Recent result on porous medium equations with nonlocal pressure

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Preliminaries

- Physical Model: a continuum (fluid or population) with density distribution $u(x, t) \geq 0$ and velocity field $v(x, t)$.
- Continuity equation $u_t = \nabla (u \cdot v)$.
- Darcy’s law: $v$ derives from a potential (fluids in porous media): $v = -\nabla P$.
- The relation between $P$ and $u$: for gasses in porous media, Leibenzon and Muskat (1930) derived a relation in the form of the state law

$$P = f(u),$$

where $f$ is a nondecreasing scalar function. $f(u)$ is linear when the flow is isothermal and is a higher power of $u$ when the flow is adiabatic, i.e. $f(u) = cu^{m-1}$ with $c > 0$ and $m > 1$.

- The linear dependence $f(u) = cu \rightarrow$ Boussinesq (1903) modelling water infiltration in an almost horizontal soil layer $\rightarrow u_t = c\Delta u^2$.
- The model $u_t = (c/m) \Delta u^m$.
- The Porous Medium Equation $u_t = \Delta u^m$. 
Porous Medium Equation / Fast Diffusion Equation

PME/FDE \[ u_t(x, t) = \Delta u^m(x, t) \quad x \in \mathbb{R}^N, \ t > 0 \]

Self Similar solutions: \[ U(x, t) = t^{-\frac{N}{N(m-1)+2}} F(|x| t^{-\frac{1}{N(m-1)+2}}) \]

Slow Diffusion
- \( m > 1 \), Profile \( \sim (R^2 - |y|^2)^{1/(m-1)} \)

Fast Diffusion
- \( m < 1 \), Profile \( \sim (R^2 + |y|^2)^{-1/(1-m)} \)
Definition of the Fractional Laplacian

Several equivalent definitions of the nonlocal operator \((-\Delta)^s\) (Laplacian of order 2s):

1. **Fourier transform** \(\widehat{(-\Delta)^s g(\xi)} = (2\pi|\xi|)^{2s}\hat{g}(\xi).\)

   [can be used for positive and negative values of \(s\)]

2. **Singular Kernel** \((-\Delta)^s g(x) = c_{N,s} \text{ P.V.} \int_{\mathbb{R}^N} \frac{g(x) - g(z)}{|x - z|^{N+2s}} dz\)

   [can be used for \(0 < s < 1\), where \(c_{N,s}\) is a normalization constant.]

3. **Heat semigroup**

   \[(-\Delta)^s g(x) = \frac{1}{\Gamma(-s)} \int_0^\infty \left( e^{t\Delta} g(x) - g(x) \right) \frac{dt}{t^{1+s}}.\]

4. **Generator of the 2s-stable Levy process:**

   \((-\Delta)^s g(x) = \lim_{h \to 0} \frac{1}{h} \mathbb{E}[g(x) - g(x + X_h)].\)
Porous medium with nonlocal pressure

- The pressure $p = (\Delta)^{-s}(u)$, $0 < s < 1$:

$$(-\Delta)^{-s}(u) = K_s * u = \int_{\mathbb{R}^N} \frac{u(y)}{|x - y|^{N-2s}} \, dy, \quad K_s(x) = C_{N,s}|x|^{-(N-2s)}.$$

- The model:

$$\partial_t u = \nabla \cdot (u \nabla p), \quad p = (-\Delta)^{-s}(u).$$

**Difficulties:** no maximum principle, no uniqueness.

**References:**

- Existence and finite speed of propagation: Caffarelli and Vázquez, ARMA 2011.
- Asymptotic behavior: Caffarelli and Vázquez, DCDS 2011.
- Regularity: Caffarelli, Soria and Vázquez, JEMS 2013.
- Exponential convergence towards stationary states in 1D: Carrillo, Huang, Santos and Vázquez, JDE 2015.
Porous Medium with nonlocal pressure

\[ \partial_t u = \nabla \cdot (u^{m-1} \nabla p), \quad p = (-\Delta)^{-s}(u). \quad (P) \]

for \( x \in \mathbb{R}^N, \ t > 0, \ N \geq 1. \) We take \( m > 1, \ 0 < s < 1 \) and \( u(x, t) \geq 0. \)

The initial data \( u(x, 0) = u_0(x) \) for \( x \in \mathbb{R}^N, \ u_0 : \mathbb{R}^N \to [0, \infty) \) is assumed to be a bounded integrable function.

