The Stein characterization of Wright-type distributions

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Idea of the Stein method

- Stein's method is a technique to quantify the error in the approximation of certain distributions by normal distribution.
- The main idea behind is to replace the characteristic function typically used to show distributional convergence with a characterizing operator.
- Nowadays the method is beyond that original purpose, applying to approximation of general random variables by distributions other than the normal (such as the Poisson, exponential, etc).
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Stein's Lemma

Lemma 1 (Stein's Lemma, Stein [1972], Stein [1986])

Let A be the operator defined by

$$Af(x) := f'(x) - xf(x).$$

1. *If $W \sim N(0,1)$, then $\mathbb{E}\big(Af(W)\big) = 0$ for all absolutely continuous functions f with $\mathbb{E}\big(|f'(W)|\big) < \infty$.*
2. *If for some r.v. Z, $\mathbb{E}\big(Af(Z)\big) = 0$ for all absolutely continuous functions f with $\|f'\| < \infty$, then $Z \sim N(0,1)$.*

The operator A is known as the characterizing operator of the standard normal distributions.

Lemma 2 (Applications)

Let Φ *be the cdf of* $N(0,1)$ *and consider the unique solution of the ODE,* $x \in \mathbb{R}$

$$f_x'(y) - yf(y) = \mathbb{1}_{\{y \le x\}} - \Phi(x). \tag{1}$$

given by

$$f_x(y) = e^{y^2/2} \int_{-\infty}^{y} e^{-t^2/2} \big(\mathbb{1}_{\{t \le x\}} - \Phi(x)\big)\, dt.$$

Then, for any r.v. Z*, evaluating* (1) *at* Z *and taking expectation gives*

$$\big|P(Z \le x) - \Phi(x)\big| = \big|\mathbb{E}\big(f_x'(Z) - Zf_x(Z)\big)\big|. \tag{2}$$

Remark: Thus we can bound the quantity

$$\big|P(Z \le x) - \Phi(x)\big|$$

by solving the Stein equation (1) and then bounding the r.h.s. of (2).

Aim: Stein' method for the class of M-Wright distributions

- For any $\alpha \in (0,1)$ we consider the distribution ν_α given by

$$d\nu_\alpha(\tau) = M_\alpha(\tau)\, d\tau,$$

where M_α is the M-Wright function. It is a special case of the Wright function $W_{\lambda,\mu}$, namely

$$\begin{aligned} M_\alpha(z) = W_{-\alpha,1-\alpha}(-z) &= \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\Gamma(-\alpha n + 1 - \alpha)} \\ &= \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{(-z)^n}{n!} \Gamma\big(\alpha(n+1)\big) \sin\big(\pi\alpha(n+1)\big). \end{aligned}$$

- Mainardi et al. [2010], Gorenflo et al. [1999], Mainardi and Tomirotti [1995], Mainardi [2010].

Properties of M_α

- Special cases:

$$M_{1/2}(z) = \frac{1}{\sqrt{\pi}} e^{-z^2/4}; \qquad M_{1/3}(z) = 3^{2/3}\mathrm{Ai}\big(3^{-1/3}z\big).$$

- Moments:

$$\int_0^\infty \tau^\delta M_\alpha(\tau)\, d\tau = \frac{\Gamma(\delta+1)}{\Gamma(\alpha\delta+1)}, \quad \delta > -1.$$

- Laplace transform:

$$\int_0^\infty e^{-s\tau} M_\alpha(\tau)\, d\tau = E_\alpha(-s) := \sum_{n=0}^{\infty} \frac{(-s)^n}{\Gamma(\alpha n+1)}.$$

- Asymptotic: $z \to \infty$

$$M_\alpha\left(\frac{z}{\alpha}\right) \simeq \frac{1}{\sqrt{2\pi(1-\alpha)}} z^{(\alpha-1/2)/(1-\alpha)} \exp\left(-\frac{1-\alpha}{\alpha} z^{1/(1-\alpha)}\right).$$

