Mittag–Leffler Analysis: Construction and Applications

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joint work with
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[Schneider'92] considered the time-fractional heat equation for $0 < \beta < 1$ and initial value $u_0 \in S(\mathbb{R})$:

$$u(t, x) = u_0(x) + \frac{1}{2} \int_0^t \frac{(t - s)^{\beta-1}}{\Gamma(\beta)} \frac{\partial^2}{\partial x^2} u(s, x) \, ds$$

$$= u_0(x) + \frac{1}{2} I_{0+}^{\beta} \left( \frac{\partial^2}{\partial x^2} u(\cdot, x) \right) (t), \quad t \geq 0, x \in \mathbb{R}. \tag{1}$$

The solution is given by the fractional Feynman-Kac formula

$$u(t, x) = \mathbb{E}(u_0(x + B_t^\beta)), \tag{2}$$

and $B_t^\beta$ is called grey Brownian motion.

For further results see e.g.: A. Kochubei, F. Mainardi.
Questions

- Can we develop an analogue to white noise analysis? Yes and No\(^1\).
- Can we give sense to the expression \( \delta_y(x + B_t^\beta), \ y \in \mathbb{R} \)? Yes! In a certain distribution space\(^1\).
- Is \( K(t, x, y) = \mathbb{E}(\delta_y(x + B_t^\beta)) \) a fractional heat kernel? Yes! By choosing the correct measure \( \mu_\beta \)^\(^2\).

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Consider the nuclear triple

\[ \mathcal{N} \subset \mathcal{H} \subset \mathcal{N}'. \]

- Real separable Hilbert space \( \mathcal{H} \), norm \( |\cdot| \) and scalar product \( (\cdot, \cdot) \).
- Nuclear space \( \mathcal{N} \), densely embedded in \( \mathcal{H} \).
- Dual space \( \mathcal{N}' \) of \( \mathcal{N} \) with dual pairing

\[ \langle \eta, \xi \rangle = (\eta, \xi), \quad \eta \in \mathcal{H}, \xi \in \mathcal{N}. \]
The Mittag-Leffler Measure

The Mittag-Leffler functions $E_\beta$, $\beta > 0$, are defined by:

$$E_\beta(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\beta n + 1)}, \quad z \in \mathbb{C}.$$ 

**Definition**

Let $\mathcal{B}$ be the cylinder $\sigma$-algebra on $\mathcal{N}'$. The Mittag-Leffler measures $\mu_\beta$, $0 < \beta \leq 1$, are the unique probability measures satisfying

$$\int_{\mathcal{N}'} e^{i\langle \omega, \xi \rangle} d\mu_\beta(\omega) = E_\beta \left( -\frac{1}{2} \langle \xi, \xi \rangle \right), \quad \xi \in \mathcal{N}.$$ 

**Moments of $\mu_\beta$:**

$$\int_{\mathcal{N}'} \langle \omega, \xi \rangle^{2n} d\mu_\beta(\omega) = \frac{(2n)!}{2^n \Gamma(\beta n + 1)} |\xi|^{2n}, \quad \xi \in \mathcal{N}, n \in \mathbb{N}.$$
Special choice: $\mathcal{N} = S(\mathbb{R})$ and $\mathcal{H} = L^2(\mathbb{R}, dx)$.

- For $0 < H < 1$ define the operator $M^H_\pm$ on $S(\mathbb{R})$ by
  
  \[ M^H_\pm f := \begin{cases} 
  K_H D^-(H-\frac{1}{2}) f, & H \in (0, \frac{1}{2}), \\
  f, & H = \frac{1}{2}, \\
  K_H I^{H-\frac{1}{2}} f, & H \in (\frac{1}{2}, 1). 
  \end{cases} \]

- Generalized grey Brownian motion for $0 < \alpha < 2$:
  
  $B^{\alpha,\beta}_t := \langle \cdot, M^{\alpha/2}_- 1_{[0,t]} \rangle$, defined as limit in $L^2(S'(\mathbb{R}), \mu_\beta)$.

- For $\beta = 1$: fractional Brownian motion with $H = \alpha/2$.

- $(B^{\alpha,\beta}_t)_{t \geq 0}$ is an $\alpha/2$-self similar stochastic process with stationary increments, see als [Mura,Mainardi’09].
Orthogonal Polynomials on $\mathcal{N}'$

Back to the general setting $\mathcal{N} \subset \mathcal{H} \subset \mathcal{N}'$ and $L^2(\mathcal{N}', \mu_\beta)$.

