\gamma(x, k) = \mu(x, k + i0\gamma),$$
$$h_\gamma(k, l) = h(k + i0\gamma, l + i0\gamma),$$

where $x, y \in \mathbb{R}^2, k \in \mathbb{R}^2, l \in \mathbb{R}^2$. The following functions

$$G^+(x, k) = G_{k/|k|}(x, k) = -\frac{i}{4}H_0^1(|x||k|),$$
$$\psi^+(x, k) = e^{ikx} \mu^+(x, k), \quad \mu^+(x, k) = \mu_{k/|k|}(x, k),$$
$$f(k, l) = h_{k/|k|}(k, l),$$

for $x, y, k, l \in \mathbb{R}^2, |k| = |l|$, are functions from the classical scattering theory.
Faddeev functions

We also define

\[ h_{\pm}(k, l) = h_{\pm \hat{k}_\perp}(k, l), \]
\[ \mu_{\pm}(x, k) = \mu_{\pm \hat{k}_\perp}(x, k), \quad \psi_{\pm}(x, k) = \psi_{\pm \hat{k}_\perp}(x, k), \]

where \( k, l, x, y \in \mathbb{R}^2, |k| = |l|, \hat{k}_\perp = |k|^{-1}(-k_2, k_1) \) for \( k = (k_1, k_2) \).

We consider, in particular, the following restriction of the function \( h \):

\[ b(k) = h(k, -\bar{k}), \quad \text{for } k \in \mathbb{C}^2, k^2 = E > 0. \]
Change of variables

We now introduce the notations

\[ z = x_1 + ix_2, \quad \bar{z} = x_1 - ix_2, \]
\[ \frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial x_1} - i \frac{\partial}{\partial x_2} \right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial}{\partial x_1} + i \frac{\partial}{\partial x_2} \right), \]

and we transform the space of parameters

\[ \lambda = E^{-1/2}(k_1 + ik_2), \quad \lambda' = E^{-1/2}(l_1 + il_2), \]

where \( x = (x_1, x_2) \in \mathbb{R}^2, k = (k_1, k_2), l = (l_1, l_2) \in \mathbb{C}^2, k^2 = l^2 = E \in \mathbb{R}_+. \]
Change of variables

In the new notations

\[ k_1 = \frac{1}{2} E^{1/2} (\lambda + \lambda^{-1}), \quad k_2 = \frac{i}{2} E^{1/2} (\lambda^{-1} - \lambda), \]
\[ l_1 = \frac{1}{2} E^{1/2} (\lambda' + \lambda'^{-1}), \quad l_2 = \frac{i}{2} E^{1/2} (\lambda'^{-1} - \lambda'), \]

\[ \exp(ikx) = \exp\left[ \frac{i}{2} E^{1/2} (\lambda z + \lambda^{-1} z) \right], \]

where \( \lambda, \lambda' \in \mathbb{C} \setminus \{0\}, \) \( z \in \mathbb{C} \) and the Schrödinger equation takes the form

\[ -4 \frac{\partial^2}{\partial z \partial \bar{z}} \psi + V(z) \psi = E \psi, \quad z \in \mathbb{C}. \]
New notations

\[ f = f(\lambda, \lambda', E), \quad h_{\pm} = h_{\pm}(\lambda, \lambda', E), \]
\[ \mu^+ = \mu^+(z, \lambda, E), \quad \psi^+ = \psi^+(z, \lambda, E), \]
\[ \mu_{\pm} = \mu_{\pm}(z, \lambda, E), \quad \psi_{\pm} = \psi_{\pm}(z, \lambda, E), \]

where \( \lambda, \lambda' \in T, \ z \in \mathbb{C}, \ E \in \mathbb{R}_+ \),

\[ \mu = \mu(z, \lambda, E), \quad \psi = \psi(z, \lambda, E), \quad b = b(\lambda, E), \]

where \( \lambda \in \mathbb{C} \setminus T, \ z \in \mathbb{C}, E \in \mathbb{R}_+ \). Here

\[ T = \{ \zeta : \zeta \in \mathbb{C}, |\zeta| = 1 \}. \]
Non-local Riemann-Hilbert problem