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Infinite vs. finite speed of propagation

Figure: $m = 1.5, s = 0.25$

Figure: $m = 2, s = 0.25$

Figure: $m = 1.5, s = 0.75$

Figure: $m = 2, s = 0.75$
New idea: existence for all \( m > 1 \) when \( u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \)

We write the problem in the form

\[
  u_t = \nabla \cdot (u^{m-1} \nabla (-\Delta)^{-1} (-\Delta)^{1-s} u)
\]

Then, formally:

\[
  \int_{\mathbb{R}^N} u_0^p(x) \, dx - \int_{\mathbb{R}^N} u(x, t)^p \, dx = C_1 \int_0^t \int_{\mathbb{R}^N} u^{m+p-2} (-\Delta)^{1-s} u \, dx \, dt \\
  \geq C_2 \int_0^t \int_{\mathbb{R}^N} \left| (-\Delta)^{\frac{1-s}{2}} u^{\frac{m+p-2}{2}} \right|^2 \, dx \, dt
\]

by the Stroock-Varoupolos Inequality.

Here \( C_1 = (p - 1)/(m + p - 2) \).

\[\text{D. Stan, F. del Teso and J.L. Vázquez,} \text{ Existence of weak solutions for a general porous medium equation with nonlocal pressure, arXiv:1609.05139.}\]
New approximation method

\[ u_t = \nabla \cdot (u^{m-1} \nabla (-\Delta)^{-1} (-\Delta)^{1-s} u) \]  \hspace{1cm} (P)

Then we approximate the operator \( \mathcal{L} = (-\Delta)^{1-s} \) by

\[ \mathcal{L}^{1-s}_\epsilon (u)(x) = C_{N,1-s} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{(|x-y|^2 + \epsilon^2)^{\frac{N+2-2s}{2}}} \, dy. \]

- **Convergence:** \( \mathcal{L}^{1-s}_\epsilon [u] \to (-\Delta)^{1-s} u \) pointwise in \( \mathbb{R}^N \) as \( \epsilon \to 0 \)

- **Generalized Stroock-Varopoulos Inequality for \( \mathcal{L}^s_\epsilon \):** Let \( u \in H^s_\epsilon (\mathbb{R}^N) \). Let \( \psi : \mathbb{R} \to \mathbb{R} \) such that \( \psi \in C^1(\mathbb{R}) \) and \( \psi' \geq 0 \). Then

\[ \int_{\mathbb{R}^N} \psi(u) \mathcal{L}^s_\epsilon [u] \, dx \geq \int_{\mathbb{R}^N} \left[ (\mathcal{L}^s_\epsilon)^{\frac{1}{2}} [\psi(u)] \right]^2 \, dx, \]

where \( \psi' = (\Psi')^2 \).
Approximating problem

We consider the approximating problem \((P_{\epsilon \delta \mu R})\)

\[
\begin{aligned}
(U_1)_t &= \delta \Delta U_1 + \nabla \cdot ((U_1 + \mu)^{m-1} \nabla (-\Delta)^{-1} \mathcal{L}_{\epsilon}^{1-s}[U_1]) \\
U_1(x, 0) &= \tilde{u}_0(x) \\
U_1(x, t) &= 0
\end{aligned}
\]

for \((x, t) \in B_R \times (0, T)\),
for \(x \in B_R\),
for \(x \in \partial B_R\), \(t \in (0, T)\),

with parameters \(\epsilon, \delta, \mu, R > 0\).

- **Existence of solutions of** \((P_{\epsilon \delta \mu R}) \rightarrow \) fixed points of the following map given by the Duhamel’s formula

\[
\mathcal{T}(v)(x, t) = e^{\delta t \Delta} u_0(x) + \int_0^t \nabla e^{\delta (t-\tau) \Delta} \cdot G(v)(x, \tau) d\tau,
\]

where \(G(v) = (v + \mu)^{m-1} \nabla (-\Delta)^{-1} \mathcal{L}_{\epsilon}^{s}[v]\) and \(e^{t \Delta}\) is the Heat Semigroup.

