PROGRESS ON THE PERTURBATIVE APPROACH TO FRACTIONAL DIFFERENTIAL EQUATION PROBLEMS

R. Spigler*

*Roma Tre University, Rome, Italy

FCPNLO 2016, Bilbao

Works done in collaboration with Moreno Concezzi and Francesca Ferrillo.

It seems, to date, that no *perturbative* approach has been developed (in the mathematical literature), to the field of fractional differential equations (fODE).

This concerning both, *regular* and *singular* perturbations.

Here we start considering *regular* perturbations to linear fODEs, subject to *initial* data.

Prototypes of such problems will concern the celebrated Bagley-Torvik equation, as well as other similar equations:

$$AD^2u + BD^{3/2}u + Cu = 0,$$

$$AD^2u + BD^{1/2}u + Cu = 0,$$

to which *two* initial data, $u(0)$ and $u'(0)$, are prescribed.

We consider the three cases for each of them, where one of the three parameters, *A*, *B*, or *C*, may vanish.

Therefore, we have for the Bagley-Torvik equation the three forms:

$$D^2 u + A D^{3/2} u + \varepsilon u = 0, \tag{1}$$

$$\varepsilon D^2 u + A D^{3/2} u + B u = 0, \tag{2}$$

$$D^2 u + \varepsilon D^{3/2} u + B u = 0, \tag{3}$$

where $\varepsilon > 0$ is a small parameter, and similarly for the model equation where $D^{3/2}$ is replaced by $D^{1/2}$:

$$D^2 u + A D^{1/2} u + \varepsilon u = 0, \tag{4}$$

$$\varepsilon\, D^2 u + A D^{1/2} u + B u = 0, \tag{5}$$

$$D^2 u + \varepsilon\, D^{1/2} u + B u = 0. \tag{6}$$

The first problem is definitely a *regular* perturbation for both, the Bagley-Torvik and the other fODE: the initial value problems for the corresponding degenerate (also called reduced) equations are well posed imposing both initial data, for both fODEs.

The third problem is expected to be also a *regular* perturbation problem, for both fODEs, since both reduced fODEs (which become integer order ODEs) are of the second order. Therefore, the associated Cauchy problems are still well posed in both cases, with two initial values.

The second problem, formally analogue to the classical [integer order derivative] analogue, seems to be of the *singular* perturbation type.

Things are instead very different when $D^{3/2}$ (Bagley-Torvik) is replaced by $D^{1/2}$.

In fact, the two reduced fODEs are now

$$AD^{3/2}u + Bu = 0, \tag{7}$$

$$AD^{1/2}u + Bu = 0, \tag{8}$$

and the point is that the Cauchy problem for equation (7) is still *well posed with two* initial data, while equation (8) requires *only one* initial value.

This means that we can still expect a "regular perturbation behavior" for equation (7), and a "singular perturbation behavior" in case of equation (8).

The Cauchy problem for an fODE like

$$D^a u + Cu = 0,$$

in fact is well posed prescribing *two data*, as long as $1 < a \leq 2$, but it is well posed prescribing *only one* initial value when $0 < a \leq 1$.

Setting $\tau := t/\delta$, where δ is a suitable function of ε, in $\varepsilon D^2 u + BD^b u + Cu = 0$, and changing variable of integration, we obtain

$$\frac{\varepsilon}{\delta^2}\,\frac{d^2U(\tau)}{d\tau^2} + \frac{\delta^{-b}B}{\Gamma(n-b)}\int_0^\tau \frac{\frac{d^nU(\sigma)}{d\sigma^n}}{(\tau-\sigma)^{b+1-n}}\,ds + CU(\tau) = 0,$$

satisfied by $U(t) := u(\delta\, t)$.

The idea of the "dominating balance" suggests to choose δ such that

$$\frac{\varepsilon}{\delta^2} = \delta^{-b},$$

that is $\delta^{2-b} = \varepsilon$, or

$$\delta = \varepsilon^{1/(2-b)}.$$

We are interested in $0 < b < 2$. Assuming $0 < \varepsilon < 1$, the "initial layer" implied by the previous relation will be "ticker" than in the classical case ($b = 1$) for $0 < b < 1$, and the corresponding reduced problem cannot be solved imposing *both* ICs.