Non-local Riemann-Hilbert problem for $\mu(z, \lambda)$:

$$\frac{\partial}{\partial \bar{\lambda}} \mu(z, \lambda, E) = \mu \left( z, -\frac{1}{\lambda}, E \right) r(\lambda, z, E), \quad \text{for } \lambda \in \mathbb{C} \setminus T,$$

$$\mu_{\pm}(z, \lambda, E) = \mu^{+}(z, \lambda, E)$$

$$+ \pi i \int_{T} \mu^{+}(z, \lambda'', E) \chi_{+} \left( \pm i \left( \frac{\lambda}{\lambda''} - \frac{\lambda''}{\lambda} \right) \right) h_{\pm}(\lambda, \lambda'', z, E) \, |d\lambda''|,$$

for $\lambda \in T$, where

$$\chi_{+}(s) = 0 \text{ for } s < 0, \quad \chi_{+}(s) = 1 \text{ for } s \geq 0.$$
Non-local Riemann-Hilbert problem

\[ r(\lambda, z, E) = \exp \left[ -\frac{i}{2} E^{1/2} \left( \lambda \bar{z} + \frac{z}{\lambda} + \bar{\lambda} z + \frac{\bar{z}}{\bar{\lambda}} \right) \right] \frac{\pi}{\bar{\lambda}} \text{sign}(\lambda \bar{\lambda} - 1) b(\lambda, E), \]

\[ h_{\pm}(\lambda, \lambda', z, E) = \exp \left[ -\frac{i}{2} E^{1/2} \left( \lambda \bar{z} + \frac{z}{\lambda} - \lambda' \bar{z} - \frac{z}{\lambda'} \right) \right] h_{\pm}(\lambda, \lambda', E) \]
Exact reconstruction formula

Reconstruction of $V$:

- $\mu(z, \lambda, E) = I + \frac{\mu_{-1}(z,E)}{\lambda} + o\left(\frac{1}{\lambda}\right), \quad \lambda \to \infty,$
- $V(z) = 2iE^{1/2} \frac{\partial}{\partial z} \mu_{-1}(z,E)$.

Exact formula:

$$V(z) = 2iE^{1/2} \frac{\partial}{\partial z} \left( \frac{1}{\pi} \int_{D_-} \mu(z, -\frac{1}{\bar{\zeta}}, E) r(\zeta, z, E) d\text{Re}\zeta d\text{Im}\zeta ight. \\
+ \frac{1}{2\pi i} \int_{T} \mu_{-}(z, \zeta, E) i\zeta |d\zeta| \bigg),$$

for $z \in \mathbb{C}$, $E$ sufficiently large and $D_- = \{ \zeta : \zeta \in \mathbb{C}, |\zeta| > 1 \}$. 
Approximate reconstruction algorithms (sketch)

Gel’fand-Calderon: from $\Phi(E)$ to $h_{\pm}$.

Reconstruction of $\psi_{\pm}(x, k)|_{\partial D}, k \in \mathbb{R}^2, k^2 = E$:

$$
\psi_{\pm}(x, k)|_{\partial D} = e^{ikx} I + \int_{\partial D} A_{\pm}(x, y, k)\psi_{\pm}(y, k)dy, \quad k \in \mathbb{R}^2, k^2 = E,
$$

where the kernel $A_{\pm}$ explicitly depends on the Dirichlet-to-Neumann data.

Reconstruction of $h_{\pm}$:

$$
h_{\pm}(k, l) = \frac{1}{(2\pi)^2} \int_{\partial D} e^{-ilx} (\Phi(E) - \Phi_0(E))\psi_{\pm}(x, k)dx,
$$

for $k, l \in \mathbb{R}^2, k^2 = l^2 = E$. $\Phi_0$ is the Dirichlet-to-Neumann map for the Laplacian.
Approximate reconstruction algorithms (sketch)

**Inverse scattering: from $f$ to $h_\pm$.**

\[
    h_\pm(\lambda, \lambda', E) - \pi i \int_T f(\lambda'', \lambda', E) \chi_+ \left( \pm i \left( \frac{\lambda}{\lambda''} - \frac{\lambda''}{\lambda} \right) \right) h_\pm(\lambda, \lambda'', E) |d\lambda''| \\
    = f(\lambda, \lambda', E), \quad (\lambda, \lambda') \in T \times T.
\]
Approximate reconstruction algorithm (sketch)

Both problems: from $h_{\pm}$ to $V_{app}$.

Reconstruction of $\tilde{\mu}^+$, an approximation to $\mu^+$:

$$
\tilde{\mu}^+(z, \lambda, E) + \int_T \tilde{\mu}^+(z, \lambda', E) B(\lambda, \lambda', z, E) |d\lambda'| = I, \ \lambda \in T, \ z \in \mathbb{C},
$$

where the kernel $B(\lambda, \lambda', z, E)$ explicitly depends on the functions $h_{\pm}$.

