Numerical methods for distributional solutions of nonlocal equations of porous medium type

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Joint work with Jørgen Endal and Espen R. Jakobsen from Norwegian University of Science and Technology (NTNU)

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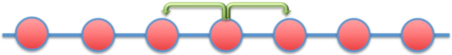
Local jump random walk



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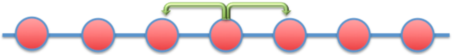

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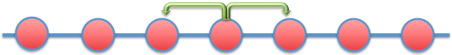

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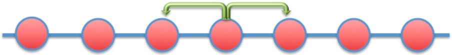

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$(\tau, h \longrightarrow 0^+)$ **Heat Equation**

$$u_t(x,t) = \Delta u(x,t) \qquad (x,t) \in \mathbb{R} \times (0,\infty)$$

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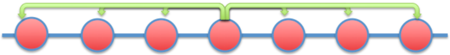

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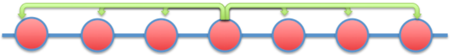

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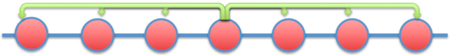

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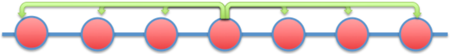

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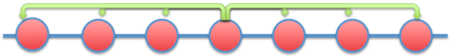

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$(\tau, h \longrightarrow 0^+)$ **Fractional Heat Equation**

$$\partial_t u(x,t) = -d_{\sigma,N}(-\Delta)^{\frac{\sigma}{2}} u(x,t) \qquad (x,t) \in \mathbb{R}\times(0,\infty)$$

Some definitions of nonlocal operators in $\mathbb{R}^N$

Let $\sigma \in (0,2)$. We define the **fractional Laplacian** as

1. Hypersingular integral:

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Physical Motivation for the Porous Medium Equation

- **Flow of a gas in a Porous Medium** (Leibenzon, 1930; Muskat 1933):

$$\begin{cases} \rho_t + \nabla \cdot (\rho \mathbf{v}) = 0 & \text{(Continuity Law)} \\ \mathbf{v} = -c\nabla p, \quad p = \phi(\rho) & \text{(Darcy's Law, 1856)} \end{cases}$$

When the pressure $p = u^{m-1}$ with $m > 1$,

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- Spreading of populations, Thin films under gravity, Kinetic limits and many more...

Local Equations of Porous Medium type

- **Porous Medium Equation:** $u_t = \Delta u^m \sim \nabla \cdot (u^{m-1} \nabla u)$, $m \in (0, \infty)$

(i) $m > 1$: Slow diffusion
Finite speed of propagation, mass conservation, C^α regularity

(ii) $m = 1$: Heat Equation
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(iii) $m < 1$: Fast Diffusion
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- **Generalized PME:** $u_t = \Delta\varphi(u), \quad \varphi$ continuous and nondecreasing
  Contains all the previous models

Vázquez, J. L. The porous medium equation. *Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, Oxford, UK, 2007.*

Non-Local Equations of Porous Medium type (I)

Powers of the Laplacian:

$$-(-\Delta)^s\psi(x) = \mathrm{P.V}\int_{\mathbb{R}^N} \left(\psi(x+y)-\psi(x)\right)\frac{dy}{|y|^{N+2s}}, \quad s\in(0,1).$$

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(P1) $u_t = -(-\Delta)^s u^m, \quad m\in(0,\infty)$

Well-posedness, smooth solutions, infinite speed of propagation

Vázquez, de Pablo, Quirós, Rodríguez, Bonforte, Stan, Brandle, Grillo, Muratori, Punzo, dT...

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(P2) $$u_t = \nabla \cdot (u^k \nabla (-\Delta)^{-s} u^m), \quad m \in (0, \infty), \quad k \in (0, \infty)$$

Conservation law, no uniqueness for dim> 1, finite/infinite speed of propagation, transformations between (P1) and (P2),...

Caffarelli, Vázquez, Karch, Biler, Monneau, Imbert, Stan, dT...

Non-Local Equations of Porous Medium type (II)

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Generalization 1: $u_t = -(-\Delta)^s \varphi(u)$ with φ continuous, non-decreasing

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($\mathcal{L}$ Bounded operator) Andreu, Mazón, Rossi, Toledo, Ignat...
($\mathcal{L} \sim -(-\Delta)^s$) de Pablo, Quirós, Rodríguez, Brandle...
(Convection term $\nabla f(u)$) Jakobsen, Alibaud, Cifani, Karlsen, Biler, Karch, Andreianov, Endal, Silvestre...