When, instead, $1 < b < 2$, the initial layer becomes thinner, but it may be unnecessary resorting to initial layer techniques, since the corresponding reduced problem can now be solved prescribing both ICs of the original problem. When b is close to 2, clearly, there is *no need* of any initial layer.

Some figures illustrate better our observations.

On the next three figures, we compare numerically the solution to the B-T fODE $D^2u + D^{3/2}u + \varepsilon u = 0$ for three values of ε, $\varepsilon = 0.1, 0.3, 0.5$, with its approximation $v(t;\varepsilon) := u_0(t) + \varepsilon u_1(t) + \varepsilon^2 u_2(t)$ (obtained upon a formal expansion in power series of ε).

Figure:

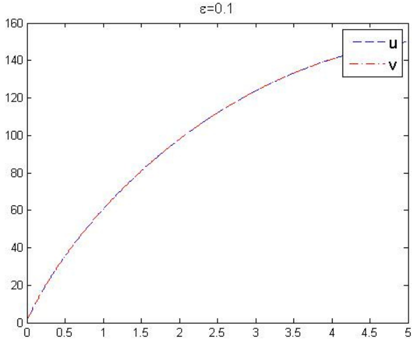


Figure:

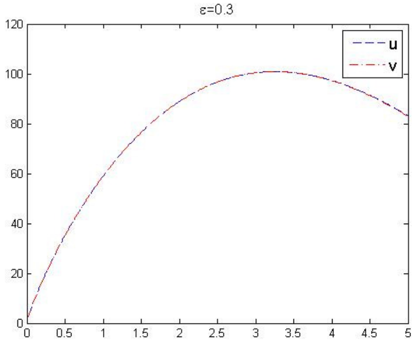
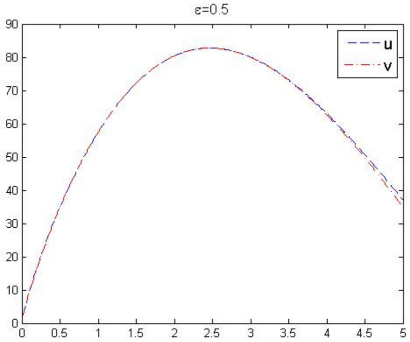


Figure:



On the next three figures, we compare numerically the solution to the fODE $\varepsilon D^2 u + D^{3/2} u + u = 0$ for three values of ε, $\varepsilon = 0.1, 0.3, 0.5$ with its approximation $v(t;\varepsilon) := u_0(t) + \varepsilon u_1(t) + \varepsilon^2 u_2(t)$ (obtained upon a formal expansion in power series of ε).

Figure:

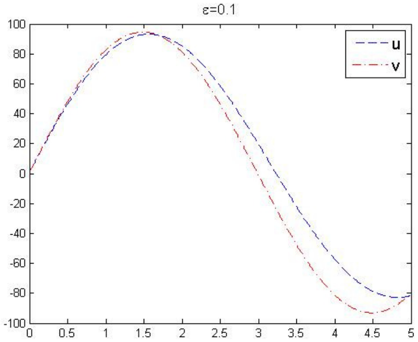


Figure:

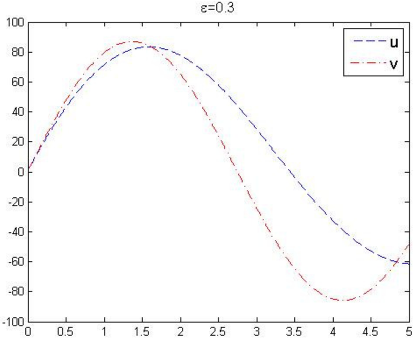
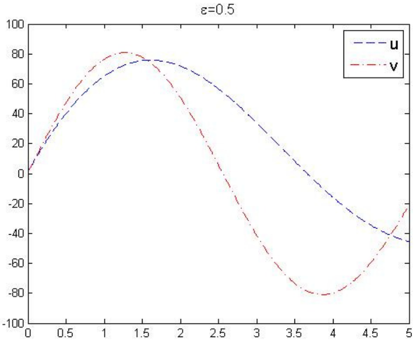


