The fractional non-homogeneous Poisson process

Mailan Trinh

University of Sussex

Joint work with Nikolai Leonenko and Enrico Scalas

November 23th 2016

Overview

Types of Poisson processes

	standard	fractional
homogeneous	(i) $(N^h_\lambda(t))$	(iii) $(N^{hf}_\alpha(t))$
inhomogeneous	(ii) $(N(t))$	(iv) $(N_\alpha(t))$

We are interested in

- Definition of the process $(X(t))$
- Marginal distribution $p_x(t)$
- Governing equation $D_t p_x(t) = A p_x(t)$

Overview

The standard (non-fractional) case

(i) The **homogeneous Poisson process** (HPP) $(N^h_\lambda(t))$ with intensity parameter $\lambda > 0$:

$$p^\lambda_x(t) := \mathbb{P}(N^h_\lambda(t) = x) = \mathrm{e}^{-\lambda t}\frac{(\lambda t)^x}{x!}, \quad x = 0, 1, 2, \dots$$

$$\frac{\mathrm{d}}{\mathrm{d}t}p^\lambda_x(t) = -\lambda(p^\lambda_x(t) - p^\lambda_{x-1}(t)), \quad p^\lambda_x(0) = \delta_0(x),\ p^\lambda_{-1}(t) \equiv 0.$$

(ii) The **inhomogeneous Poisson process** (NHPP) $(N(t))$ with intensity $\lambda(t) : [0, \infty) \longrightarrow [0, \infty)$ and rate function $\Lambda(s, t) = \int_s^t \lambda(u)\mathrm{d}u$. For $x = 0, 1, 2, \dots$

$$p_x(t, v) := \mathbb{P}\{N(t + v) - N(v) = x\} = \frac{\mathrm{e}^{-\Lambda(v,t+v)}\Lambda(v, t + v)^x}{x!},$$

$$\frac{\mathrm{d}}{\mathrm{d}t}p_x(t, v) = -\lambda(t + v)(p_x(t, v) - p_{x-1}(t, v))$$

$$p_x(0, v) = \delta_0(x),\ p_{-1}(t, v) \equiv 0.$$

Overview

The (inverse) α-stable subordinator

Let $L_\alpha = \{L_\alpha(t), t \geq 0\}$, be an **$\alpha$-stable subordinator** with Laplace transform

$$\mathbb{E}\left[\exp(-sL_\alpha(t))\right] = \exp(-ts^\alpha), \quad 0 < \alpha < 1, s \geq 0$$

and $Y_\alpha = \{Y_\alpha(t), t \geq 0\}$, be an **inverse α-stable subordinator** defined by

$$Y_\alpha(t) = \inf\{u \geq 0 : L_\alpha(u) > t\}.$$

Let $h_\alpha(t, \cdot)$ denote the density of the distribution of $Y_\alpha(t)$. See for example Meerschaert and Scheffler (2004) for properties.

The fractional case

(iii) The **fractional homogeneous Poisson process** (FHPP) $(N_\alpha^{hf}(t))$ is defined as $N_\alpha^{hf}(t) := N_\lambda^h(Y_\alpha(t))$ for $t \geq 0, 0 < \alpha < 1$.

Theorem (Beghin and Orsingher (2009, 2010))

The marginal distribution of the FHPP

$$p_x^\alpha(t) = \mathbb{P}\{N_\lambda(Y_\alpha(t)) = x\} = \int_0^\infty \mathrm{e}^{-\lambda u} \frac{(\lambda u)^x}{x!} h_\alpha(t,u) \mathrm{d}u$$
$$= (\lambda t^\alpha)^x E_{\alpha,\alpha x+1}^{x+1}(-\lambda t^\alpha), \ x = 0, 1, 2, \ldots \quad \textit{satisfies}$$

$$D_t^\alpha p_x^\alpha(t) = -\lambda(p_x^\alpha(t) - p_{x-1}^\alpha(t)), \quad p_x^\alpha(0) = \delta_0(x), \ p_{-1}^\alpha(0) \equiv 0.$$

Recall the **Caputo derivative** and its Laplace transform

$$D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\mathrm{d}f(\tau)}{\mathrm{d}\tau} \frac{\mathrm{d}\tau}{(t-\tau)^\alpha}, \quad 0 < \alpha < 1$$
$$\mathcal{L}\{D_t^\alpha f\}(s) = s^\alpha \mathcal{L}\{f\}(s) - s^{\alpha-1} f(0^+),$$

Concerning consistency

	standard	fractional
homogeneous	(i) $(N^h_\lambda(t))$	(iii) $(N^{hf}_\alpha(t))$
inhomogeneous	(ii) $(N(t))$	(iv) $(N_\alpha(t))$

$\lambda(t) = \lambda$ const.