The operator $\mathcal{L}^{\mu}$ (I)

Fractional Laplacian: $\nu(dy) = \dfrac{dy}{|y|^{N+2s}}$ and $s \in (0,1)$

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- **Step 2**: Assume $\psi \in C_b^2(\mathbb{R}^N)$

$$|(-\Delta)^s\psi(x)| \leq \|D^2\psi\|_{L^\infty(\mathbb{R}^N)} \int_{B_1} |y|^2\nu(dy) + 2\|\psi\|_{L^\infty(\mathbb{R}^N)} \int_{\mathbb{R}^N\setminus B_1} \nu(dy)$$

ANSWER: $|(-\Delta)^s\psi(x)| \leq C_\psi \int_{\mathbb{R}^N} \min\{|y|^2, 1\}\nu(dy)$

The operator $\mathcal{L}^\mu$ (II)

(A_μ) μ is any symmetric Radon measure s.t. $\int \min\{|y|^2, 1\}\mu(dy) < \infty$

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3. $\Delta_h\psi(x)=\frac{\psi(x+h)+\psi(x-h)-2\psi(x)}{h^2}$ with $\mu(dy)=\frac{\delta_h(dy)+\delta_{-h}(dy)}{h^2}$

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Bonforte, M., Figalli, A. and Vázquez, J. L. Sharp global estimates for local and nonlocal porous medium-type equations in bounded domains. *preprint*.

3. $\Delta_h \psi(x) = \frac{\psi(x+h)+\psi(x-h)-2\psi(x)}{h^2}$ with $\mu(dy) = \frac{\delta_h(dy)+\delta_{-h}(dy)}{h^2}$
4. Powers of Δ_h: Ciaurri, Roncal, Stinga, Torrea, Varona

Main results (I): Well-posedness

(IVP)
$$\begin{cases} u_t - \mathcal{L}^{\mu}[\varphi(u)] = 0 & Q_T = \mathbb{R}^N \times (0,T), \\ u(x,0) = u_0(x) & \mathbb{R}^N. \end{cases}$$

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Theorem (Uniqueness - dT, Endal, Jakobsen. Adv. Math. 2017)

There is at most one distributional solution u of (IVP) such that $u \in L^\infty(\mathbb{R}^N)$ and $u - u_0 \in L^1(\mathbb{R}^N)$.

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Theorem (Existence - dT, Endal, Jakobsen. Adv. Math. 2017)

If $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, then there exists a (unique) distributional solution of (IVP) and

$$u \in L^1(Q_T) \cap L^\infty(Q_T) \cap C((0,T), L^1_{\text{loc}}(\mathbb{R}^N)).$$

Main results (II): A priori estimates

(IVP)
$$\begin{cases} u_t - \mathcal{L}^\mu[\varphi(u)] = 0 & Q_T = \mathbb{R}^N \times (0,T), \\ u(x,0) = u_0(x) & \mathbb{R}^N. \end{cases}$$

Theorem (dT, Endal, Jakobsen. Adv. Math. 2017)

Let $u_0, \hat{u}_0 \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$. *Then*

(a) (L^1-contr.) $\int_{\mathbb{R}^N} (u(x,t) - \hat{u}(x,t))^+ \mathrm{d}x \leq \int_{\mathbb{R}^N} (u_0(x) - \hat{u}_0(x))^+ \mathrm{d}x$;

(b) (Comp. principle) *If* $u_0 \leq \hat{u}_0$ *a.e. in* $\mathbb{R}^N$, *then* $u \leq \hat{u}$ *a.e. in* Q_T;

(c) ($L^{1/\infty}$-bound) $\|u(\cdot,t)\|_{L^{1/\infty}(\mathbb{R}^N)} \leq \|u_0\|_{L^{1/\infty}(\mathbb{R}^N)}$;

(d) (Mass conservation) *If* $|\varphi(r)| \leq L|r|$ *for* $|r| \leq \delta$, *then*

$$\int_{\mathbb{R}^N} u(x,t)\mathrm{d}x = \int_{\mathbb{R}^N} u_0(x)\mathrm{d}x.$$

Main results (III): Continuous dependence

Narutal question: Under which conditions

$$\begin{cases} \partial_t u_n - \mathcal{L}^{\mu_n}[\varphi_n(u_n)] & = 0 \\ u_n(x,0) & = u_0(x) \end{cases} \overset{n\to\infty}{\longrightarrow} \begin{cases} \partial_t u - \mathcal{L}[\varphi(u)] & = 0 \\ u(x,0) & = u_0(x). \end{cases}$$

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Theorem (Cont. dependence - dT, Endal, Jakobsen. Adv. Math. 2017)

Let $u_0 \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$. *Assume:*

(i) $\mathcal{L}^{\mu_n}[\psi] \overset{n\to\infty}{\longrightarrow} \mathcal{L}[\psi]$ *in* $L^1(\mathbb{R}^N)$ *for all* $\psi \in C_c^\infty(\mathbb{R}^N)$;

(ii) $\varphi_n \overset{n\to\infty}{\longrightarrow} \varphi$ *locally uniformly.*

Then

(Distributional solutions) $u_n \overset{n\to\infty}{\longrightarrow} u$ *in* $C([0,T]; L^1_{\text{loc}}(\mathbb{R}^N))$.

