Jump Processes in a Potential Landscape

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an early motivation: study spectral properties of relativistic Schrödinger operator

\[ H = \left(-\Delta + m^2\right)^{1/2} - m + V, \quad m \geq 0 \]

giving the Hamiltonian in the relativistic Schrödinger equation

\[
i\hbar \frac{\partial \Psi(x, t)}{\partial t} = \left(-\hbar^2 c^2 \Delta + m^2 c^4\right)^{1/2} - mc^2 \right) \Psi(x, t) + V(x) \Psi(x, t)
\]

Weder 1974, Herbst 1977, Lieb, Daubechies 1983, Carmona, Masters, Simon 1990, ...

recently much interest in

- non-local equations: Bass, Caffarelli, Silvestre, Vázquez, Kaßmann, Cabré, Ros-Oton, Valdinoci, Kuusi, ...

- Markov processes generated by pseudo-differential operators: Jacob, Farkas, Hoh, Schilling, Kolokoltsov ...

- potential theory: Wrocław group, Bañuelos, Chen, Song, Kim, Vondraček, ...

NB: many qualitative differences between

\[-\Delta \leftrightarrow BM \leftrightarrow PDE\]

vs

non-local operators \(\leftrightarrow\) jump processes \(\leftrightarrow\) integro-differential equations
an early motivation: study spectral properties of relativistic Schrödinger operator

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Motion in a potential landscape

.. back to original motivation: \( H = (-\Delta + m^2)^{1/2} - m + V \)

- use \textit{Feynman-Kac type formula}

\[
(e^{-tH}f)(x) = \mathbb{E}^{x}[e^{-\int_0^t V(X_s)ds}f(X_t)]
\]

where \((X_t)_{t \geq 0}\) is a Lévy process

- suppose \textit{ground state} \( \varphi_0 \) exists, ie,

\[
H \varphi_0 = \lambda_0 \varphi_0, \quad \varphi_0 \in \text{Dom} \, H, \quad \lambda_0 = \inf \text{Spec} \, H
\]

- suppose also \textit{local time} \((L^x_t)_{t \geq 0}\) exists (else use occupation measure); then

\[
\int_0^t V(X_s)ds = \int_{\mathbb{R}^d} V(y)L^y_t dy
\]

- so spectral problem translates into probability problem

\[
\varphi_0(x) = e^{\lambda_0 t} \mathbb{E}^{x}[e^{-\int_{\mathbb{R}^d} V(y)L^y_t dy} \varphi_0(X_t)]
\]

hence interested in properties of paths in a potential landscape

position-dependent transitions, penalised/reinforced by \( V \)
Feynman-Kac semigroup $T_tf(x) = \mathbb{E}^x[e^{-\int_0^t V(X_s)ds}f(X_t)]$ not Markov

if ground state (GS) $\varphi_0$ exists, then strictly positive, unique

by using $T_t\varphi_0 = e^{-\lambda_0 t}\varphi_0$ and changing measure

$$\frac{e^{\lambda_0 t}}{\varphi_0(x)} T_t\varphi_0(x) = \frac{e^{\lambda_0 t}}{\varphi_0(x)} \int_{\mathbb{R}^d} u_t(x,y)\varphi_0(y)dy = \int_{\mathbb{R}^d} \tilde{u}_t(x,y)dP(y) = \tilde{T}_t1_{\mathbb{R}^d}(x) = 1_{\mathbb{R}^d}(x)$$

gives Markov semigroup on $L^2(\mathbb{R}^d, dP)$

$$\tilde{T}_tf(x) = \frac{e^{\lambda_0 t}}{\varphi_0(x)} T_t(\varphi_0 f)(x)$$

with

$$\tilde{u}_t(x,y) = \frac{e^{\lambda_0 t}u_t(x,y)}{\varphi_0(x)\varphi_0(y)} \quad \text{and} \quad dP(x) = \varphi_0^2(x)dx$$
Ground state-transformed (GST) processes

infinitesimal generator of \( \tilde{T}_t = e^{t\tilde{L}} \)

\[
(\tilde{L}f)(x) = \frac{1}{2} \Delta f(x) + \nabla \log \varphi_0(x) \cdot \nabla f(x)
\]