$\alpha = 1$

Definition of the FHPP

(iv) The **fractional non-homogenous Poisson process** (FNPP) could be defined in the following way:
Recall that the NPP can be expressed via the HPP:

$$N(t) = N_1^h(\Lambda(t))$$

and the FHPP is defined as

$$N_\alpha^{hf}(t) := N_\lambda^h(Y_\alpha(t))$$

Analogously define $N_\alpha(t) := N(Y_\alpha(t)) = N_1^h(\Lambda(Y_\alpha(t)))$

The marginal distribution of the FNPP

Recall that the result for the FHPP used the 1-dim. marginal

$$p_x^\alpha(t) = \mathbb{P}\{N_\lambda^h(Y_\alpha(t)) = x\},$$

whereas for the NPP we would like to say something about the increment

$$p_x(t, v) := \mathbb{P}\{N(t+v) - N(v) = x\}.$$

It is difficult to work with the increments of the subordinated process $(N_\alpha(t))$ directly:

$$N_\alpha(t+v) - N_\alpha(v) = N_1^h(\Lambda(Y_\alpha(v), Y_\alpha(t+v))).$$

Instead, we will consider the subordinated increment process:

$$I(t, v) := N_1^h(\Lambda(t+v)) - N_1^h(\Lambda(v)),$$
$$I_\alpha(t, v) := I(Y_\alpha(t), v) = N_1^h(\Lambda(Y_\alpha(t)+v)) - N_1^h(\Lambda(v)).$$

The governing equation for the FNPP

We can define the marginals

$$f_x^\alpha(t,v) := \mathbb{P}\{N_1^h(\Lambda(Y_\alpha(t)+v)) - N_1^h(\Lambda(v)) = x\}, \quad x = 0,1,2,\ldots$$
$$= \int_0^\infty p_x(u,v) h_\alpha(t,u) \mathrm{d}u$$

Theorem (Leonenko et al. (2017))

Let $I_\alpha(t,v)$ be the fractional increment process. Then, its marginal distribution satisfies the following fractional differential-integral equations ($x = 0,1,\ldots$)

$$D_t^\alpha f_x^\alpha(t,v) = \int_0^\infty \lambda(u+v)[-p_x(u,v) + p_{x-1}(u,v)] h_\alpha(t,u) \mathrm{d}u,$$

with initial condition $f_x^\alpha(0,v) = \delta_0(x)$ and $f_{-1}^\alpha(0,v) \equiv 0$.

Proof

We have the marginals

$$f_x^\alpha(t,v) = \int_0^\infty p_x(u,v) h_\alpha(t,u)\mathrm{d}u$$

In contrast to the FHPP case, we cannot calculate the Laplace transform $\tilde{\cdot}$ explicitly:

$$\tilde{f}_x^\alpha(r,v) = \int_0^\infty p_x(u,v) \tilde{h}_\alpha(r,u)\mathrm{d}u$$

Instead, consider the Fourier-Laplace transform $\bar{\cdot}$ of f_x^α:

$$\bar{f}_y^\alpha(r,v) = \int_0^\infty \hat{p}_y(u,v) \tilde{h}_\alpha(r,u)\mathrm{d}u$$

Proof

We have the marginals

$$f_x^\alpha(t,v) = \int_0^\infty p_x(u,v) h_\alpha(t,u) \mathrm{d}u$$

In contrast to the FHPP case, we cannot calculate the Laplace transform $\tilde{\cdot}$ explicitly:

$$\tilde{f}_x^\alpha(r,v) = \int_0^\infty p_x(u,v) \tilde{h}_\alpha(r,u) \mathrm{d}u$$

Instead, consider the Fourier-Laplace transform $\bar{\cdot}$ of f_x^α:

$$\bar{f}_y^\alpha(r,v) = \int_0^\infty \underbrace{\hat{p}_y(u,v)}_{\downarrow} \underbrace{\tilde{h}_\alpha(r,u)}_{\uparrow} \mathrm{d}u$$

and integrate by parts.