- **Existence of solutions of** \((P)\)

\[
(P_{\epsilon \delta \mu R})_{\epsilon \rightarrow 0} \rightarrow (P_{\delta \mu R})_{R \rightarrow \infty} \rightarrow (P_{\delta \mu})_{\mu \rightarrow 0} \rightarrow (P_{\delta})_{\epsilon \rightarrow 0} \rightarrow (P).
\]
Existence of weak solutions for $m > 1$

**Theorem.** Let $1 < m < \infty$, $N \geq 1$, and let $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and nonnegative.

Main results:

- **Existence of a weak solution** $u \geq 0$ of Problem (PMFP) with initial data $u_0$.

- **Conservation of mass:** For all $0 < t < T$ we have
  \[ \int_{\mathbb{R}^N} u(x, t) \, dx = \int_{\mathbb{R}^N} u_0(x) \, dx. \]

- **$L^\infty$ estimate:** $\| u(\cdot, t) \|_\infty \leq \| u_0 \|_\infty$, $\forall 0 < t < T$.

- **$L^p$ energy estimate:** For all $1 < p < \infty$ and $0 < t < T$ we have
  \[ \int_{\mathbb{R}^N} u^p(x, t) \, dx + \frac{4p(p-1)}{(m+p-1)^2} \int_0^t \int_{\mathbb{R}^N} \left| (-\Delta)^{\frac{1-s}{2}} u \frac{m+p-1}{2} \right|^2 \, dx \, dt \leq \int_{\mathbb{R}^N} u_0^p(x) \, dx. \]

- **Second energy estimate:** For all $0 < t < T$ we have
  \[ \frac{1}{2} \int_{\mathbb{R}^N} \left| (-\Delta)^{-\frac{s}{2}} u(t) \right|^2 \, dx + \int_0^t \int_{\mathbb{R}^N} u^{m-1} \left| \nabla (-\Delta)^{-s} u(t) \right|^2 \, dx \, dt \leq \frac{1}{2} \int_{\mathbb{R}^N} \left| (-\Delta)^{-\frac{s}{2}} u_0 \right|^2 \, dx. \]
Smoothing effect

**Theorem**

Let $u \geq 0$ be a weak solution of Problem (PMFP) with $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $u_0 \geq 0$, as constructed before. Then

$$\|u(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} \leq C_{N,s,m,p} t^{-\gamma_p} \|u_0\|_{L^p(\mathbb{R}^N)}^{\delta_p}$$

for all $t > 0$,

where $\gamma_p = \frac{N}{(m-1)N+2p(1-s)}$, $\delta_p = \frac{2p(1-s)}{(m-1)N+2p(1-s)}$.

$\Rightarrow$ Existence of weak solutions for only $u_0 \in M^+(\mathbb{R}^N)$.

$\Rightarrow$ Existence of weak solutions for only $u_0 \in L^1(\mathbb{R}^N)$. 
Existence for measure data

Let $1 < m < \infty$, $N \geq 1$ and $\mu \in \mathcal{M}^+(\mathbb{R}^N)$. Then there exists a weak solution $u \geq 0$ of Problem (P) s.t. the smoothing effect holds for $p = 1$ in the following sense:

$$\|u(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} \leq C_{N,s,m} t^{-\gamma} \mu(\mathbb{R}^N)^{\delta} \quad \text{for all} \quad t > 0,$$

where $\gamma = \frac{N}{(m-1)N + 2(1-s)}$, $\delta = \frac{2(1-s)}{(m-1)N + 2(1-s)}$. Moreover,

- **Regularity:**
  $$u \in L^\infty((\tau, \infty) : L^1(\mathbb{R}^N)) \cap L^\infty(\mathbb{R}^N \times (\tau, \infty)) \cap L^\infty((0, \infty) : \mathcal{M}^+(\mathbb{R}^N)) \quad \text{for all} \quad \tau > 0$$

- **Conservation of mass:** For all $0 < t < T$ we have
  $$\int_{\mathbb{R}^N} u(x, t) \, dx = \int_{\mathbb{R}^N} d\mu(x).$$

- **$L^p$ energy estimate:** For all $1 < p < \infty$ and $0 < \tau < t < T$ we have
  $$\int_{\mathbb{R}^N} u^p(x, t) \, dx + \frac{4p(p-1)}{(m+p-1)^2} \int_{\tau}^{t} \int_{\mathbb{R}^N} \left| (-\Delta)^{\frac{1-s}{2}} \frac{u^{m+p-1}}{2} \right|^2 \, dx \, dt \leq \int_{\mathbb{R}^N} u^p(x, \tau) \, dx.$$

