Numerical methods for distributional solutions of nonlocal equations of porous medium type

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Joint work with Jørgen Endal and Espen R. Jakobsen from Norwegian University of Science and Technology (NTNU)
1 Introduction

2 Nonlocal Porous Medium Equations

3 General Nonlocal Porous Medium Equation

4 Numerical methods
Introduction

- Probabilistic Motivation to Nonlocal Diffusion
- The Porous Medium Equation: physical applications.

Nonlocal Porous Medium Equations

- Related local models
- Related non-local models
- Nonlocal operators

General Nonlocal Porous Medium Equation

- Well-Posedness
- A priori estimates

Numerical methods

- An example for nonlocal equations
- An example for local equations
- General theory of finite differences
Probabilistic Motivation to Nonlocal Diffusion

• $h\mathbb{Z} = \{hz : z \in \mathbb{Z}\}$: Set of possible states of a jumping particle.
Probabilistic Motivation to Nonlocal Diffusion

- $h\mathbb{Z} = \{hz : z \in \mathbb{Z}\}$: Set of possible states of a jumping particle.
- $u(x, t)$: Probability of a particle to be at $x \in h\mathbb{Z}$ at time $t \in \tau\mathbb{N}$. 
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Local jump random walk
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$$u(x, t + \tau) = \frac{1}{2}u(x + h, t) + \frac{1}{2}u(x - h, t)$$
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Local jump random walk

$$u(x, t + \tau) = \frac{1}{2} u(x + h, t) + \frac{1}{2} u(x - h, t)$$

$$2\tau = h^2 \implies \frac{u(x, t + \tau) - u(x, t)}{\tau} = \frac{u(x + h, t) + u(x - h, t) - 2u(x, t)}{h^2}$$
**Probabilistic Motivation to Nonlocal Diffusion**

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  u(x, t + \tau) &= \frac{1}{2} u(x + h, t) + \frac{1}{2} u(x - h, t) \\
  &\quad \uparrow 2\tau = h^2 \frac{u(x, t + \tau) - u(x, t)}{\tau} = \frac{u(x + h, t) + u(x - h, t) - 2u(x, t)}{h^2}
\end{align*}
\]

\((\tau, h \rightarrow 0^+)\) **Heat Equation**

\[
u_t(x, t) = \Delta u(x, t) \quad (x, t) \in \mathbb{R} \times (0, \infty)
\]
Probabilistic Motivation

- $h\mathbb{Z} = \{hz : z \in \mathbb{Z}\}$: Set of possible states of a jumping particle.
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**Long jump random walk**
Introduction

Probabilistic Motivation to Nonlocal Diffusion

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**Long jump random walk**

\[
\begin{align*}
    u(x, t + \tau) &= \sum_{k \in \mathbb{Z}^N} K(k)u(x + hk, t) \quad \text{with} \quad K(y) = c_{\sigma,N}|y|^{-(N+\sigma)}.
\end{align*}
\]
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    \tau = h^\sigma \quad \frac{u(x, t + \tau) - u(x, t)}{\tau} &= h^N \sum_{k \in \mathbb{Z}} K(hk)(u(x + hk, t) - u(x, t))
\end{align*}
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\[ u(x, t + \tau) = \sum_{k \in \mathbb{Z}^N} K(k)u(x + hk, t) \text{ with } K(y) = c_{\sigma,N}|y|^{-(N+\sigma)}. \]

\[
\frac{\tau = h^\sigma u(x, t + \tau) - u(x, t)}{\tau} = h^N \sum_{k \in \mathbb{Z}} K(hk)(u(x + hk, t) - u(x, t))
\]

$(\tau, h \to 0^+)$ **Fractional Heat Equation**

\[
\partial_t u(x, t) = c_{\sigma,N} \int_{\mathbb{R}^N} \frac{u(x + y, t) - u(x, t)}{|y|^{n+\sigma}} \, dy \quad (x, t) \in \mathbb{R} \times (0, \infty)
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Probabilistic Motivation