Proof: Integration by parts

$$\bar{f}_y^{\alpha}(r,v) = \int_0^\infty \underbrace{\exp(\Lambda(v,u+v)(e^{iy}-1))}_{\downarrow} \underbrace{r^{\alpha-1}e^{-ur^\alpha}}_{\uparrow} \, du.$$

$$= \boxed{\frac{1}{r^\alpha}\left[r^{\alpha-1}\right.} +$$

$$\left.(e^{iy}-1)\int_0^\infty \boxed{\lambda(u+v)} \underbrace{\exp(\Lambda(v,u+v)(e^{iy}-1))}_{=\hat{p}_y(u,v)} \underbrace{r^{\alpha-1}e^{-ur^\alpha}}_{\tilde{h}_\alpha(r,u)} du\right].$$

Recall the Laplace transform of the Caputo derivative

$$\widetilde{D_t^\alpha f}(r) = r^\alpha \widetilde{f}(r) - r^{\alpha-1} f(0^+)$$

Taking the Caputo derivative cancels the terms in $\boxed{}$.

Proof: Taking the time derivative

$$r^{\alpha}\bar{f}_y^{\alpha}(r,v) - r^{\alpha-1} = (e^{iy}-1)\int_0^{\infty}\lambda(u+v)\hat{p}_y(u,v)\tilde{h}_{\alpha}(r,u)\mathrm{d}u.$$

Invert the Laplace transform

$$D_t^{\alpha}\hat{f}_y^{\alpha}(t,v) = (e^{iy}-1)\int_0^{\infty}\lambda(u+v)\hat{p}_y(u,v)h_{\alpha}(t,u)\mathrm{d}u$$

and finally invert the characteristic function to end up with the result

$$D_t^{\alpha}f_x^{\alpha}(t,v) = \int_0^{\infty}\lambda(u+v)[-p_x(u,v)+p_{x-1}(u,v)]h_{\alpha}(t,u)\mathrm{d}u.$$

□

Governing equation for 1-dim. marginals

$$
\begin{aligned}
f_x^\alpha(t,v) &:= \mathbb{P}\{N_1^h(\Lambda(Y_\alpha(t)+v)) - N_1^h(\Lambda(v)) = x\}, \quad x = 0,1,2,\dots \\
f_x^\alpha(t,0) &= \mathbb{P}\{N_1^h(\Lambda(Y_\alpha(t)+0)) - N_1^h(\Lambda(0)) = x\} \\
&= \mathbb{P}\{N_1^h(\Lambda(Y_\alpha(t))) = x\}
\end{aligned}
$$

Corollary

Let $\{N_\alpha(t), t \geq 0\}$, $0 < \alpha < 1$ be a FNPP. Then, its marginal distributions satisfy the following fractional differential-integral equations:

$$
D_t^\alpha f_x^\alpha(t,0) = \int_0^\infty \lambda(u)[-p_x(u,0) + p_{x-1}(u,0)]h_\alpha(t,u)\mathrm{d}u,
$$

with initial condition $f_x^\alpha(0) = \delta_0(x)$ and $f_{-1}^\alpha(0) \equiv 0$.

Special cases

	standard	fractional
homogeneous	(i) $(N^h_\lambda(t))$	(iii) $(N^{hf}_\alpha(t))$
inhomogeneous	(ii) $(N(t))$	(iv) $(N_\alpha(t))$

$\lambda(t) = \lambda$ const.

$\alpha = 1$

Special cases

	standard	fractional
homogeneous	(i) $(N^h_\lambda(t))$	(iii) $(N^{hf}_\alpha(t))$
inhomogeneous	(ii) $(N(t))$	(iv) $(N_\alpha(t))$

$\lambda(t) = \lambda$ const.

$\alpha = 1$

Special cases: FNPP $\Rightarrow$ FHPP

For constant λ we have

$$f_x^\alpha(t,0) = \int_0^\infty \frac{e^{-u\lambda}(\lambda u)^x}{x!} h_\alpha(t,u)\mathrm{d}u = p_x^\alpha(t),$$

i.e. f_x^α coincides with the marginal probabilities of the FHPP and from previous corollary

$$\begin{aligned} D_t^\alpha f_x^\alpha(t,0) &= \lambda \int_0^\infty [-p_x(u,0) + p_{x-1}(u,0)] h_\alpha(t,u)\mathrm{d}u \\ &= -\lambda f_x^\alpha(t,0) + \lambda f_{x-1}^\alpha(t,0) \qquad (1) \end{aligned}$$

Special cases

	standard	fractional
homogeneous	(i) $(N_\lambda^h(t))$	(iii) $(N_\alpha^{hf}(t))$
inhomogeneous	(ii) $(N(t))$	(iv) $(N_\alpha(t))$

$\lambda(t) = \lambda$ const.