Figure:



On the next three figures, we compare numerically the solution to the fODE $D^2 u + \varepsilon D^{1/2} u + u = 0$ for three values of ε, $\varepsilon = 0.1, 0.3, 0.5$ with its approximation $v(t;\varepsilon) := u_0(t) + \varepsilon u_1(t) + \varepsilon^2 u_2(t)$ (obtained upon a formal expansion in power series of ε).

Figure:

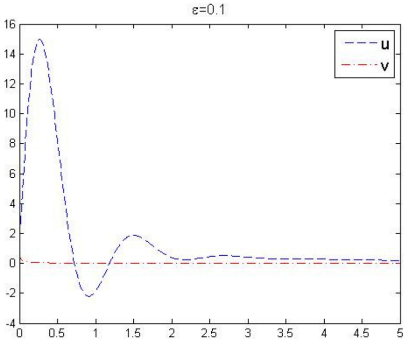


Figure:

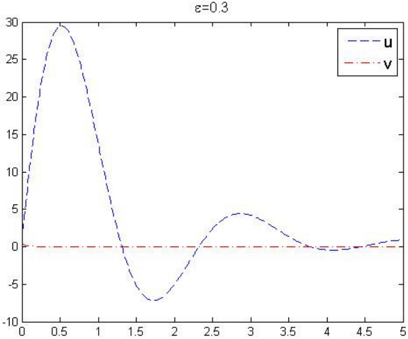
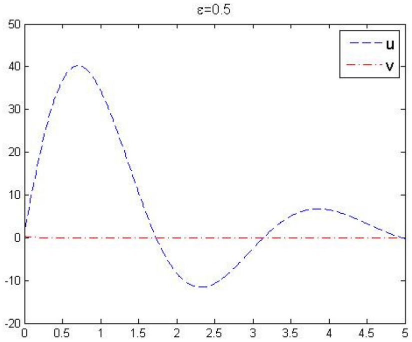


Figure:



On the next three figures, we compare numerically the solution to the fODE $\varepsilon D^2 u + D^{1/2} u + u = 0$ for three values of ε, $\varepsilon = 0.1, 0.3, 0.5$ with its approximation $v(t; \varepsilon) := u_0(t) + \varepsilon u_1(t) + \varepsilon^2 u_2(t)$ (obtained upon a formal expansion in power series of ε).

Figure:

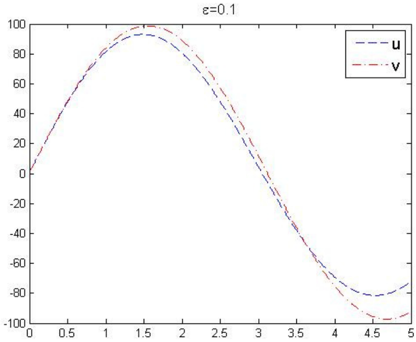


Figure:

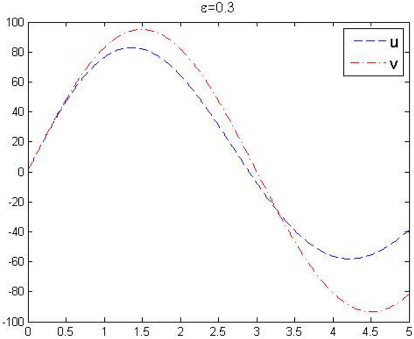
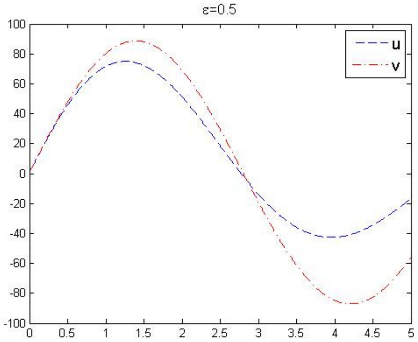


Figure:



A bit of theory

The idea is to transfom the given fODE problem into an equivalent *integral equation*.