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\begin{align*}
\tau = h \sigma & \quad \Rightarrow \quad \frac{u(x, t + \tau) - u(x, t)}{\tau} = h^N \sum_{k \in \mathbb{Z}} K(hk)(u(x + hk, t) - u(x, t)) \\
\text{(Fractional Heat Equation)}
\end{align*}
\]

\[
\partial_t u(x, t) = -d_{\sigma, N} (-\Delta)^{\frac{\sigma}{2}} u(x, t) \quad (x, t) \in \mathbb{R} \times (0, \infty)
\]
Some definitions of nonlocal operators in $\mathbb{R}^N$

Let $\sigma \in (0, 2)$. We define the **fractional Laplacian** as

1. Hypersingular integral:

$$(-\Delta)^{\frac{\sigma}{2}} u(x) := d_{\sigma,N} \text{P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+\sigma}} dy$$
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2. Fourier symbol:

\[
(-\Delta)^{\frac{\sigma}{2}} u(x) := \mathcal{F}^{-1}(|\xi|^\sigma (\mathcal{F}u)(\xi))(x).
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3. Heat-Semigroup:

$$(-\Delta)^{\sigma/2} u(x) := \frac{1}{\Gamma(-\sigma/2)} \int_0^\infty \left(e^{t\Delta} u(x) - u(x)\right) \frac{dt}{t^{1+\sigma/2}}$$
Let $\sigma \in (0, 2)$. We define the power $\sigma$ of an elliptic operator $L$ as

1. **Hypersingular integral:**

   $$(-\Delta)^{\frac{\sigma}{2}} u(x) := d_{\sigma,N} \text{P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+\sigma}} \, dy$$

2. **Fourier symbol:**

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3. **Heat-Semigroup:**

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Physical Motivation for the Porous Medium Equation

• **Flow of a gas in a Porous Medium** ([Leibenzon, 1930; Muskat 1933]):

\[
\begin{aligned}
\rho_t + \nabla \cdot (\rho \mathbf{v}) &= 0 \quad \text{(Continuity Law)} \\
\mathbf{v} = -c\nabla p, \quad p &= \phi(\rho) \quad \text{(Darcy’s Law, 1856)}
\end{aligned}
\]

When the pressure \( p = u^{m-1} \) with \( m > 1 \),

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\rho_t = c \nabla \cdot (\rho \nabla \rho^{m-1}) = \tilde{c} \Delta \rho^m. \quad \text{(Porous Medium Equation)}
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- **Plasma radiation.** (Zeldovich-Raizer \( \sim 1950 \)) \( m > 4 \)
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- **Spreading of populations, Thin films under gravity, Kinetic limits and many more...**
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- Related local models
- Related non-local models
- Nonlocal operators

3 General Nonlocal Porous Medium Equation
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- An example for nonlocal equations
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Local Equations of Porous Medium type

- **Porous Medium Equation:** \( u_t = \Delta u^m \sim \nabla \cdot (u^{m-1} \nabla u), \ m \in (0, \infty) \)

  (i) \( m > 1 \): Slow diffusion  
  Finite speed of propagation, mass conservation, \( C^\alpha \) regularity

  (ii) \( m = 1 \): Heat Equation  
  Infinite speed of propagation, mass conservation, \( C^\infty \) regularity

  (iii) \( m < 1 \): Fast Diffusion  
  Infinite speed of propagation, conservation-extinction of mass
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- **Stefan Problems:** 
  \[ u_t = \Delta (u - 1)^+ = \begin{cases} 
  0 & u \leq 1 \\
  \Delta u & u > 1 
  \end{cases} \]

  Phase transitions, free boundaries, strongly degenerate...
Local Equations of Porous Medium type

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  Phase transitions, free boundaries, strongly degenerate...