**Question**

Find polynomials $I_n(\xi), \xi \in \mathcal{N}'$, on $\mathcal{N}'$ with the following properties:

- $I_n(\xi) = p_{n,\xi}(\langle \cdot, \xi \rangle)$ for some polynomial $p_{n,\xi}$ of degree $n \in \mathbb{N}$.
- $I_n(\xi) \perp I_m(\xi)$ in $L^2(\mathcal{N}', \mu_\beta)$, $n \neq m$.
- $\left( I_n(\xi), I_m(\eta) \right)_{L^2(\mu_\beta)} = C \langle \xi, \eta \rangle^n \delta_{n,m} \quad \xi, \eta \in \mathcal{N}, C \in \mathbb{R}$. 

For $\beta = 1$: $p_{n,\xi}$ are the Hermite polynomials $H_n$.

For $\beta \neq 1$: Applying Gram-Schmidt orthogonalization to the monomials, one finds $I_n(\xi), n \in \mathbb{N},$ fulfilling (i) and (ii). But there is no such system fulfilling (i), (ii) and (iii) at the same time.

$\Rightarrow$ Use biorthogonal polynomials.
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**Answer**

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\[\implies \text{Use biorthogonal polynomials}\]
In [KSWY’98]: Let $\mu$ be a probability measure on $\mathcal{N}'$. Under certain conditions the following holds:

- There exists a $\mathbb{P}$-system: For each polynomial $\varphi \in \mathcal{P}(\mathcal{N}')$ on $\mathcal{N}'$

\[
\varphi(x) = \sum_{n=0}^{N} \langle P_n^\mu(x), \varphi^{(n)} \rangle, \quad \varphi^{(n)} \in \mathcal{N}_c^{\otimes n}, x \in \mathcal{N}'.
\]

- There exists a $\mathbb{Q}$-system: For each "dual polynomial" $\Phi \in \mathcal{P}_{\mu,\beta}'(\mathcal{N}')$

\[
\Phi = \sum_{n=0}^{\infty} Q_n^\mu(\Phi^{(n)}), \quad \Phi^{(n)} \in (\mathcal{N}_c^{\otimes n})'.
\]

- The biorthogonality relation holds:

\[
\langle \langle Q_n^\mu(\Phi^{(n)}), \langle P_m^\mu, \varphi^{(m)} \rangle \rangle \rangle_\mu = \delta_{m,n} n! \langle \Phi^{(n)}, \varphi^{(m)} \rangle.
\]
Compare to the Gaussian case \((\beta = 1)\) and the Hermite polynomials \(H_n\).
Consider the normalized exponential for \(\xi \in \mathcal{N}_C, x \in \mathcal{N}':\)

\[
e_{\mu_\beta}(\xi; x) := \frac{e^{\langle x, \xi \rangle}}{\int_{\mathcal{N}'} e^{\langle x, \xi \rangle} d\mu_\beta(x)}
= \sum_{n=0}^{\infty} \frac{1}{n!} \langle P_{n}^{\mu_\beta}(x), \xi^\otimes n \rangle
\]
Compare to the Gaussian case \((\beta = 1)\) and the Hermite polynomials \(H_n\). Consider the normalized exponential for \(\xi \in \mathcal{N}_C, x \in \mathcal{N}'\):

\[
e_{\mu_\beta}(\xi; x) := \frac{e^{\langle x, \xi \rangle}}{\int_{\mathcal{N}'} e^{\langle x, \xi \rangle} d\mu_\beta(x)} = \exp\left(\langle x, \xi \rangle - \frac{1}{2}\langle \xi, \xi \rangle\right)
\]

\[
= \sum_{n=0}^{\infty} \frac{1}{n!} \langle P_{\mu_\beta}^n(x), \xi \otimes^n \rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{|\xi|^n}{2^{n/2}} H_n \left(\frac{\langle x, \xi \rangle}{\sqrt{2}|\xi|}\right).
\]
Compare to the Gaussian case ($\beta = 1$) and the Hermite polynomials $H_n$.

