Bernstein-gamma functions and exponential functionals of Lévy processes

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joint work with P. Patie[2]

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ϕ is a Bernstein function that is $\phi \in \mathcal{B}$ iff

$$\phi(\mathrm{z}) = \mathrm{m} + \delta \mathrm{z} + \int_0^\infty \left(1 - \mathrm{e}^{-\mathrm{zy}}\right) \mu(\mathrm{dy}), \;\; \mathrm{z} \in \{\zeta \in \mathbb{C} : \; \mathrm{Re}(\zeta) \geq 0\},$$

where $\mathrm{m}, \delta \geq 0$; $\int_0^\infty (1 \wedge \mathrm{y})\, \mu(\mathrm{dy}) < \infty$.

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where $\mathrm{m}, \delta \geq 0$; $\int_0^\infty (1 \wedge \mathrm{y})\, \mu(\mathrm{dy}) < \infty$.

The unique solution in the space of Mellin transforms of positive random variables[3] to the recurrent equation

$$\mathrm{f}(\mathrm{z}+1) = \phi(\mathrm{z})\mathrm{f}(\mathrm{z}) \ \text{ on } \mathrm{z} \in \mathbb{C}_{(0,\infty)} = \{\zeta \in \mathbb{C} : \ \mathrm{Re}(\zeta) > 0\} \qquad (0.1)$$

we denote by W_ϕ and call a Bernstein-gamma function.

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1. W_ϕ appear crucially in the spectral studies of the generalized Laguerre semigroups and the positive self-similar Markov processes as instances of non-selfadjoint Markov semigroups.
2. W_ϕ are related to the "phenomenon of self-similarity" the same way the Gamma function appears in the study of some diffusions
3. Amongst W_ϕ are some well-known special functions, e.g. the Barnes-Gamma function, the q-gamma function
4. W_ϕ participates in the computation of the Mellin transform of the exponential functionals of Lévy processes

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Quantities of $\phi \in \mathcal{B}$ that describe the analytic structure of W_ϕ

We use $A_{(a,b)}$ (resp. $M_{(a,b)}$) to denote the holomorphic (resp. meromorphic) functions on the complex strip $\mathbb{C}_{(a,b)} = \{z \in \mathbb{C} : \mathrm{Re}(z) \in (a,b)\}$.

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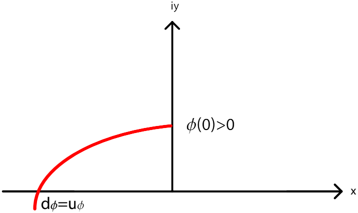
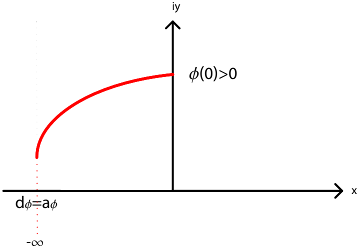
If $\phi \in \mathcal{B}$ then $\phi \in A_{(0,\infty)}$ and set

$$a_\phi = \inf_{u<0} \left\{\phi \in A_{(u,\infty)}\right\} \in [-\infty, 0],$$

$$u_\phi = \sup_{u\leq 0} \{\phi(u) = 0\} \in [-\infty, 0],$$

$$d_\phi = \sup_{u\leq 0} \{\phi(u) = 0 \text{ or } \phi(u) = -\infty\} \in [a_\phi, 0].$$





Theorem

For any $\phi \in \mathcal{B}$

$$W_\phi(z) = \frac{1}{\phi(z)} e^{-\gamma_\phi z} \prod_{k=1}^{\infty} \frac{\phi(k)}{\phi(k+z)} e^{\frac{\phi'(k)}{\phi(k)} z} \in A_{(d_\phi, \infty)} \cap M_{(a_\phi, \infty)}.$$

Morever

- W_ϕ solves $f(z+1) = \phi(z) f(z)$, $f(1) = 1$, on $\mathbb{C}_{(d_\phi, \infty)}$,
- $W_\phi(z+1) = \mathbb{E}\left[Y_\phi^z\right]$ for some positive random variable Y_ϕ.

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When $\phi(z) = z$, $d_\phi = 0$, $a_\phi = -\infty$, $W_\phi(z) = \Gamma(z)$.