- **Generalized PME:** \[ u_t = \Delta \varphi(u), \quad \varphi \text{ continuous and nondecreasing} \]

  Contains all the previous models

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Non-Local Equations of Porous Medium type (I)

Powers of the Laplacian:

\[-(\Delta)^s \psi(x) = \text{P.V.} \int_{\mathbb{R}^N} (\psi(x+y) - \psi(x)) \frac{dy}{|y|^{N+2s}}, \quad s \in (0, 1).\]
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(P1) \[u_t = -(-\Delta)^s u^m, \quad m \in (0, \infty)\]

Well-posedness, smooth solutions, infinite speed of propagation

Vázquez, de Pablo, Quirós, Rodríguez, Bonforte, Stan, Brandle, Grillo, Muratori, Punzo, dT...
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(P2) \[u_t = \nabla \cdot (u^k \nabla (-\Delta)^{-s} u^m), \quad m \in (0, \infty), \quad k \in (0, \infty)\]

Conservation law, no uniqueness for \(\text{dim} > 1\), finite/infinite speed of propagation, transformations between (P1) and (P2),...

Caffarelli, Vázquez, Karch, Biler, Monneau, Imbert, Stan, dT...
Non-Local Equations of Porous Medium type (II)

\[ (P1) \quad u_t = -(-\Delta)^s u^m, \quad m \in (0, \infty) \]

**Generalization 1:** \( u_t = -(-\Delta)^s \varphi(u) \) with \( \varphi \) continuous, non-decreasing

Non-local Stefan problems...

Vázquez, de Pablo, Quirós, Rodríguez
Non-Local Equations of Porous Medium type (II)

\[(P1)\]

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Non-local Stefan problems...

Vázquez, de Pablo, Quirós, Rodríguez

**Generalization 2:** \(u_t = \mathcal{L}[\varphi(u)]\)

\[-(-\Delta)^s \equiv\text{Generator of } 2s - \text{stable Lévy processes}\]

(Goal) \(\mathcal{L} \equiv\text{Generator of ANY symmetric pure jump Lévy process}\)
Nonlocal Porous Medium Equations

Related non-local models

Non-local Equations of Porous Medium type (II)

\[(P1) \quad u_t = -(-\Delta)^s u^m, \quad m \in (0, \infty)\]

**Generalization 1:** \( u_t = -(-\Delta)^s \varphi(u) \) with \( \varphi \) continuous, non-decreasing

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**Generalization 2:** \( u_t = \mathcal{L}[\varphi(u)] \)

\[-(-\Delta)^s \equiv \text{Generator of } 2s - \text{stable Lévy processes}\]

(\text{Goal}) \quad \mathcal{L} \equiv \text{Generator of ANY symmetric pure jump Lévy process}

(\mathcal{L} \text{ Bounded operator}) \quad \text{Andreu, Mazón, Rossi, Toledo, Ignat...}

(\mathcal{L} \sim -(-\Delta)^s) \quad \text{de Pablo, Quirós, Rodríguez, Brandle...}

(\text{Convection term } \nabla f(u)) \quad \text{Jakobsen, Alibaud, Cifani, Karlsen, Biler, Karch, Andreianov, Endal, Silvestre...}
The operator $\mathcal{L}^\mu (I)$

**Fractional Laplacian:** $\nu(dy) = \frac{dy}{|y|^{N+2s}}$ and $s \in (0, 1)$

$$-(-\Delta)^s \psi(x) = \text{P.V} \int_{\mathbb{R}^N} (\psi(x + y) - \psi(x)) \nu(dy).$$
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**Question:** IS $-(-\Delta)^s$ WELL DEFINED?
The operator $\mathcal{L}^\mu$ (I)

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**Step 1:** Since $\nu(dy)$ symmetric

$$\quad -(-\Delta)^s \psi(x) = \int_{\mathbb{R}^N} (\psi(x+y) - \psi(x) - \nabla \psi(x) \cdot y \mathbb{1}_{\{|y|<1\}}) \nu(dy).$$
The operator $\mathcal{L}^\mu (I)$