- **Second energy estimate:** For all $0 < \tau < t < T$ we have
  $$\frac{1}{2} \int_{\mathbb{R}^N} \left| (-\Delta)^{-\frac{s}{2}} u(t) \right|^2 \, dx + \int_{\tau}^{t} \int_{\mathbb{R}^N} u^{m-1} \left| \nabla (-\Delta)^{-s} u(t) \right|^2 \, dx \, dt \leq \frac{1}{2} \int_{\mathbb{R}^N} \left| (-\Delta)^{-\frac{s}{2}} u(\tau) \right|^2 \, dx.$$
Finite speed of propagation for $m \geq 2$

**Theorem**

Assume that $u_0$ has compact support and $u(x, t)$ is bounded for all $x, t$. Then $u(\cdot, t)$ is compactly supported for all $t > 0$.

If $0 < s < 1/2$ and

$$u_0(x) \leq U_0(x) := a(|x| - b)^2,$$

then there is a constant $C$ large enough s.t.

$$u(x, t) \leq U(x, t) := a(Ct - (|x| - b))^2.$$

For $1/2 \leq s < 2 \Rightarrow C = C(t)$ is an increasing function of $t$.

**Consequence: Free Boundaries!**

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**Figure:** $u_0 \leq U_0$

**Figure:** $u(x, t) \leq U(x, t)$
Theorem. Let \( m \in (1, 2) \), \( s \in (0, 1) \) and \( N = 1 \). Let \( u \) be the solution of Problem (PMFP) with initial data \( u_0 \geq 0 \) radially symmetric and monotone decreasing in \( |x| \). Then \( u(x, t) > 0 \) for all \( t > 0, x \in \mathbb{R} \).

Idea of the proof: Prove that \( v(x, t) = \int_{-\infty}^{x} u(y, t) dy > 0 \) for \( t > 0, x \in \mathbb{R} \).

The integrated problem

\[
\partial_t v = -|v_x|^{m-1}(\Delta)^{1-s} v \quad (IP)
\]

The initial data is given by \( v_0(x) = \int_{-\infty}^{x} u_0(y) dy \).

Initial data \( v_0(x) \) satisfies:
\( v_0(x) = 0 \) for \( x < -\eta \),
\( v_0(x) = M \) for \( x > \eta \),
\( v'_0(x) \geq 0 \) for \( x \in (-\eta, \eta) \).

Figure: Typical initial data for models (P) and (IP).
Fractional Porous Medium Equation

\[ U_t + (-\Delta)^s U^m = 0 \quad \text{(FPME)} \]

Porous Medium with Fractional Pressure

\[ V_t = \nabla \cdot (V^{\tilde{m}-1} \nabla (-\Delta)^{-\tilde{s}} V) \quad \text{(PMFP)} \]

Self Similar Solutions

\[ U(x, t) = t^{-\alpha_1} F_1(t^{-\beta_1} x) \quad \text{with} \]
\[ \alpha_1 = N\beta_1, \quad \beta_1 = \frac{1}{N(m-1)+2s}, \]
\[ (-\Delta)^s F_1^m = \beta_1 \nabla \cdot (y F_1). \quad \text{(P1)} \]

\[ V(x, t) = t^{-\alpha_2} F_2(t^{-\beta_2} x) \quad \text{with} \]
\[ \alpha_2 = N\beta_2, \quad \beta_2 = \frac{1}{N(\tilde{m}-1)+2-2s}, \]
\[ \nabla \cdot (F_2^{\tilde{m}-1} \nabla (-\Delta)^{-\tilde{s}} F_2) = -\beta_2 \nabla \cdot (y F_2). \quad \text{(P2)} \]

J. L. Vázquez. *Barenblatt solutions and asymptotic behaviour for a nonlinear fractional heat equation of porous medium type.* JEMS 2014.