Theorem

If $a > 0, b \in \mathbb{R}$, $z = a + ib$. Then

$$|W_\phi(z)| = \frac{\sqrt{\phi(1)}}{\sqrt{\phi(a)\phi(1+a)|\phi(z)|}} e^{G_\phi(a) - A_\phi(z)} \underbrace{e^{-E_\phi(z) - R_\phi(a)}}_{\text{error term}},$$

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where

$$A_\phi(z) = \int_0^{|b|} \arg(\phi(a + iu))\,du,$$

$$G_\phi(z) = \int_1^{1+a} \ln\phi(u)du = G_\phi(a)$$

and $\frac{1}{|b|} A_\phi(a + ib) \in \left[0, \frac{\pi}{2}\right]$.

A Lévy process $\xi = (\xi_s)_{sgeq0}$ is a stochastic process with the properties:

1. for every $t \geq 0$, $(\xi_{t+s} - \xi_t)_{s\geq 0} \stackrel{w}{=} (\xi_s)_{s\geq 0}$, that is ξ has stationary increments
2. for every $t \geq 0$, $(\xi_{t+s} - \xi_t)_{s\geq 0}$ is independent of $(\xi_s)_{t\geq s\geq 0}$, that is ξ has independent increments
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We say that ξ is a killed Lévy process if there exists independent $e_q \sim \text{Exp}(q)$ such that $\xi_s = \infty$, $s \geq e_q$.

A killed Lévy process is determined by its Lévy-Khintchine exponent via

$$\log \mathbb{E}\left[e^{z\xi_1}\right] = \Psi(z) = \frac{\sigma^2}{2}z^2 + bz + \int_{-\infty}^{\infty} \left(e^{zr} - 1 - zr1_{|r|<1}\right) \Pi(dr) - q,$$

where

- $b \in \mathbb{R}$ is linear drift,
- $\sigma^2 \geq 0$ is the variance of the Brownian component,
- Π is a sigma-finite measure describing the structure of the jumps satisfying $\int_{-\infty}^{\infty} \min\left\{x^2, 1\right\} \Pi(dx) < \infty$,
- $q \geq 0$ is the killing rate.

Denote by

$$\overline{\mathcal{N}} = \left\{ \Psi : \Psi(z) = \frac{\sigma^2}{2} z^2 + bz + \int_{-\infty}^{\infty} \left(e^{zr} - 1 - zr 1_{|r|<1} \right) \Pi(dr) - q \right\}$$

the set of all Lévy-Khintchine exponents of possibly killed Lévy processes.

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The random variables

$$I_\Psi = \int_0^{e_q} e^{-\xi_s} ds, \; e_q \sim \mathrm{Exp}(q); e_0 = \infty$$

are called exponential functionals of Lévy processes and

$$I_\Psi < \infty \iff \Psi \in \mathcal{N} = \left\{ \Psi \in \overline{\mathcal{N}} : \, q > 0 \text{ or } \lim_{s \to \infty} \xi_s = \infty \right\} \subsetneq \overline{\mathcal{N}}.$$

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2. Further important studies have been made by Carmona et al.[4] and Maulik et al.[5]
3. Most recent contributions belong to Kuznetsov et al.

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1. Prove that $\mathcal{M}_{I_\Psi}(z+1) = \mathbb{E}\left[I_\Psi^z\right]$ solves

$$f(z+1) = \frac{-z}{\Psi(-z)} f(z) \text{ on } \{z \in i\mathbb{R} \setminus \{0\} : \Psi(-z) \neq 0\} \qquad (0.2)$$

in some meaningful sense.

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The Wiener-Hopf factorization gives that

$$\Psi(-z) = -\phi_+(z)\phi_-(-z), \text{ at least for } z \in i\mathbb{R}$$

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$$f(z+1) = -\frac{-z}{\Psi(-z)} f(z) = \frac{z}{\phi_+(z)} \frac{1}{\phi_-(-z)} f(z), \tag{0.3}$$

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Strategy to solve $f(z+1) = \frac{z}{\phi_+(z)} \frac{1}{\phi_-(-z)} f(z)$

If

$$f_1(z+1) = \frac{z}{\phi_+(z)} \quad f_1(z)$$
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is solved on a common complex domain then $f_1(z)f_2(z)$ is a solution to $f(z+1) = \frac{-z}{\Psi(-z)} f(z)$ on this domain.