Fractional Laplacian: $\nu(dy) = \frac{dy}{|y|^{N+2s}}$ and $s \in (0, 1)$

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  $$-(\Delta)^s \psi(x) = \int_{\mathbb{R}^N} (\psi(x+y) - \psi(x) - \nabla \psi(x) \cdot y 1_{\{|y|<1\}}) \nu(dy).$$

- **Step 2:** Assume $\psi \in C^2_b(\mathbb{R}^N)$

  $$|-(\Delta)^s \psi(x)| \leq \|D^2 \psi\|_{L^\infty(\mathbb{R}^N)} \int_{B_1} |y|^2 \nu(dy) + 2\|\psi\|_{L^\infty(\mathbb{R}^N)} \int_{\mathbb{R}^N \setminus B_1} \nu(dy).$$

**Answer:** $|-(\Delta)^s \psi(x)| \leq C_\psi \int_{\mathbb{R}^N} \min\{|y|^2, 1\} \nu(dy)$
The operator $\mathcal{L}^\mu (\Pi)$

$(A_\mu) \mu$ is any symmetric Radon measure s.t. $\int \min\{|y|^2, 1\} \mu(dy) < \infty$

$$
\mathcal{L}^\mu[\psi](x) := \text{P.V} \int_{\mathbb{R}^N} (\psi(x + y) - \psi(x)) \mu(dy) \quad \forall \psi \in C^2_b(\mathbb{R}^N),
$$
The operator $\mathcal{L}^\mu (\mu)$

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Examples:

1. $-(-\Delta)^s$ with $\mu(dy) = \frac{dy}{|y|^{N+2s}}$. Many authors in many contexts.
The operator $\mathcal{L}^\mu$ (II)

$(A_\mu) \mu$ is any symmetric Radon measure s.t. $\int \min\{|y|^2, 1\} \mu(dy) < \infty$

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Examples:

1. $-(\Delta)^s$ with $\mu(dy) = \frac{dy}{|y|^{N+2s}}$. Many authors in many contexts.

2. $L^s$ where $L$ is any symmetric unif. elliptic op. with $\mu(dy) = K(y)dy$

$$K(y) = \frac{1}{\Gamma(-s)} \int_0^\infty H_L(y,t) \frac{dt}{t^{1+s}} \quad \text{and} \quad H_L \equiv \text{Heat Kernel}$$

The operator $\mathcal{L}^\mu$ (II)

$(A_\mu) \ \mu$ is any symmetric Radon measure s.t. \[ \int \min\{|y|^2, 1\} \mu(dy) < \infty \]

$\mathcal{L}^\mu[\psi](x) := \text{P.V.} \int_{\mathbb{R}^N} (\psi(x+y) - \psi(x)) \mu(dy) \quad \forall \psi \in C_0^2(\mathbb{R}^N),$

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3. $\Delta_h \psi(x) = \frac{\psi(x+h)+\psi(x-h)-2\psi(x)}{h^2}$ with $\mu(dy) = \frac{\delta_h(dy)+\delta_{-h}(dy)}{h^2}$
The operator $\mathcal{L}^\mu$ (II)

$(A_\mu) \ \mu$ is any symmetric Radon measure s.t. $\int \min\{|y|^2, 1\} \mu(dy) < \infty$

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\forall \psi \in C^2_b(\mathbb{R}^N),

Examples:

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3. $\Delta_h \psi(x) = \frac{\psi(x+h)+\psi(x-h)-2\psi(x)}{h^2}$ with $\mu(dy) = \frac{\delta_h(dy)+\delta_{-h}(dy)}{h^2}$

4. Powers of $\Delta_h$: Ciaurri, Roncal, Stinga, Torrea, Varona
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Main results (I): Well-posedness