Differential equations

- Ordinary differential equation (subject to certain $q-1$ initial conditions at $z=0$):

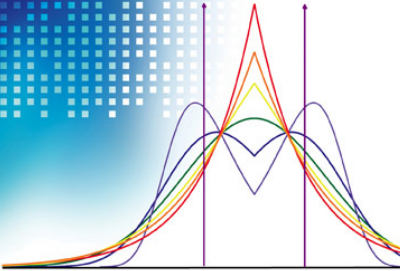
$$\frac{d^{q-1}}{dz^{q-1}} M_{1/q}(z) + \frac{(-1)^q}{q} z M_{1/q}(z) = 0,$$

or

$$\frac{d^{1/\alpha-1}}{dz^{1/\alpha-1}} M_\alpha(z) + e^{\pm i\pi/\alpha} \alpha z M_\alpha(z) = 0.$$

Fractional Calculus and Waves in Linear Viscoelasticity

Francesco Mainardi



An Introduction to Mathematical Models

Imperial College Press

The Stein operator for $M_{1/3}$

Question: Can we derive the Stein operator and the Stein equation which characterize the M-Wright class $MW(\alpha)$, $0 < \alpha < 1$?

The Stein operator for $M_{1/3}$

Proposition 3 (Characterization of $MW(1/3)$)

Let $\mathcal{A}_{1/3}$ be the operator defined on a class of functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$\big(\mathcal{A}_{1/3} f\big)(x) := f''(x) - \frac{1}{3} x f(x), \quad x \in \mathbb{R}.$$

1. *If $Y_{1/3} \sim MW(1/3)$, then $\mathbb{E}\big((\mathcal{A}_{1/3} f)(Y_{1/3})\big) = 0$ for all C^2 functions f such that $\frac{f'(0)}{\Gamma(2/3)} - \frac{f(0)}{\Gamma(1/3)}$, $\mathbb{E}\big(\big|f''(Y_{1/3})\big|\big) < \infty$ and $\mathbb{E}\big(\big|Y_{1/3} f(Y_{1/3})\big|\big) < \infty$.*
2. *If X is a random variable with $\mathbb{E}\big((\mathcal{A}_{1/3} f)(X)\big) = 0$ for any C^2 function f such that $\mathbb{E}(|(\mathcal{A}_{1/3} f)(Y_{1/3})|) < \infty$ and*

$$\lim_{x\to\infty} e^{-\frac{2}{\sqrt{3^3}} x^{3/2}} x^{-1/4} f'(x) = 0, \quad \lim_{x\to\infty} e^{-\frac{2}{\sqrt{3^3}} x^{3/2}} x^{1/4} f(x) = 0 \quad (3)$$

then $X \sim MW(1/3)$.

The Stein operator for $M_{1/3}$

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2. *If* X *is a random variable with* $\mathbb{E}\big((\mathcal{A}_{1/3}f)(X)\big)=0$ *for any* C^2 *function* f *such that* $\mathbb{E}(|(\mathcal{A}_{1/3}f)(Y_{1/3})|)<\infty$ *and*

$$\lim_{x\to\infty} e^{-\frac{2}{\sqrt{3^3}}x^{3/2}}x^{-1/4}f'(x) = 0, \quad \lim_{x\to\infty} e^{-\frac{2}{\sqrt{3^3}}x^{3/2}}x^{1/4}f(x)=0 \quad (3)$$

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$$\lim_{x\to\infty} e^{-\frac{2}{\sqrt{3^3}}x^{3/2}}x^{-1/4}f'(x) = 0, \quad \lim_{x\to\infty} e^{-\frac{2}{\sqrt{3^3}}x^{3/2}}x^{1/4}f(x)=0 \quad (3)$$
then $X\sim MW(1/3)$.