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f_1, f_2 are expressed in terms of the Bernstein-gamma functions $W_{\phi_\pm}$ that solve

$$f_\pm(z+1) = \phi_\pm(z) f_\pm(z).$$

Theorem

Let $\Psi \in \overline{\mathcal{N}}$. Then

$$\mathcal{M}_\Psi(z) = \frac{\Gamma(z)}{W_{\phi_+}(z)} W_{\phi_-}(1-z) \in A_{\left(a_{\phi_+} 1_{\left\{d_{\phi_+}=0\right\}},\, 1-d_{\phi_-}\right)} \cap M_{\left(a_{\phi_+},\, 1-a_{\phi_-}\right)}$$

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Also, if $\Psi \in \mathcal{N}$, that is $I_\Psi < \infty$, then

$$\mathbb{E}\left[I_\Psi^{z-1}\right] = \mathcal{M}_{I_\Psi}(z) = \phi_-(0)\mathcal{M}_\Psi(z) = \frac{\Gamma(z)}{W_{\phi_+}(z)} \phi_-(0) W_{\phi_-}(1-z).$$

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To obtain information for the law of I_Ψ via Mellin inversion we need to understand the decay of $\mathcal{M}_{I_\Psi}(z) = \phi_-(0)\mathcal{M}_\Psi(z)$ along lines $a + i\mathbb{R}$.

Theorem

Let $\Psi \in \overline{\mathcal{N}}$. Then exists $N_\Psi \in (0,\infty]$ such that for any $a \in \left(0, 1 - d_{\phi_-}\right)$

$$\lim_{|b|\to\infty} |b|^\eta \left|\mathcal{M}_\Psi\left(a + ib\right)\right| = 0 \iff \eta \in \left(0, N_\Psi\right)$$

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and

$$N_\Psi < \infty \iff \Psi \text{ corresponds to } \xi_t = \delta_+ t + \sum_{j=1}^{N_t} X_j,\ \delta_+ > 0.$$

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$$N_\Psi < \infty \iff \Psi \text{ corresponds to } \xi_t = \delta_+ t + \sum_{j=1}^{N_t} X_j,\ \delta_+ > 0.$$

Therefore, by Mellin inversion

$$p_\Psi(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} x^{-a-ib}\mathcal{M}_{I_\Psi}(a+ib)db \in C_0^{\lfloor N_\Psi\rfloor - 1}\left(\mathbb{R}^+\right),$$

where p_Ψ is the density of I_Ψ.

Smoothness has been investigated by a number of authors in different contexts, including the works of Carmona et al.[6] and Bertoin et. al.[7].

[6] P. Carmona, F. Petit and M. Yor (1997) Exponential functionals and principal values related to Brownian motion. Bibl. Rev.Mat. Iber.

[7] J. Bertoin, A. Lindner and R. Maller (2008) On continuity properties of the law of integrals of Lévy processes. Séminaire de probabilitès, 137–159

The behaviour of $p_\Psi^{(n)}$ at infinity

Theorem

If $\exists\, \theta_\Psi < 0 : \Psi(\theta_\Psi) = 0$ and $\left|\Psi'\left(\theta_\Psi^+\right)\right| < \infty$ (Cramer's condition) then

$$\lim_{x\to\infty} x^{-\theta_\Psi+n+1} p_\Psi^{(n)}(x) = C > 0 \tag{0.4}$$

provided $N_\Psi > n+1$.

[8] A. Kuznetsov (2011) On extrema of stable processes. Ann. Probab. 39(3):1027–1060
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Application: If ξ is a stable Lévy process of index α and $S_1 = \sup_{s\leq 1} \xi_s$ then it is known that $S_1 \stackrel{d}{=} I_\Psi^{-1/\alpha}$ for some $\Psi \in \mathcal{N}$. Then the statements of Kuznetsov[8] and Doney et al.[9] for the asymptotic behaviour at zero of the density of S_1 are immediate corollaries of the theorem above.

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Another result of ours recovers the asymptotic behaviour at infinity of the density of S_1.

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The value of an Asian option is given by

$$C(S_0, T, K, r) = e^{-rT}\mathbb{E}\left[\left(\underbrace{\frac{S_0}{T}\int_0^T e^{-\xi_s}ds}_{\text{average price}} - K\right)^+\right]$$

where $a^+ = \max\{a, 0\}$ and $K \geq 0$ is the strike price.