(IVP) \[
\begin{align*}
\frac{u_t}{t} - \mathcal{L}^\mu[\varphi(u)] &= 0 \\
u(x, 0) &= u_0(x)
\end{align*}
\]
\[Q_T = \mathbb{R}^N \times (0, T), \quad \mathbb{R}^N.\]
Main results (I): Well-posedness

\[
\begin{aligned}
(IVP) \quad \left\{ 
\begin{array}{ll}
 u_t - \mathcal{L}^\mu[\varphi(u)] = 0 & \quad Q_T = \mathbb{R}^N \times (0, T), \\
 u(x, 0) = u_0(x) & \quad \mathbb{R}^N.
\end{array}
\right.
\end{aligned}
\]


There is at most one distributional solution \( u \) of (IVP) such that \( u \in L^\infty(\mathbb{R}^N) \) and \( u - u_0 \in L^1(\mathbb{R}^N) \).
Main results (I): Well-posedness

\[
\begin{aligned}
(\text{IVP}) & \quad \left\{ \begin{array}{ll}
\frac{\partial u}{\partial t} - \mathcal{L}^\mu \phi(u) = 0 \\
u(x, 0) = u_0(x)
\end{array} \right. & Q_T = \mathbb{R}^N \times (0, T), \\
\end{aligned}
\]


*There is at most one distributional solution \( u \) of (IVP) such that \( u \in L^\infty(\mathbb{R}^N) \) and \( u - u_0 \in L^1(\mathbb{R}^N) \).*


*If \( u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \), then there exists a (unique) distributional solution of (IVP) and*

\[
u \in L^1(Q_T) \cap L^\infty(Q_T) \cap C((0, T), L^1_{\text{loc}}(\mathbb{R}^N)).\]
Main results (II): A priori estimates

\[ (IVP) \quad \begin{cases} u_t - L^\mu[\varphi(u)] = 0 & Q_T = \mathbb{R}^N \times (0, T), \\ u(x, 0) = u_0(x) & \mathbb{R}^N. \end{cases} \]


Let \( u_0, \hat{u}_0 \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N) \). Then

(a) (\( L^1 \)-contr.) \( \int_{\mathbb{R}^N} (u(x, t) - \hat{u}(x, t))^+ \, dx \leq \int_{\mathbb{R}^N} (u_0(x) - \hat{u}_0(x))^+ \, dx \); 

(b) (Comp. principle) If \( u_0 \leq \hat{u}_0 \) a.e. in \( \mathbb{R}^N \), then \( u \leq \hat{u} \) a.e. in \( Q_T \); 

(c) (\( L^{1/\infty} \)-bound) \( \|u(\cdot, t)\|_{L^{1/\infty}(\mathbb{R}^N)} \leq \|u_0\|_{L^{1/\infty}(\mathbb{R}^N)} \); 

(d) (Mass conservation) If \( |\varphi(r)| \leq L|r| \) for \( |r| \leq \delta \), then

\[ \int_{\mathbb{R}^N} u(x, t) \, dx = \int_{\mathbb{R}^N} u_0(x) \, dx. \]
Main results (III): Continuous dependence

**Natural question:** Under which conditions

\[
\begin{align*}
\frac{\partial}{\partial t} u_n - \mathcal{L}^{\mu_n} [\varphi_n(u_n)] &= 0 \\
&\quad \text{as } n \to \infty \\
u_n(x, 0) &= u_0(x)
\end{align*}
\]

\[
\begin{align*}
\frac{\partial}{\partial t} u - \mathcal{L} [\varphi(u)] &= 0 \\
u(x, 0) &= u_0(x)
\end{align*}
\]
Narutal question: Under which conditions

\[
\begin{cases}
  \partial_t u_n - \mathcal{L}^{\mu_n}[\varphi_n(u_n)] = 0 \\
  u_n(x,0) = u_0(x)
\end{cases}
\text{ \text{\scriptsize{\textit{\textbf{n}}}} \rightarrow \infty}
\begin{cases}
  \partial_t u - \mathcal{L}[\varphi(u)] = 0 \\
  u(x,0) = u_0(x).
\end{cases}
\]