Technical lemma

Lemma 4 (Stein equation)

Let $h : \mathbb{R}_+ \longrightarrow \mathbb{R}$ be a bounded measurable function and denote by $\tilde{h}(x) := h(x) - \mathbb{E}\big(h(Y_{1/3})\big)$. Then the function

$$f_h(x) = -3\pi \left[\mathrm{Ai}\left(\frac{x}{3^{1/3}}\right) \int_0^x \mathrm{Bi}\left(\frac{t}{3^{1/3}}\right) \tilde{h}(t)\, dt \right. \tag{4}$$

$$\left. - \mathrm{Bi}\left(\frac{x}{3^{1/3}}\right) \int_x^\infty \mathrm{Ai}\left(\frac{t}{3^{1/3}}\right) \tilde{h}(t)\, dt \right] \tag{5}$$

solves the Stein equation

$$\boxed{(\mathcal{A}_{1/3} f_h)(x) = \tilde{h}(x).} \tag{6}$$

In addition, the solution f_h satisfies $\|f_h\|_\infty \leq C_1 \|\tilde{h}\|_\infty$ and $\|f_h'\|_\infty \leq C_2 \|\tilde{h}\|_\infty$ for certain constants $C_1, C_2 > 0$.

Proof of the lemma 4. I

1. Let w_1 and w_2 be the two independent solutions of the homogeneous Stein equation (6).

$$f''(x) - \frac{1}{3} x f(x) = 0.$$

$$w_1(x) = M_{1/3}(x) = 3^{2/3} \mathrm{Ai}\left(\frac{x}{3^{1/3}}\right) \quad (7)$$

$$w_2(x) = 3^{2/3} \mathrm{Bi}\left(\frac{x}{3^{1/3}}\right). \quad (8)$$

The solution of the Stein equation is given by

$$f_h(x) = -w_1(x) \int_0^x \frac{w_2(t)\tilde{h}(t)}{W(t)}\,dt - w_2(x) \int_x^\infty \frac{w_1(t)\tilde{h}(t)}{W(t)}\,dt, \quad (9)$$

Proof of the lemma 4. II

where $W(t) = \sqrt[3]{3}/2\pi$ is the Wronskian of w, w_2. The solution follows.

2. To show the boundedness of f_h and f_h' we use the asymptotic of the Airy functions and its derivative

$$\begin{aligned} \mathrm{Ai}'(x) &= \frac{x}{\sqrt{3}\pi} K_{2/3}\left(\frac{2}{3}x^{3/2}\right), \\ \mathrm{Bi}'(x) &= \frac{x}{\sqrt{3}}\left[I_{2/3}\left(\frac{2}{3}x^{3/2}\right) + I_{-2/3}\left(\frac{2}{3}x^{3/2}\right)\right], \end{aligned}$$

where I_ν,K_ν are the modified Bessel functions with asymptotic given by

$$K_\nu(x) \simeq \sqrt{\frac{\pi}{2x}} e^{-x}, \qquad I_\nu(x) \simeq \frac{e^x}{\sqrt{2\pi x}}, \quad x \to \infty.$$

Proof of Proposition 3. I

1. If $Y_{1/3} \sim MW(1/3)$, then $\mathbb{E}\big((\mathcal{A}_{1/3}f)(Y_{1/3})\big) = 0$. It is clear that

$$\mathbb{E}\big((\mathcal{A}f)(Y_{1/3})\big) = \int_0^\infty \Big(f''(x) - \frac{1}{3}xf(x)\Big)M_{1/3}(x)\,dx$$

and an integration by parts yields **??**

$$\mathbb{E}\big((\mathcal{A}f)(Y_{1/3})\big) = \int_0^\infty \Big(M''_{1/3}(x) - \frac{1}{3}xM_{1/3}(x)\Big)f(x)\,dx + f'(x)M_{1/3}(x)\Big|_0^\infty - f(x)M'_{1/3}(x)\Big|_0^\infty.$$

The result follows by the asymptotic for $M_{1/3}, M'_{1/3}$ and the ODE for $M_{1/3}$.