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Putting $I_\Psi(T) = \int_0^T e^{-\xi_s} ds$, it suffices to evaluate

$$f(T, K) = \mathbb{E}\left[(I_\Psi(T) - K)^+\right].$$

Putting $\Psi_q(z) = \Psi(z) - q \in \mathcal{N}$ then the Laplace transform of $f(T, K)$ in T is given by

$$\int_0^\infty e^{-qT} f(T, K) dT = \int_0^\infty e^{-qT} \mathbb{E}\left[(I_\Psi(T) - K)^+\right] dT$$
$$= \frac{1}{q}\mathbb{E}\left[\left(\int_0^{e_q} e^{-\xi_s} ds - K\right)^+\right] = \frac{1}{q}\mathbb{E}\left[\left(I_{\Psi_q} - K\right)^+\right]$$

[10] D.Hackmann and A. Kuznetsov (2014). Asian options and meromorphic Lévy processes. Finance Stoch. 18(4):825–844

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Then Mellin transform in K yields that

$$\widehat{f}(q,z) = \frac{1}{q}\int_0^\infty K^{z-1}\mathbb{E}\left[\left(I_{\Psi_q} - K\right)^+\right] dK = \frac{1}{q}\frac{\mathcal{M}_{I_{\Psi_q}}(z+2)}{z(z+1)},\ \mathrm{Re}(z) \in (-2,-1)$$

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Attempt to obtain $f(T, K)$ via numerical or analytical inversion in z and q. In special case Hackmann et al.[10] propose an algorithm based on

$$I_{\Psi_q} \stackrel{d}{=} \bigotimes_{k=0}^{\infty} \mathrm{Beta}\left(\alpha(k,q), \beta(k,q)\right).$$

[10] D.Hackmann and A. Kuznetsov (2014). Asian options and meromorphic Lévy processes. Finance Stoch. 18(4):825–844

Theorem

For any $\Psi \in \mathcal{N}$ we have that

$$I_\Psi \stackrel{d}{=} \bigotimes_{k=0}^{\infty} Y_k,$$

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Can we tackle

$$\mathrm{f}(\mathrm{T},\mathrm{K}) = \mathrm{C}\int_{\zeta} \frac{\mathrm{e}^{\zeta \mathrm{T}}}{\zeta} \int_{\mathrm{z}=\mathrm{a}-\mathrm{i}\infty}^{\mathrm{a}+\mathrm{i}\infty} \mathrm{K}^{-\mathrm{z}} \frac{\mathcal{M}_{\mathrm{I}_{\Psi_{\zeta}}}(\mathrm{z}+2)}{\mathrm{z}(\mathrm{z}+1)} \mathrm{dz}\mathrm{d}\zeta ?$$

If $I_\Psi = \int_0^\infty e^{-\xi_s} ds = \infty$, i.e. $\Psi \in \overline{\mathcal{N}} \setminus \mathcal{N}$, then we consider the asymptotic behaviour of the laws of $I_\Psi(t) = \int_0^t e^{-\xi_s} ds$.

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$$\lim_{t\to\infty} \frac{\mathbb{E}\left[I_\Psi^{-a}(t) f(I_\Psi(t))\right]}{\kappa_-\left(\frac{1}{t}\right)} = \int_0^\infty f(x)\vartheta_a(dx),$$

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If $\mathbb{E}[\xi_1] = 0, \mathbb{E}\left[\xi_1^2\right] < \infty$ then $\kappa_-(r) \overset{0}{\sim} Cr^{\frac{1}{2}}$.

Applications:

- A diffusion X in a Lévy random potential has generator $\frac{1}{2}e^{\xi_x}\frac{\partial}{\partial x}e^{-\xi_x}\frac{\partial}{\partial x}$ and

$$\mathbb{P}\left(\sup_{s>0} X_s > t\right) = \int_0^\infty \mathbb{E}\left[\frac{a}{a+I_\Psi(t)}\right]h(a)da = \int_0^\infty \mathbb{E}\left[f_a(I_\Psi(t))\right]h(a)da.$$

If $\mathbb{E}\left[\xi_1^2\right] = 1$ and $\mathbb{E}\left[\xi_1\right] = 0$ then $\mathbb{E}\left[\frac{a}{a+I_\Psi(t)}\right] \sim c(a)t^{-1/2}$.

- For continuous state branching process Z in random environment in a Lévy random environment ξ

$$\mathbb{P}_z\left(Z_t < \infty\right) = \int \mathbb{P}_z\left(Z_t < \infty | \xi = \omega\right)\mathbb{P}\left(\xi \in \omega\right)$$
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Thank you!