Let \( u_0 \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N) \). Assume:

(i) \( \mathcal{L}^{\mu_n}[\psi] \xrightarrow{n \rightarrow \infty} \mathcal{L}[\psi] \) in \( L^1(\mathbb{R}^N) \) for all \( \psi \in C_\infty^\prime(\mathbb{R}^N) \);

(ii) \( \varphi_n \xrightarrow{n \rightarrow \infty} \varphi \) locally uniformly.

Then

\( \text{\textbf{(Distributional solutions)}} \quad u_n \xrightarrow{n \rightarrow \infty} u \quad \text{in} \quad C([0,T]; L^1_{\text{loc}}(\mathbb{R}^N)). \)
1 Introduction
- Probabilistic Motivation to Nonlocal Diffusion
- The Porous Medium Equation: physical applications.

2 Nonlocal Porous Medium Equations
- Related local models
- Related non-local models
- Nonlocal operators

3 General Nonlocal Porous Medium Equation
- Well-Posedness
- A priori estimates

4 Numerical methods
- An example for nonlocal equations
- An example for local equations
- General theory of finite differences
Numerical methods

An example for nonlocal equations

Numerical approximation (I): Midpoint quadrature rule

\[
\partial_t u - L^\mu [\varphi(u)] = 0, \quad L^\mu [\psi](x) := \int_{|z| > 0} (\psi(x + z) - \psi(x)) \mu(dz).
\]


\[
u_h \rightarrow u \text{ in } C([0,T]; L^1_{loc} (\mathbb{R}^N)).
\]

Proof:
1. \(L^\nu h\) is in "our class" of operators. Just need to take \(L^\nu h\) with \(\nu_h(dz) = \sum_{\alpha \neq 0} \mu(z^\alpha + R_h) \delta_z^\alpha(dz)\).
2. \(L^\mu [\psi] n \rightarrow \infty \rightarrow L^\mu [\psi]\) in \(L^1(\mathbb{R}^N)\) for all \(\psi \in C^\infty_c(\mathbb{R}^N)\).
3. Use the continuous dependence Theorem.
Numerical approximation (I): Midpoint quadrature rule

\[ \partial_t u - \mathcal{L}^\mu[\varphi(u)] = 0, \quad \mathcal{L}^\mu[\psi](x) := \int_{|z|>0} (\psi(x + z) - \psi(x)) \mu(dz). \]

Discrete version: \( R_h = \frac{h}{2}[-1, 1]^N, \ h > 0, \ z_\alpha, x_\alpha \in h\mathbb{Z}^N \) (grid parameter)

\[ \partial_t u_h - \mathcal{L}_h^\mu[\varphi(u_h)] = 0, \quad \mathcal{L}_h^\mu[\psi](x_\alpha) := \sum_{\alpha \neq 0} (\psi(x_\alpha + z_\alpha) - \psi(x_\alpha)) \mu(z_\alpha + R_h) \]
Numerical approximation (I): Midpoint quadrature rule

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\[ u_h \to u \text{ in } C([0, T]; L^1_{loc}(\mathbb{R}^N)) \]

**Proof:**
1. \( \mathcal{L}_h^\mu \) is in ”our class” of operators. Just need to take \( \mathcal{L}^{\nu_h} \) with

\[ \nu_h(dz) = \sum_{\alpha \neq 0} \mu(z_\alpha + R_h) \delta_{z_\alpha}(dz). \]

2. \( \mathcal{L}_h^\mu[\psi] \xrightarrow{n \to \infty} \mathcal{L}^\mu[\psi] \) in \( L^1(\mathbb{R}^N) \) for all \( \psi \in C^\infty_c(\mathbb{R}^N) \)