Proof of Proposition 3. II

2. If X is a r.v. with $\mathbb{E}\big((\mathcal{A}_{1/3}f)(X)\big) = 0$ + conditions, then $X \sim MW(1/3)$.
   As f_h satisfies the Stein equation (6)

   $$(\mathcal{A}f_h)(x) = f_h''(x) - \frac{1}{3}xf_h(x) = \tilde{h}(x),$$

   taking expectation

   $$\mathbb{E}\big((\mathcal{A}f_h)(X)\big) = \mathbb{E}\big(\tilde{h}(X)\big) = \mathbb{E}\big(h(X)\big) - \mathbb{E}\big(h(Y_{1/3})\big).$$

   By hypothesis $\mathbb{E}((\mathcal{A}f_h)(X)) = 0$, then

   $$0 = \mathbb{E}\big((\mathcal{A}f_h)(X)\big) = \mathbb{E}\big(h(X)\big) - \mathbb{E}\big(h(Y_{1/3})\big)$$

   it follows that $X \sim MW(1/3)$.

Mixtures of M-Wright functions I

- Consider the M-mixture

$$p_{1/3}(x) := \int_0^\infty \rho_s(x) M_{1/3}(s)\, ds, \qquad \rho_s(x) := \frac{1}{\sqrt{2\pi s}} e^{-x^2/2s}.$$

- Using the ODE: $M_{1/3}(s) = \frac{3}{s} M''_{1/3}(s)$ and an IbP:

$$p_{1/3}(x) = \frac{45}{4} \int_0^\infty \frac{1}{s^3} \rho_s(x) M_{1/3}(s)\, ds + \frac{15}{2} x^2 \int_0^\infty \frac{1}{s^4} \rho_s(x) M_{1/3}(s)\, ds + \frac{3}{4} x^4 \int_0^\infty \frac{1}{s^5} \rho_s(x) M_{1/3}(s)\, ds.$$

- Now we have to write each term

$$\int_0^\infty \frac{1}{s^k} \rho_s(x) M_{1/3}(s)\, ds = F(p_{1/3}(x), p'_{1/3}(x), \ldots, p^{(k)}_{1/3}(x)), \quad k = 3, 4, 5.$$

- The resulting ODE for $p_{1/3}$ is

$$\frac{3}{4}x^4 p^{(5)}_{1/3}(x) - 15x^3 p^{(4)}_{1/3}(x) + 90x^2 p'''_{1/3}(x) - 225x p''_{1/3}(x) + 225 p'_{1/3}(x) + x^5 p_{1/3}(x) = 0.$$

- The corresponding Stein operator is of the form

$$(\mathcal{A}f)(x) = \sum_{i=0}^{5} A_i(x) f^{(i)}(x),$$

with coefficients A_i to be determined. If Z has distribution $p_{1/3}$, then

$$\mathbb{E}(\mathcal{A}f(Z)) = \int_{\mathbb{R}} (\mathcal{A}f)(x) p_{1/3}(x)\, dx = 0.$$

Making IbP, we obtain

$$\begin{aligned} A_5(x) &= -\frac{3}{4}x^4 \\ A_4(x) &= -30x^3 \\ A_3(x) &= -360x^2 \\ A_2(x) &= -1485x \\ A_1(x) &= -1665 \\ A_0(x) &= x^5. \end{aligned}$$

Open questions

1. More general case $\beta \in \{1/n,\ n \in \mathbb{N}\}$

$$\frac{d^{n-1}}{dz^{n-1}} M_{1/n}(z) + \frac{(-1)^n}{n} z M_{1/n}(z) = 0.$$

Characterizing operator and Stein equation?

2. For $0 < \beta < 1$:

$$\frac{d^{1/\beta - 1}}{dz^{1/\beta - 1}} M_\beta(z) + e^{\pm i\pi/\beta} \beta z M_\beta(z) = 0.$$

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