An *outer* expansion in powers of ε can be done, then an *inner* expansion after the aforementioned *stretching* of the time variable.

Manipulations are more involved compared to the analogous classical (integer derivative) case. This especialy with *variable coefficients*.

Instead of performing the typical *matching*, one can combine such expansion procedures with a numerical treatment.

Let first list **some facts** concerning linear homogenous fODEs:

1. We *cannot* say that the full set of solutions to a given fODE is a finite-dimensional vector space.

2. Linear combinations of two or more solutions are still solutions, hence the full set of solutions *contains* a vector space of them.

3. Therefore, we should merely rest on the existence and uniquenss of solutions to the Cauchy problem.

4. if the [classical] Wronskian of two or more solutions is nonzero at some point, then such solutions are linearly independent.

5. Given the singularly perturbed problem

$$\varepsilon\, D_t^{\alpha} u + B\, D_t^{\beta} u + C\, u = 0,$$

we can construct a solution of the form

$$u_1(t,\varepsilon) \sim \sum_{j=0}^{\infty} a_j(t)\,\varepsilon^j, \quad \varepsilon \to 0^+,$$

and a solution to the *stretched* equation

$$D_\tau^{\alpha} u + B\, D_\tau^{\beta} u + C\,\eta\, u = 0,$$

where $\tau := t/\delta(\varepsilon)$, $\delta(\varepsilon) := \varepsilon^{1/(\alpha-\beta)}$, $\eta = \eta(\varepsilon) := \varepsilon^{\beta/(\alpha-\beta)}$, of the form

$$u_2(\tau,\eta) \sim \sum_{j=0}^{\infty} \tilde{a}_j(\tau)\,\eta^j, \;\; \varepsilon \to 0^+.$$

Then, we have constructed two linearly independent solutions, u_1, and u_2 (check their Wronskian).

Note: a second solution of the form

$$\left(\sum_{j=0}^{\infty} a_j(t)\, \varepsilon^j \right) \exp\left\{ -\frac{B\,t}{\varepsilon} \right\},$$

as it can be done in the classical case ($\alpha = 2$, $\beta = 1$) does *not* exist now.

One could expect this since the typical scale associated to fractional differential operators follows a power rather than an exponential law.

Indeed, the second solution above follows a certain stretched scale.

Consider the regularly perturbed equation

$$D_t^{\alpha} u + B\, D_t^{\beta} u + \varepsilon\, C\, u = 0,$$

subjet to initial data. In the special case $1 < \alpha \leq 2$, $0 < \beta \leq 1$, with $\alpha - \beta = 1$, we can write it as

$$D^{\beta} \left(u' + B\, u \right) + \varepsilon C u = 0,$$

and then, applying the operator J^{β} to both sides,

$$u' + B\, u = u'(0) + B\, u(0) - \varepsilon\, C\, J^{\beta} u.$$

Expanding u in powers of ε with coefficients $a_j(t)$, we obtain the equations

$$a_0' + B\, a_0 = u'(0) + B\, u(0),$$

and

$$a_j' + B\, a_j = -\frac{C}{\Gamma(\beta)} \int_0^t \frac{a_{j-1}(s)}{(t-s)^{1-\beta}}\, ds + u'(0) + B\, u(0), \quad j = 1, 2, ...$$