3. Use the continuous dependence Theorem.
Numerical approximation (II): Local problems

\[ \partial_t u - \Delta [\varphi(u)] = 0 \]

Discrete version: \( h > 0, \ x_j \in h \mathbb{Z} \) (grid parameter)

\[ \partial_t u_h - \Delta_h [\varphi(u_h)] = 0, \ \Delta_h [\psi](x_j) := \frac{\psi(x_j + h) + \psi(x_j - h) - 2\psi(x_j)}{h^2} \]


\[ u_h \rightarrow u \text{ in } C([0, T]; L^1_{\text{loc}}(\mathbb{R}^N)) \]
Numerical approximation (II): Local problems

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\[ u_h \rightarrow u \text{ in } C([0, T]; L^1_{loc}(\mathbb{R}^N)) \]

**Remarks:**

1. The classical theory of finite differences for nonlinear local problems is a consequence of our general nonlocal theory.

2. Usually, finite difference requires higher order regularity. We just need \( u \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \) to be a distributional solution.
Local case:

\[
\Delta \psi(x_j) := \frac{\partial^2 \psi}{\partial x^2}(x_j),
\]

\[
\Delta_h \psi(x_j) := \frac{1}{h^2} \left( \psi(x_{j+1}) - \psi(x_j) \right) + \frac{1}{h^2} \left( \psi(x_{j-1}) - \psi(x_j) \right)
\]
Numerical approximation (III): Finite Differences

Local case:

\[ \Delta \psi(x_j) := \frac{\partial^2 \psi}{\partial x^2}(x_j), \]
\[ \Delta_h \psi(x_j) := \left( \psi(x_{j+1}) - \psi(x_j) \right) \frac{1}{h^2} + \left( \psi(x_{j-1}) - \psi(x_j) \right) \frac{1}{h^2} \]

Nonlocal case:

\[ \mathcal{L}^\mu[\psi](x_j) := \int_{|z| > 0} (\psi(x_j + z) - \psi(x)) \mu(dz), \]
\[ \mathcal{L}^\mu_h[\psi](x_j) := \sum_{i \neq 0} (\psi(x_j + z_i) - \psi(x_j)) \omega_i, \quad \omega_j \geq 0 \]
Numerical methods

General theory of finite differences

Numerical approximation (III): Finite Differences

Local case:

\[ \Delta \psi(x_j) := \frac{\partial^2 \psi}{\partial x^2}(x_j), \]
\[ \Delta_h \psi(x_j) := \frac{1}{h^2} \left( \psi(x_{j+1}) - \psi(x_j) \right) + \frac{1}{h^2} \left( \psi(x_{j-1}) - \psi(x_j) \right) \]

Nonlocal case:

\[ \mathcal{L}_\mu^\psi(x_j) := \int_{|z|>0} (\psi(x_j + z) - \psi(x)) \mu(dz), \]
\[ \mathcal{L}_h^\psi(x_j) := \sum_{i \neq 0} (\psi(x_j + z_i) - \psi(x_j)) \omega_i, \quad \omega_j \geq 0 \]

Remark: The operator \( \mathcal{L}_h^\mu \) is included in our theory if \( \omega_j = \omega_{-j} \) and
\[ \sum_i \min\{|z_i|^2, 1\} \omega_i < +\infty. \]

It is enough to consider the operator \( \mathcal{L}^\nu \) with \( \nu(dz) := \sum_{i \neq 0} \delta_{z_i}(dz) \omega_i \).
Some examples included in our theory (I)

\[ \mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 = \int_{|z|<h} \cdots + \int_{|z|>h} \cdots \sim \mathcal{L}_{1,h} + \mathcal{L}_{2,h} = \mathcal{L}_h \]

Sample operator: \( \mathcal{L} = (-\Delta)^{\frac{\sigma}{2}} \) for \( \sigma \in (0, 2) \).