Consider the normalized exponential for $\xi \in \mathcal{N}_\mathbb{C}, x \in \mathcal{N}':$

$$e_{\mu\beta}(\xi; x) := \frac{e^{\langle x, \xi \rangle}}{\int_{\mathcal{N}'} e^{\langle x, \xi \rangle} d\mu_{\beta}(x)} = \exp (\langle x, \xi \rangle - 1/2 \langle \xi, \xi \rangle)$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \langle P_{n}^{\mu\beta}(x), \xi \otimes^n \rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{|\xi|^n}{2^{n/2}} H_n \left( \frac{\langle x, \xi \rangle}{\sqrt{2} |\xi|} \right).$$

By calculation:

The system $\left( \langle P_n^{\mu\beta}(\cdot), \xi \otimes^n \rangle \right)_{n \in \mathbb{N}}$ is different from the orthogonal system $\left( I_n(\xi) \right)_{n \in \mathbb{N}}$ obtained via Gram-Schmidt orthogonalization.
The following chain of spaces can be constructed ([KSWY’98]):

\[(\mathcal{N})^{1} \subset L^{2}(\mathcal{N}', \mu_{\beta}) \subset (\mathcal{N})^{-1}_{\mu_{\beta}}.\]

For \(\Phi \in (\mathcal{N})^{-1}_{\mu_{\beta}}\) we can define its \(T_{\mu_{\beta}}\)-transform (Fourier transform) by:

\[T_{\mu_{\beta}} \Phi(\xi) = \langle \langle \Phi, \exp(i\langle \cdot, \xi \rangle) \rangle \rangle_{\mu_{\beta}},\]

for \(\xi\) from a suitable neighborhood of \(0 \in \mathcal{N}_{\mathbb{C}}\).

**Theorem (Kondratiev, Streit, Westerkamp, Yan)**

\[
\Phi = T^{-1}_{\mu_{\beta}}F \in (\mathcal{N})^{-1}_{\mu_{\beta}}
\]

for some \(F : \mathcal{N}_{\mathbb{C}} \to \mathbb{C}\)

if and only if

\(F\) is holomorphic in a neighborhood of \(0 \in \mathcal{N}_{\mathbb{C}}\).
Donsker’s Delta I

Theorem

Donsker’s delta $\delta_a(\langle \cdot, \eta \rangle)$ for $0 \neq \eta \in \mathcal{H}$ and $a \in \mathbb{R}$ exists as a weak integral in $(\mathcal{N}^\mu_\beta)^{-1}$ via the integral representation

$$\delta_a(\langle \cdot, \eta \rangle) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix(\langle \cdot, \eta \rangle - a)} \, dx, \quad \eta \in \mathcal{H}.$$  

This means that

$$(T_{\mu_\beta} \delta_a(\langle \cdot, \eta \rangle))(\xi) = \frac{1}{2\pi} \int_{\mathbb{R}} \left( T_{\mu_\beta} e^{ix(\langle \cdot, \eta \rangle - a)} \right)(\xi) \, dx$$

- Choose $\eta = M^\alpha_{\alpha/2}[0,t]$ (grey noise case), then we have $\delta_a(B_t^{\alpha,\beta})$.
- The complex-scaled Donsker’s delta $\delta_a(i^{\alpha/2} \langle \cdot, \eta \rangle)$ for $0 < \alpha < 2$ also exists.
The weak integral and the Pettis integral:

\[
\langle \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix(\langle \cdot, \eta \rangle - a)} \, dx, F \rangle_{\mu_\beta} = \frac{1}{2\pi} \int_{\mathbb{R}} \langle e^{ix(\langle \cdot , \eta \rangle - a)} , F \rangle_{\mu_\beta} \, dx
\]

holds for

- **Pettis integral**: For all \( F \in (\mathcal{N})_{\mu_\beta}^1 \).
- **Weak integral**: For all \( F = \exp(i\langle \cdot , \xi \rangle) \), \( \xi \) from a suitable neighborhood of \( 0 \in \mathcal{N}_\mathbb{C} \), i.e. for all \( F \) from a dense subset \( D \).
The weak integral and the Pettis integral:

\[
\left\langle \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix(\langle \cdot, \eta \rangle - a)} \, dx, F \right\rangle_{\mu_\beta} = \frac{1}{2\pi} \int_{\mathbb{R}} \left\langle e^{ix(\langle \cdot, \eta \rangle - a)}, F \right\rangle_{\mu_\beta} \, dx
\]

holds for

- **Pettis integral**: For all \( F \in (\mathcal{N})_{\mu_\beta}^1 \).