$\alpha = 1$

Special cases

	standard	fractional
homogeneous	(i) $(N^h_\lambda(t))$	(iii) $(N^{hf}_\alpha(t))$
inhomogeneous	(ii) $(N(t))$	(iv) $(N_\alpha(t))$

$\lambda(t) = \lambda$ const.

$\alpha = 1$

Special cases: FNPP $\Rightarrow$ NPP

For $\alpha = 1$, we have $\tilde{h}_1(s,u) = e^{-us}$, i.e. $h_1(t,u) = \delta(t-u)$. Formally, we get

$$f_x^1(t,v) = \int_0^\infty p_x(u,v)\delta(t-u)\mathrm{d}u = p_x(t,v)$$

and

$$\begin{aligned} D_t^1 p_x(t,v) &= D_t^1 f_x^1(t,v) \\ &= \int_0^\infty \lambda(u+v)[-p_x(u+v) + p_{x-1}(u,v)]\delta(t-u)\mathrm{d}u \\ &= \lambda(t+v)[-p_x(t,v) + p_{x-1}(t,v)] \end{aligned}$$

Overview

Moments (I)

Moments of the Poisson distribution with parameter $\Lambda(t)$ are given by

$$\mathbb{E}[[N(t)]^k] = \sum_{i=1}^{k} \Lambda(t)^i \left\{ \begin{array}{c} k \\ i \end{array} \right\},$$

where the Stirling Numbers of second kind are given by

$$\left\{ \begin{array}{c} k \\ i \end{array} \right\} = \frac{1}{i!} \sum_{j=0}^{i} (-1)^{i-j} \left(\begin{array}{c} i \\ j \end{array} \right) j^k.$$

Using the tower property of the conditional expectation we get

$$\mathbb{E}[[N(Y_\alpha(t))]^k] = \mathbb{E}[\mathbb{E}[[N(Y_\alpha(t))]^k | Y_\alpha(t)]] = \int_0^\infty \mathbb{E}[[N(x)]^k] h_\alpha(t,x) \mathrm{d}x$$

$$= \int_0^\infty \sum_{i=1}^{k} \Lambda(x)^i \left\{ \begin{array}{c} k \\ i \end{array} \right\} h_\alpha(t,x) \mathrm{d}x = \mathbb{E}\left[\sum_{i=1}^{k} \Lambda(Y_\alpha(t))^i \left\{ \begin{array}{c} k \\ i \end{array} \right\} \right].$$

Moments (II)

In particular, we can calculate the expectation and the variance of the FNPP

$$\mathbb{E}[N(Y_\alpha(t))] = \mathbb{E}[\Lambda(Y_\alpha(t))],$$

$$\begin{aligned}\mathrm{Var}[N(Y_\alpha(t))] &= \mathbb{E}[[N(Y_\alpha(t))]^2] - \mathbb{E}[[N(Y_\alpha(t))]]^2 \\ &= \mathbb{E}[\Lambda(Y_\alpha(t))] + \mathrm{Var}[\Lambda(Y_\alpha(t))].\end{aligned}$$

Watanabe characterisation for the FHPP: *Fractional Poisson fields and Martingales* Aletti et al. (2016).

Covariance structure

The covariance of the inhomogeneous Poisson process with rate function $\Lambda(t)$ is given by $\mathrm{Cov}(N(s), N(t)) = \Lambda(0, s \wedge t)$.