\[
+ \int_{0<|z|\leq 1} \frac{\varphi_0(x+z) - \varphi_0(x)}{\varphi_0(x)} z \cdot \nabla f(x) \nu(z) \, dz
\]

\[
+ \int_{\mathbb{R}^d\setminus\{0\}} (f(x+z) - f(x) - z \cdot \nabla f(x) \1_{\{|z|\leq 1\}}(z)) \frac{\varphi_0(x+z)}{\varphi_0(x)} \nu(z) \, dz,
\]

leading to ground state-transformed (GST) process

\[
X_t = X_0 + B_t + \int_0^t \nabla \log \varphi_0(X_s) \, ds + \int_0^t \int_{|z|\leq 1} \frac{\varphi_0(X_s+z) - \varphi_0(X_s)}{\varphi_0(X_s)} z\nu(z) \, dz \, ds
\]

\[
+ \int_0^t \int_{|z|\leq 1} \int_0^\infty \1_{\{w \leq \frac{\varphi_0(X_s+z)}{\varphi_0(X_s)}\}} z\tilde{N}(ds, dz, dw)
\]

\[
+ \int_0^t \int_{|z|>1} \int_0^\infty \1_{\{w \leq \frac{\varphi_0(X_s+z)}{\varphi_0(X_s)}\}} zN(ds, dz, dw)
\]

NB: potential gives rise to drift and bias in jump rates, enters through GS
goal: derive properties of GST process wrt input parameters \( \nu \) and \( V \) (or \( \varphi_0 \))
Ground states: existence

- **spectrally**
  
  $\varphi_0$ exists $\iff$ $\text{Spec}_d H \neq \emptyset$

  generally $\text{Spec}(-L) = \text{Spec}_{\text{ess}}(-L) = [0, \infty)$

  this may change on adding $V$

  - if $V$ *confining*, ie, $\lim_{|x| \to \infty} V(x) = \infty$, then

    $$\text{Spec} H = \text{Spec}_d H = \{0 < \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots \to \infty\}$$

  - if $V$ *decaying*, ie, $\lim_{|x| \to \infty} V(x) = 0$, then $\text{Spec} H = \text{Spec}_{\text{ess}} H \sqcup \text{Spec}_d H$

    mechanism: variational principle – potential energy dominates kinetic energy

- **probabilistically**

  mechanism: sufficient mean total sojourn time in local neighbourhoods

  GS formation is a recurrence-type phenomenon
Ground states: explicit solutions

- \( H = (-d^2/dx^2)^{1/2} + x^2 \) on \( L^2(\mathbb{R}) \)  
  L-Małecki, JDE 253, 2012

\[
\text{Spec } H = \{-a'_n, -a_n : n \in \mathbb{N}\}, \quad \text{with } \text{Ai}(-a_n) = 0 = \text{Ai}'(-a'_n)
\]

\[
\hat{\varphi}_n(y) = \begin{cases} 
\frac{\text{Ai}(|y|+a'_n)}{\sqrt{-2a'_n\text{Ai}(a'_n)}} & n = 1, 3, 5, \ldots \\
\frac{\text{sgn}(y)\text{Ai}(|y|+a_n)}{\sqrt{2\text{Ai}'(a_n)}} & n = 2, 4, 6, \ldots
\end{cases}
\]

- \( H = (-d^2/dx^2)^{1/2} + x^4 \) on \( L^2(\mathbb{R}) \)  
  Durugo-L, 2015

\[
\text{Spec } H = \text{zeroes of special functions related to Fresnel } \text{si, ci integrals}
\]

\[
\hat{\varphi}_n = \text{related to 4th order Airy function } \text{Ai}_4(y) = \frac{1}{\pi} \int_0^\infty \cos \left( \frac{t^5}{5} + yt \right) dt
\]

**NB:** in either case detailed spectral information (spectral gaps, heat trace, Weyl th), uniform boundedness, analyticity, full asymptotic expansion of eigenfunctions + subtle spectral shift effect in quartic case
Ground states: asymptotic estimates