Reconstruction of $\tilde{\mu}_-$, an approximation to $\mu_-$:

$$
\tilde{\mu}_-(z, \lambda, E) = \tilde{\mu}^+(z, \lambda, E)
+ \pi i \int_T \tilde{\mu}^+(z, \lambda'', E) \chi_+ \left( -i \left( \frac{\lambda}{\lambda''} - \frac{\lambda''}{\lambda} \right) \right) h_-(\lambda, \lambda'', z, E) |d\lambda''|,
$$

for $\lambda \in T, \ z \in \mathbb{C}$. 
Approximate reconstruction algorithm (sketch)

Reconstruction of the approximated potential $V_{\text{appr}}$:

$$V_{\text{appr}}(z, E) = 2iE^{1/2} \frac{\partial}{\partial z} \left( \frac{1}{2\pi i} \int_T \tilde{\mu} - (z, \zeta, E) i\zeta |d\zeta| \right).$$

Remarks:

- $V_{\text{appr}}$ depends non-linearly on $\Phi(E)$;
- We provided also a linearized algorithm in the paper;
- We have the convergence result:

$$\|V_{\text{appr}}(\cdot, E) - V(\cdot)\|_{L^\infty(D)} \leq O(E^{-(m-2)/2}).$$
Lipschitz stability (Gel’fand-Calderon)

Let $\Phi_{V,0}(x, y, E)$, $x, y \in \partial D$ denote the Schwartz kernel of the operator $\Phi(E) - \Phi_0(E)$, and let

$$\delta = \| \Phi'_{V,0}(\cdot, \cdot, E) - \Phi_{V,0}(\cdot, \cdot, E) \|_{L^2(\partial D \times \partial D)} \leq \delta_1(V, E, D, n).$$

Then

$$\varepsilon = \| V'_{appr} - V_{appr} \|_{L^\infty(D)} \leq \eta_1(V, E, D, n) \delta,$$

where in particular

$$\delta_1(V, E, D, n) \geq \delta_0,$$

$$\eta_1(V, E, D, n) \leq \eta_1^0,$$

as $\| \Phi_{V,0}(\cdot, \cdot, E) \|_{L^2(\partial D \times \partial D)} \to 0$, for some positive (sufficiently small) $\delta_1^0$ and (sufficiently big) $\eta_1^0$, where $\delta_1^0$ and $\eta_1^0$ are independent of $V$ and $E$ for fixed $D$ and $n$. 
Lipschitz stability (inverse scattering)

Let
\[ \delta = \| f - f' \|_{L^2(\mathcal{M}_E)} \leq \delta_2(V, E, n). \]

Then
\[ \epsilon = \| V_{\text{appr}} - V'_{\text{appr}} \|_{L^\infty(\mathbb{R}^2)} \leq \eta_2(V, E, n) \delta, \]

where in particular,
\[ \delta_2(V, E, n) \geq \delta_2^0, \]
\[ \eta_2(V, E, n) \leq \eta_2^0 E, \]
as \[ \| f \|_{L^2(\mathcal{M}_E)} \to 0, \]
for some positive (sufficiently small) \( \delta_2^0 \) and (sufficiently big) \( \eta_2^0 \), where \( \delta_2^0 \) and \( \eta_2^0 \) are independent of \( V \) and \( E \) for fixed \( n \).
Numerical Results (with S. Siltanen and M. Lassas)

- $D$ the unit disk

- Truncated Fourier basis to approximate the Dirichlet-to-Neumann map (using FEM): $\phi^{(n)}(\theta) = \frac{1}{\sqrt{2\pi}} e^{i n \theta}$, $n = -N, ..., N$. Generally $N \leq 20$.

- All integral equations are solved representing operators as matrix in truncated Fourier basis.

- The scattering data $h(k, l)$ are calculated for $k, l$ which vary over 128 points on the circle of radius $\sqrt{E}$. 
Phantom
Reconstruction of the potential from Dirichlet-to-Neumann at fixed energy $E = 100$ using three different version of the algorithm.
Reconstruction of the 20 times higher contrast potential from Dirichlet-to-Neumann at fixed energy $E = 100$ using three different version of the algorithm.
Reconstruction of the 20 times higher contrast potential from noisy Dirichlet-to-Neumann at fixed energy $E = 100$ using partially linearized algorithm. From left to right: original potential, 1% noise,
Reconstruction of the 20 times higher contrast potential from noisy Dirichlet-to-Neumann at fixed energy $E = 100$ using partially linearized algorithm. From left to right: original potential, 3.5% noise.
Resolution test

Reconstruction from boundary data at $E = 10, 20, 30, 100, 200, 300, 500$. 
Future Work and References

Future work: 3D reconstruction with the same method (in the multi-channel case).