- **Weak integral**: For all \( F = \exp(i\langle \cdot, \xi \rangle) \), \( \xi \) from a suitable neighborhood of \( 0 \in \mathcal{N}_\mathbb{C} \), i.e. for all \( F \) from a dense subset \( D \).

**Approximation of Donsker’s delta**:

- \( \Phi_n := \frac{1}{2\pi} \int_{-n}^{n} e^{ix(\langle \cdot, \eta \rangle - a)} \, dx, \ n \in \mathbb{N} \),

  is a Bochner integral in the Hilbert space \( L^2(\mathcal{N}', \mu_\beta) \).

- \( (\Phi_n)_{n\in\mathbb{N}} \) converges to \( \delta_a(\langle \cdot, \eta \rangle) \) for \( n \to \infty \) in the strong topology of the distribution space \( (\mathcal{N})_{\mu_\beta}^{-1} \).
Theorem

Define for $0 < \alpha < 1$, $x, y \in \mathbb{R}$ and $t > 0$ the integral kernel

$$K(t, x, y) := \mathbb{E}_{\mu_\alpha} (\delta_y (x + B_t^{\alpha, \alpha}))$$

Then $u(t, x) = \int_{\mathbb{R}} K(t, x, y) u_0(y) \, dy$, solves the time-fractional heat equation

$$u(t, x) = u_0(x) + \frac{1}{2} \left( l_0^\alpha \frac{\partial^2}{\partial x^2} u(\cdot, x) \right)(t)$$

under the condition that $u_0 \in L^1(\mathbb{R}, dx) \cap L^2(\mathbb{R}, dx)$ and for some $\varepsilon > 0$

$$\int_{\mathbb{R}} \left| (\mathcal{F}^{-1} u_0)(\lambda) \lambda \right|^{1+\varepsilon} \, d\lambda < \infty \quad \text{and} \quad \int_{\mathbb{R}} \left| (\mathcal{F}^{-1} u_0)(\lambda) \lambda^2 \right|^{1+\varepsilon} \, d\lambda < \infty.$$

A corresponding result for $B_t^{\alpha, \beta}$ with $\alpha \neq \beta$ also exists. In this case $K$ gives the solution to the time-stretched fractional heat equation.
Consider the \textit{time-fractional heat equation with potential}:

\begin{align*}
  u(t, x) &= u_0(x) + \frac{1}{2} \left( l_0^\alpha \left( \frac{\partial^2}{\partial x^2} u(\cdot, x) + V(x)u(\cdot, x) \right) \right)(t),
\end{align*}

for a continuous potential $V$, which is bounded from above.

\textbf{Question}

\textit{How to find $\Psi(t, x, y) \in (\mathcal{N})_{\mu_\alpha}^{-1}$ such that}

\begin{align*}
  K(t, x, y) &= \mathbb{E}_{\mu_\alpha}(\Psi(t, x, y))
\end{align*}

\textit{is a Green’s function.}
For $0 < \alpha < 1$, $t > 0$ and $x \in \mathbb{R}$ the time-fractional Schrödinger equation has the form:

$$u(t, x) = u_0(x) + \frac{1}{2} i^\alpha \left( l_0^\alpha + \frac{\partial^2}{\partial x^2} u(\cdot, x) \right)(t).$$

**Theorem**

Let $K(t, x, y) = \mathbb{E}_{\mu_\alpha} \left( \delta_y(x + i^{\alpha/2} B_t^\alpha) \right)$. Then $K$ is a Green’s function to the time-fractional Schrödinger equation.
Can we develop an analogue to white noise analysis?  
Yes and No.

Can we give sense to the expression $\delta_y(x + B_t^\beta)$, $y \in \mathbb{R}$?  
Yes! In a certain distribution space.

Is $K(t, x, y) = \mathbb{E}(\delta_y(x + B_t^\beta))$ a fractional heat kernel?  
Yes!
References

W.R. Schneider.
Grey Noise.

Generalized functions in infinite dimensional analysis.

A. Mura and F. Mainardi.
A class of self-similar stochastic processes with stationary increments to model anomalous diffusion in physics.


Generalized functions in infinite dimensional analysis.

A. Mura and F. Mainardi.
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M. Grothaus, F. Jahnert, F. Riemann and J.L. da Silva.
Mittag–Leffler analysis I: Construction and characterization.

M. Grothaus and F. Jahnert.
Mittag–Leffler analysis II: Application to the fractional heat equation.

Thank you for your attention!