GS decay at infinity = tails of stationary distribution

- **Case 1: isolated eigenvalues**  
  Kaleta-L, Ann Probab 43, 2106; Potential Anal 2016

  roughly, for many cases — with \( \tau_B(x) = \inf\{t > 0 : X_t \not\in \tau_B(x)\} \)

\[
\varphi_0(x) \asymp \mathbb{E}^x \left[ \int_0^{\tau_B(x)} e^{-\int_0^t V(X_s)ds} dt \right] \nu(x), \quad \text{for } |x| \text{ large enough}
\]

this means that decay depends on how soon the process leaves unit balls far out

implies, roughly,

\[ \rightarrow \text{ confining potentials: } \varphi_0 \asymp \frac{\nu}{V} \text{ (balance of jump freedom and soft killing)} \]

\[ \rightarrow \text{ decaying potentials: } \varphi_0 \asymp \nu \text{ (more subtle)} \]

**NB:** \( V \) growing quickly enough for given \( \nu \) \( \Rightarrow \) FK semigroup IUC

\[
\left| \frac{e^{\lambda_0 t}u(t, x, y)}{\varphi_0(x)\varphi_0(y)} - 1 \right| \leq Ce^{-(\lambda_1 - \lambda_0)t}, \quad t > 0
\]

implies exponential relaxation to equilibrium, GS domination of e.f., ergodicity, ...
this mechanism works when all multiple large jumps are dominated by single large jumps, expressed by *jump-paring property*

\[
\int_{|x-y|>1} \nu(|x-y|)\nu(|y|)dy \leq C \nu(|x|), \quad |x| \geq 1
\]

when jump-paring fails to hold, decay slows down, frequent multiple re-entries into local neighbourhoods build up backlogs

for an idea, choose \(\nu(z) \asymp e^{-c|z|^{\beta}} |z|^{-\delta}, \beta > 0\)

\(--\) jump-paring holds holds for subexp/part of exp, fails otherwise

\(--\) “phase transition” in GS decay rates around critical exponent \(\delta = \frac{d+1}{2}\)
Case 2: behaviour at spectral edge  

\[
Y_\kappa(x) = _2F_1\left(1, \frac{1}{2} + \kappa, \frac{1}{2}, -x^2\right), \quad \kappa > 0
\]

\[
V_\kappa(x) = -\frac{2}{\sqrt{\pi}} \frac{\Gamma\left(\frac{1}{2} + \kappa\right)}{\Gamma(\kappa)} \frac{Y_\kappa(x)}{\sqrt{1 + x^2}}
\]

\[
\varphi_\kappa(x) = \frac{1}{(1 + x^2)^\kappa}
\]

and

\[
V_\kappa(x) = \begin{cases} 
O(1/|x|) & \text{if } 0 < \kappa < \frac{1}{2} \\
O(\log |x|/|x|) & \text{if } \kappa = \frac{1}{2} \\
O(1/|x|^{2-2\kappa}) & \text{if } \frac{1}{2} < \kappa < 1
\end{cases}
\]

\[
\tilde{V}_\kappa(x) = \begin{cases} 
O(1/|x|) & \text{if } \frac{1}{2} < \kappa < \frac{3}{2}, \kappa \neq 1 \\
O(1/|x|^2) & \text{if } \kappa = 1 \\
O(\log |x|/|x|) & \text{if } \kappa = \frac{3}{2} \\
O(1/|x|^{4-2\kappa}) & \text{if } \frac{3}{2} < \kappa < 2
\end{cases}
\]

both families satisfy \((-d^2/dx^2)^{1/2}u + Vu = 0\), with \(u \in L^2\) for some \(\kappa\) only.
let \( H = (-\Delta + m^2)^{1/2} + V \)

if \( m > 0 \) large enough, \( d = 1 \) or 3, then there exists \( V(x) = O(1/|x|) \) s.t.

\[
H\varphi = \lambda \varphi, \quad \varphi \in \text{Dom}H,
\]

where

\[
\lambda = \sqrt{1 + m^2} - m \in \text{Spec}_p H
\]

converging in non-relativistic limit to

Neumann-Wigner (1929)

Moses-Tuan (1959)