If $0 < \beta \leq 1 < \alpha \leq 2$, things are similar, though a bit more elaborated (especially if the coefficients are allowed to vary with t.
But we can construct also a solution like u_1 above also for the singularly perturbed equation. With constant coefficients, we obtain successively

$$\varepsilon D_t^{\alpha} u + B\, D_t^{\beta} u + C\, u = 0,$$

$$B\, J^{\beta} D^{\beta} u + C\, J^{\beta} u = -\varepsilon\, J^{\beta} D^{\alpha} u,$$

$$B\, u = B\, u_0 - C\, J^{\beta} u - \varepsilon\, J^{\beta-\alpha} J^{\alpha} D^{\alpha} u,$$

$$u = u_0 - \frac{C}{B}\, J^{\beta} u - \frac{\varepsilon}{B}\, J^{\beta-\alpha}[u - u(0) - u'(0)t].$$

Expanding u in powers of ε yields,

$$a_0 = u_0 - \frac{C}{B} J^{\beta} a_0,$$

$$a_j = -\frac{C}{B} J^{\beta} a_j + \frac{1}{B} J^{\beta-\alpha}[a_{j-1} - u(0) - u'(0)t], \quad j = 1, 2, ...$$

i.e., Volterra integral equations with weakly singular kernels.

One can then show existence of all a_j's *and* their boundedness. Hence the analytical validity of the power series expansion.

A treatment very similar can be done for the *stretched* equation, in the variable τ, (but) expanding in powers of $\eta := \varepsilon^{\beta/(\alpha-\beta)}$ (see above).

Note that $\eta = \varepsilon$ when $\alpha = 2$, $\beta = 1$.

The “fractional Cattaneo” model of heat conduction

A long-standing debate concerns the validity of the Fourier law in thermodynamics.

To go beyond the *infinite speed* of propagation predicted by the heat equation, the hyperbolic Cattaneo (or Maxwell - Cattaneo - Vernotte) model equation was proposed.

Since the solution to the latter may be affected by oscillations and perhaps negative values of the absolute temperature, Fabrizio et al. recently proposed a *fractional* Cattaneo model equation.

This, however, is not only characterized [again] by an infinite speed of heat transfer, but may still be affected by oscillations and negative values.

This depends:

(1) on the *initial profile* of the tempearture,

(2) on being the *fractional order*, α, close or not to 2, and

(3) on being the *relaxation parameter*, τ, very small or not.

We solved numerically both, the Cattaneo

$$\tau\, u_{tt} + u_t = D\, u_{xx}$$

and the “fractional Cattaneo”

$$\tau\, D_t^{\alpha} u + u_t = D\, u_{xx}$$

model equations, along with the basic “heat equation”, $u_t = D\, u_{xx}$, under the same initial and boundary data.

Typically, the relaxation parameter, τ, is very small (ranging from picoseconds or less, but even up to minutes), while the fractional order α is less but close to 2.

We found that, when τ is sufficiently small, no oscillations (and no negative values of the absolute tempertaure) are observed in the Cattaneo hyperbolic model. The “threshold” for τ depends on the specific initial profile prescribed for the temperature.

On the other hand, when α is sufficiently close to 2, oscillations are exhibited also by the fractional Cattaneo model.

Consider the problem

$$\begin{cases} \varepsilon^2 u_{tt} + u_t = u_{xx}, \quad (x,t) \in \Omega \\ u(x,0) = \varphi(x) \quad 0 \le x \le L \\ u_t(x,0) = \psi(x) \quad 0 \le x \le L \\ u(0,t) = u(L,t) = 0 \quad 0 < t \le T, \end{cases}$$

where $\Omega := \{(x,t) \in \mathbf{R}^2 : 0 < x < L, 0 < t \le T\}$.