**Theorem. Transformation of self similar solutions**

If \( m > N/(N + 2s) \), \( s \in (0, 1) \) and \( F_1 \) is a solution to the profile equation (P1) then

\[ F_2(x) = (\beta_1/\beta_2)^{1-m} (F_1(x))^m \]

is a solution to the profile equation (P2) if we put \( \tilde{m} = (2m - 1)/m \) and \( \tilde{s} = 1 - s \).
FPME: The profile $F_1(y)$ is a smooth and positive function in $\mathbb{R}^N$, it is a radial function, it is monotone decreasing in $r = |y|$ and has a definite decay rate as $|y| \to \infty$, that depends on $m$. For $m > N/(N + 2s)$, $F_1(y) \sim |y|^{-(N+2s)}$ for large $|y|$. [Vázquez, JEMS 2014]

Consequence:

PMFP: $F_2 > 0$ and $F_2(x) \sim C|x|^{-(N+2-2\bar{s})/(2-\bar{m})}$ if $\bar{m} \in ((N - 2 + 2\bar{s})/N, 2)$.

\[ \implies \text{Infinite Propagation for Self-Similar Solutions of the PMFP in } \mathbb{R}^N, \quad N \geq 1, \quad m < 2. \]

Similar results are proved for smaller values of $m$. 
Explicit Solutions

Fractional Porous Medium Equation

\[ U_t + (-\Delta)^s U^m = 0 \]  \hspace{1cm} (FPME)

\[ \forall s \in (0, 1) \rightarrow m = \frac{N+2-2s}{N+2s} > m_c = \frac{N-2s}{N} \]

\[ u(x, t) = at^{-N\beta_1} \left( R^2 + |xt^{-\beta_1}|^2 \right)^{-(N+2s)/2} \]


Porous Medium with Fractional Pressure

(II) \[ V_t = \nabla \cdot (V \nabla (-\Delta)^{-\tilde{s}} V^\tilde{m}-1), \tilde{m} > 1 \]

Biler, Imbert, Karch [2013]:

\[ v(x, t) = at^{-N\beta_2} \left( R^2 - |xt^{-\beta_2}|^2 \right)^{(1-\tilde{s})/(\tilde{m}-1)} \]

Huang [2014]: \[ \forall \tilde{s} \in (0, 1) \rightarrow \tilde{m} = \frac{N+6s-2}{N+2s} < 2 \]

\[ v(x, t) = at^{-N\beta_2} \left( R^2 + |xt^{-\beta_2}|^2 \right)^{-(N+2\tilde{s})/2} \]
Explicit Solutions

Fractional Porous Medium Equation

\[
U_t + (-\Delta)^s U^m = 0 
\]
(FPME)

\[
\forall s \in (0, 1) \rightarrow m = \frac{N+2-2s}{N+2s} > m_c = \frac{N-2s}{N}
\]

\[
u(x, t) = at^{-N\beta_1} \left(R^2 + |xt^{-\beta_1}|^2\right)^{-(N+2s)/2}
\]


Porous Medium with Fractional Pressure

(II) \[
V_t = \nabla \cdot (V \nabla (-\Delta)^{-s} V^{m-1}), \quad \tilde{m} > 1
\]

Biler, Imbert, Karch [2013]:

\[
\nu(x, t) = at^{-N\beta_2} \left(R^2 - |xt^{-\beta_2}|^2\right)^{(1-\tilde{s})/\tilde{m}-1}
\]

(I) \[
V_t = \nabla \cdot (V^{\tilde{m}-1} \nabla (-\Delta)^{-s} V), \quad m < 2
\]

Huang [2014]: \[
\forall \tilde{s} \in (0, 1) \rightarrow \tilde{m} = \frac{N+6\tilde{s}-2}{N+2\tilde{s}} < 2
\]

\[
\nu(x, t) = at^{-N\beta_2} \left(R^2 + |xt^{-\beta_2}|^2\right)^{-(N+2\tilde{s})/2}
\]
Nonlocal/Local. Self-Similar Solutions

\[ V_t = \nabla \cdot (V \nabla (-\Delta)^{-s} V^{m-1}), \; m > 1 \]
\[ V_t = \nabla \cdot (V^{m-1} \nabla (-\Delta)^{-s} V), \; \tilde{m} \geq 2 \]

\[ U_t + (-\Delta)^s U^m = 0 \]
\[ V_t = \nabla \cdot (V^{m-1} \nabla (-\Delta)^{-s} V), \; \tilde{m} < 2 \]

PME/FDE \[ u_t = \Delta u^m \]
- \( m > 1 \), Profile \( \sim (R^2 - |y|^2)^{1/(m-1)} \)
- \( m < 1 \), Profile \( \sim (R^2 + |y|^2)^{-1/(1-m)} \)
Gracias!

Eskerrik asko!