1. Small Jumps:
   a) Trivial discretization: \( \mathcal{L}_{1,h}[\psi] \equiv 0 \). Error = \( O(h^{2-\sigma}) \)
Numerical methods
General theory of finite differences

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   b) Diffusion Correction: Error = \( O(h^{4-\sigma}) \)

   \[
   \mathcal{L}_{1,h}[\psi](x) = \frac{1}{2} \frac{\psi(x+h)+\psi(x-h)-2\psi(x)}{h^2} \int_{|z|<h} |z|^2 \mu(dz)
   \]

Huang, Oberman, Endal, Jakobsen, dT...
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   Huang, Oberman, Endal, Jakobsen, dT...

   c) Regularization of the measure. \( \mu_h = \mu \ast \rho_h \)
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\[ \mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 = \int_{|z|<h} + \int_{|z|>h} \cdots \sim \mathcal{L}_{1,h} + \mathcal{L}_{2,h} = \mathcal{L}_h \]

2. Long Jumps:
   a) Midpoint quadrature rule for \( \mathcal{L}_2 \): Error = min\{O(h), O(h^{2-\sigma})\}
Some examples included in our theory (I)

\[ \mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 = \int_{|z|<h} \cdots + \int_{|z|>h} \cdots \sim \mathcal{L}_{1,h} + \mathcal{L}_{2,h} = \mathcal{L}_h \]

2. Long Jumps:
   a) Midpoint quadrature rule for \( \mathcal{L}_2 \): Error \( = \min\{O(h), O(h^{2-\sigma})\} \)

   b) Linear and quadratic interpolation: Error \( l = O(h^{2-\sigma}), \) Error \( q = O(h^{3-\sigma}) \)

   \[ \mathcal{P}_k^\psi(z) \equiv \text{Lagrange interpolant of order } k \text{ of } \psi, \quad \mathcal{L}_{2,h}[\psi] = \mathcal{L}_2[\mathcal{P}_k^\psi]. \]

Huang, Oberman, Jakobsen, Droniou...
Some examples included in our theory (I)

\[ \mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 = \int_{|z|<h} \cdots + \int_{|z|>h} \cdots \sim \mathcal{L}_{1,h} + \mathcal{L}_{2,h} = \mathcal{L}_h \]

2. Long Jumps:
   a) Midpoint quadrature rule for \( \mathcal{L}_2 \): Error = \( \min\{O(h), O(h^{2-\sigma})\} \)

   b) Linear and quadratic interpolation: Error\(_l = O(h^{2-\sigma}), \text{Error}_{q} = O(h^{3-\sigma}) \)

\[ \mathcal{L}_2,h[\psi] = \mathcal{L}_2[\mathcal{P}^k_{\psi}] \]

Huang, Oberman, Jakobsen, Droniou…

   c) Powers of the discrete laplacian: Error = \( \min\{O(h), O(h^{2-\sigma})\} \)

\[ \mathcal{L}_2,h[\psi] = -(-\Delta_h)^{\frac{\sigma}{2}} \psi \quad \text{with} \quad \Delta_h = \frac{\psi(x+h) + \psi(x-h) - 2\psi(x)}{h^2} \]

Ciaurri, Roncal, Stinga, Torrea, Varona.
Conclusions and Consequences

* Discrete version of local operators can be seen as nonlocal operators.
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* Classical theory of finite differences actually works for distributional solutions of (possibly) nonlinear local problem.
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* The class of operators given by

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\mathcal{L}^\mu[\psi](x) := \int_{|z|>0} (\psi(x + z) - \psi(x)) \mu(dz)
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includes many discretizations of themselves.
Conclusions and Consequences

* Discrete version of local operators can be seen as nonlocal operators.

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\mathcal{L}^\mu[\psi](x) := \int_{|z|>0} (\psi(x+z) - \psi(x)) \mu(dz)
\]

includes many discretizations of themselves.

* Finite differences theory for nonlocal porous medium type problems is a consequence of the continuous dependence theory.
Thank you