Upon the substitution

$$\begin{cases} t = s\delta(\varepsilon) \\ x = \xi\alpha(\varepsilon) \end{cases},$$

we obtain

$$\frac{\varepsilon^2}{\delta^2} u_{ss} + \frac{1}{\delta} u_s = \frac{D}{\alpha^2} u_{\xi\xi},$$

and, in the spirit of the so-called “dominating balance”, we choose $\delta(\varepsilon)$ such that the coefficients of u_{ss} and of u_s are equal, i.e.,

$$\delta(\varepsilon) = \varepsilon^2.$$

Therefore, we have

$$u_{ss} + u_s = \varepsilon^2 \frac{D}{\alpha^2} u_{\xi\xi},$$

and choosing

$$\alpha = \sqrt{D}\varepsilon,$$

we finally obtain

$$u_{ss} + u_s = u_{\xi\xi}.$$

Correspondingly, the original problem is transformed into the equivalent problem

$$\begin{cases} u_{ss} + u_s = u_{\xi\xi} \\ u(\xi, 0) = \varphi(\varepsilon\sqrt{D}\xi) \\ u_t(\xi, 0) = \psi(\varepsilon\sqrt{D}\xi) \\ u(0, s) = u\left(\frac{L}{\varepsilon\sqrt{D}}, s\right) = 0, \end{cases}$$

where

$$(\xi, s) \in \Omega' = \left\{(\xi, s) \in \mathbf{R}^2 : 0 \leq \alpha\xi \leq L, 0 \leq \varepsilon^2 s \leq T\right\}$$

$$= \left\{(\xi, s) \in \mathbf{R}^2 : 0 \leq \xi \leq \frac{L}{\varepsilon\sqrt{D}}, 0 \leq s \leq \frac{T}{\varepsilon^2}\right\}.$$

We are now able to solve numerically this problem. The equation is no longer singularly perturbed, but the price we pay for this is that the domain Ω' is now much larger compared to the original one, Ω.

Since the term $\varepsilon^2 u_{tt}$ should be negligible for $t > 0$, we can solve such a problem numerically only for times close to 0, say for $0 < t \leq t_0$, for some $t_0 > 0$, and then merely solve *the heat equation* for $t_0 \leq t \leq T$, starting with the computed initial value, $u(x, t_0)$, i.e.,

$$w_t = w_{xx}$$

with the initial condition

$$w(x, 0) = u\left(\xi, s = \varepsilon^{-2} t_0\right)\Big|_{\xi=\xi(x)} .$$

When $\tau = 10^{-4}$, the solution to the Cattaneo model was obtained by means of the *asymptotic-numerical* method, which involves a boundary (actually an “initial”) layer.

Figure: Solution of the Cattaneo equation with $\tau = 10^{-4}$

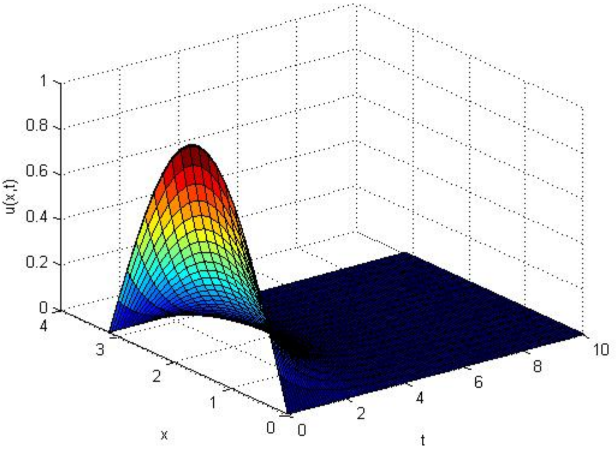
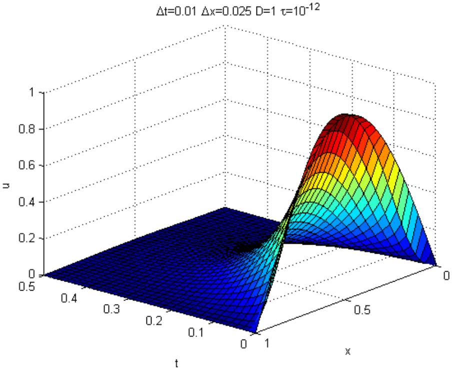


Figure: Solution of the Cattaneo equation with $\tau = 10^{-4}$



The same procedure can be used (and was used) for the *fractional* Cattaneo problem.
Results are similar.

THANK YOU FOR YOUR ATTENTION!