Proposition

By the law of total covariance, one finds:

$$
\begin{aligned}
&Cov[N(Y_\alpha(s)), N(Y_\alpha(t))] \\
&\qquad = \mathbb{E}[\Lambda(0, Y_\alpha(s \wedge t))] + Cov[\Lambda(Y_\alpha(s)), \Lambda(Y_\alpha(t))]
\end{aligned}
$$

Remark

The two-point cumulative distribution function of the inverse stable subordinator $Y_\alpha(t)$ can be computed using the fact that (see Leonenko et al., 2013)

$$
\begin{aligned}
&\mathbb{P}(Y_\alpha(s) > x, Y_\alpha(t) > y) \\
&\qquad = \int_{v=0}^{t} \frac{\alpha}{v} y h_\alpha(s, y) \int_{u=0}^{s-v} \frac{\alpha}{u} (x - y) h_\alpha(t, x - y) \, \mathrm{d}u \mathrm{d}v.
\end{aligned}
$$

Overview

Arrival times: standard case

Let $T_n = \min\{t \in [0, \infty) : N(t) = n\}$ be the arrival times of a NPP. The distribution of T_n is called **Erlang Distribution**:

$$F_{T_n}^{\mathrm{NPP}}(t) = \mathbb{P}(T_n \leq t) = \mathbb{P}(N(t) \geq n)$$
$$= 1 - \mathrm{e}^{-\Lambda(t)} \sum_{x=0}^{n-1} \frac{(\Lambda(t))^x}{x!}. \qquad (2)$$

Erlang distribution: fractional case

$$F_{T_n}^{\mathrm{FNPP}}(t) = \mathbb{P}(T_n \leq t) = \mathbb{P}(N_\alpha(t) \geq n) = \sum_{x=n}^{\infty} f_x^\alpha(t)$$

$$= \sum_{x=n}^{\infty} \int_0^\infty \frac{\mathrm{e}^{-\Lambda(u)}\Lambda(u)^x}{x!} h_\alpha(t,u)\mathrm{d}u$$

$$= \int_0^\infty h_\alpha(t,u) \sum_{x=n}^{\infty} \frac{\mathrm{e}^{-\Lambda(u)}\Lambda(u)^x}{x!}\mathrm{d}u \tag{3}$$

$$= \int_0^\infty h_\alpha(t,u) \left[1 - \sum_{x=0}^{n-1} \frac{\mathrm{e}^{-\Lambda(u)}\Lambda(u)^x}{x!}\right] \mathrm{d}u$$

$$= \int_0^\infty h_\alpha(t,u) F_{T_n}^{\mathrm{NPP}}(u)\mathrm{d}u \tag{4}$$

For the homogeneous case (see Mainardi et al. (2004))

$$F_{T_n}^{\mathrm{FHPP}}(t) = \int_0^\infty h_\alpha(t,u) F_{T_n}^{\mathrm{HPP}}(u)\mathrm{d}u = 1 - \sum_{x=0}^{n-1} \frac{(\lambda t^\alpha)^x}{x!} E_\alpha^{(x)}(-\lambda t^\alpha)$$

Overview

Future research

- Other possible definitions of FNPP: $N_1(Y_\alpha(\Lambda(t)))$ (see for example Maheshwari and Vellaisamy (2016))
- Other related stochastic processes: fractional non-homogeneous compound Poisson, Skellam processes...
- Functional limit for $\alpha \to 1$ and diffusive limits.
- Long-term behaviour of the covariance structure

Thank you for your attention!

Aletti, G., N. Leonenko, and E. Merzbach (2016). Fractional poisson fields and martingales. arXiv:1601.08136v2 [math.PR].

Beghin, L. and E. Orsingher (2009). Fractional Poisson processes and related planar random motions. *Electron. J. Probab. 14*, no. 61, 1790–1827.

Beghin, L. and E. Orsingher (2010). Poisson-type processes governed by fractional and higher-order recursive differential equations. *Electron. J. Probab. 15*, no. 22, 684–709.

Leonenko, N., E. Scalas, and M. Trinh (2017). The fractional non-homogeneous Poisson process. *Statistics and Probability Letters 120*, 147–156.

Leonenko, N. N., M. M. Meerschaert, and A. Sikorskii (2013). Correlation structure of fractional Pearson diffusions. *Comput. Math. Appl. 66*(5), 737–745.

Maheshwari, A. and P. Vellaisamy (2016). Non-homogeneous fractional poisson processes. arXiv:1607.06016.

Mainardi, F., R. Gorenflo, and E. Scalas (2004). A fractional generalization of the Poisson processes. *Vietnam J. Math. 32*(Special Issue), 53–64.

Meerschaert, M. M. and H.-P. Scheffler (2004). Limit theorems for continuous-time random walks with infinite mean waiting times. *J. Appl. Probab. 41*(3), 623–638.