Numerical approximation (I): Midpoint quadrature rule

$$\partial_t u - \mathcal{L}^\mu[\varphi(u)] = 0, \quad \mathcal{L}^\mu[\psi](x) := \int_{|z|>0} (\psi(x+z) - \psi(x))\, \mu(\mathrm{d}z).$$

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Discrete version: $R_h = \frac{h}{2}[-1,1)^N$, $h > 0$, $z_\alpha, x_\alpha \in h\mathbb{Z}^N$ (grid parameter)

$$\partial_t u_h - \mathcal{L}_h^\mu[\varphi(u_h)] = 0, \; \mathcal{L}_h^\mu[\psi](x_\alpha) := \sum_{\alpha \neq 0} (\psi(x_\alpha + z_\alpha) - \psi(x_\alpha))\, \mu\left(z_\alpha + R_h\right)$$

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Theorem (dT, Endal, Jakobsen. Adv. Math. 2017)

$$u_h \to u \ \textit{in} \ C([0,T]; L^1_{\mathrm{loc}}(\mathbb{R}^N))$$

Proof: 1. $\mathcal{L}_h^\mu$ is in "our class" of operators. Just need to take $\mathcal{L}^{\nu_h}$ with

$$\nu_h(\mathrm{d}z) = \sum_{\alpha\neq 0} \mu\left(z_\alpha + R_h\right)\delta_{z_\alpha}(\mathrm{d}z).$$

2. $\mathcal{L}_h^\mu[\psi] \stackrel{n\to\infty}{\longrightarrow} \mathcal{L}^\mu[\psi]$ in $L^1(\mathbb{R}^N)$ for all $\psi \in C_c^\infty(\mathbb{R}^N)$
3. Use the continuous dependence Theorem.

Numerical approximation (II): Local problems

$$\partial_t u - \Delta[\varphi(u)] = 0$$

Discrete version: $h > 0$, $x_j \in h\mathbb{Z}$ (grid parameter)

$$\partial_t u_h - \Delta_h[\varphi(u_h)] = 0, \ \Delta_h[\psi](x_j) := \frac{\psi(x_j + h) + \psi(x_j - h) - 2\psi(x_j)}{h^2}$$

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Remarks:

1. The classical theory of finite differences for nonlinear local problems is a consequence of our general nonlocal theory.
2. Usually, finite difference requires higher order regularity. We just need $u \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ to be a distributional solution.

Numerical approximation (III): Finite Differences

Local case:

$$\Delta\psi(x_j) := \frac{\partial^2\psi}{\partial x^2}(x_j),$$

$$\Delta_h\psi(x_j) := (\psi(x_{j+1}) - \psi(x_j))\,\frac{1}{h^2} + (\psi(x_{j-1}) - \psi(x_j))\,\frac{1}{h^2}$$

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Nonlocal case:

$$\mathcal{L}^\mu[\psi](x_j) := \int_{|z|>0} \left(\psi(x_j + z) - \psi(x)\right)\mu(\mathrm{d}z),$$

$$\mathcal{L}_h^\mu[\psi](x_j) := \sum_{i\neq 0} \left(\psi(x_j + z_i) - \psi(x_j)\right)\omega_i, \quad \omega_j \geq 0$$

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REMARK: The operator $\mathcal{L}_h^\mu$ is included in our theory if $\omega_j = \omega_{-j}$ and

$$\sum_i \min\{|z_i|^2, 1\} w_i < +\infty.$$

It is enough to consider the operator $\mathcal{L}^\nu$ with $\nu(\mathrm{d}z) := \sum_{i\neq 0} \delta_{z_i}(\mathrm{d}z)\omega_i$

Some examples included in our theory (I)

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 = \int_{|z|<h} \cdots + \int_{|z|>h} \cdots \sim \mathcal{L}_{1,h} + \mathcal{L}_{2,h} = \mathcal{L}_h$$

Sample operator: $\mathcal{L} = (-\Delta)^{\frac{\sigma}{2}}$ for $\sigma \in (0,2)$.

1. Small Jumps:
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   c) Regularization of the measure. $\mu_h = \mu * \rho_h$

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   $\mathcal{P}_\psi^k(z) \equiv$Lagrange interpolant of order k of ψ, $\quad \mathcal{L}_{2,h}[\psi] = \mathcal{L}_2[\mathcal{P}_\psi^k]$.

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   c) Powers of the discrete laplacian: Error$= \min\{O(h), O(h^{2-\sigma})\}$

   $\mathcal{L}_{2,h}[\psi] = -(-\Delta_h)^{\frac{\sigma}{2}}\psi$ with $\Delta_h = \frac{\psi(x+h)+\psi(x-h)-2\psi(x)}{h^2}$

   Ciaurri, Roncal, Stinga, Torrea, Varona.

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* Finite differences theory for nonlocal porous medium type problems is a consequence of the continuous dependence theory.

Thank you