Nonlocal evolution equations

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Few words about local diffusion problems

\[ \begin{cases} 
    u_t - \Delta u = 0 & \text{in } \mathbb{R}^d \times (0, \infty), \\
    u(0) = u_0. 
\end{cases} \]

For any \( u_0 \in L^1(\mathbb{R}) \) the solution \( u \in C([0, \infty), L^1(\mathbb{R}^d)) \) is given by:

\[ u(t, x) = (G(t, \cdot) * u_0)(x) \]

where

\[ G(t, x) = (4\pi t)^{1/2} \exp\left(-\frac{|x|^2}{4t}\right) \]

Smoothing effect

\[ u \in C^\infty((0, \infty), \mathbb{R}^d) \]

Decay of solutions, \( 1 \leq p \leq q \leq \infty \):

\[ \|u(t)\|_{L^q(\mathbb{R})} \lesssim t^{-\frac{d}{2}\left(\frac{1}{p} - \frac{1}{q}\right)} \|u_0\|_{L^p(\mathbb{R})} \]
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\]
Asymptotics

Theorem

For any \( u_0 \in L^1(\mathbb{R}^d) \) and \( p \geq 1 \) we have

\[
t^{\frac{d}{2}(1 - \frac{1}{p})} \|u(t) - MG_t\|_{L^p} \to 0,
\]

where \( M = \int u_0 \).

Proof:

\[
(G_t*u_0)(x) - G_t(x) \int u_0 = \frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} \left( \exp\left(-\frac{|x-y|^2}{4t}\right) - \exp\left(-\frac{|x|^2}{4t}\right) \right) u_0(y)
\]
Refined asymptotics

Zuazua & Duoandikoetxea, CRAS '92
For all $\varphi \in L^p(\mathbb{R}^d, 1 + |x|^k)$

$$u(t, \cdot) \sim \sum_{|\alpha| \leq k} \frac{(-1)^{|\alpha|}}{\alpha!} \left( \int u_0(x)x^\alpha dx \right) D^\alpha G(t, \cdot) \quad \text{in } L^q(\mathbb{R}^d)$$

for some $p, q, k$

- Similar thinks on Heisenberg group: L. I. & Zuazua JEE 2013
A linear nonlocal problem


\[
\begin{align*}
    u_t(x, t) &= J * u - u(x, t) = \int_{\mathbb{R}^d} J(x - y)u(y, t) \, dy - u(x, t), \\
    &= \int_{\mathbb{R}^d} J(x - y)(u(y, t) - u(x, t)) \, dy \\
    u(x, 0) &= u_0(x),
\end{align*}
\]

where \( J : \mathbb{R}^N \to \mathbb{R} \) be a nonnegative, radial function with \( \int_{\mathbb{R}^N} J(r) \, dr = 1 \).
Models

Gunzburger’s papers
1. Analysis and approximation of nonlocal diffusion problems with volume constraints
2. A nonlocal vector calculus with application to nonlocal boundary value problems

Case 1: \( s \in (0, 1) \),

\[
\frac{c_1}{|y - x|^{d+2s}} \leq J(x, y) \leq \frac{c_2}{|y - x|^{d+2s}}
\]

Case 2: essentially \( J \) is a nice, smooth function
Heat equation and nonlocal diffusion

- **Similarities**
  - bounded stationary solutions are constant
  - a maximum principle holds for both of them

- **Difference**
  - there is no regularizing effect in general

The fundamental solution can be decomposed as

$$ e^{-t}\delta_0(x) + v(x,t), $$

with $v(x,t)$ smooth

$$ S(t)\varphi = e^{-t}\varphi + v * \varphi = \text{smooth as initial data} + \text{smooth part} $$

= no smoothing effect
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$$S(t)\varphi = e^{-t} \varphi + v \ast \varphi = \text{smooth as initial data + smooth part}$$
$$= \text{no smoothing effect}$$
Asymptotic Behaviour

- If \( \hat{J}(\xi) = 1 - A|\xi|^2 + o(|\xi|^2) \), \( \xi \sim 0 \), the asymptotic behavior is the same as the one for solutions of the heat equation

\[
\lim_{t \to +\infty} t^{d/\alpha} \max_x |u(x, t) - v(x, t)| = 0,
\]

where \( v \) is the solution of \( v_t(x, t) = A\Delta v(x, t) \) with initial condition \( v(x, 0) = u_0(x) \).

- The asymptotic profile is given by

\[
\lim_{t \to +\infty} \max_y \left| t^{d/2} u(y t^{1/2}, t) - \left( \int_{\mathbb{R}^d} u_0 \right) G_A(y) \right| = 0,
\]

where \( G_A(y) \) satisfies \( \hat{G}_A(\xi) = e^{-A|\xi|^2} \).
Other results on the linear problem


I.L. Ignat, J.D. Rossi, A. San Antolin, JDE 2012.

A nonlinear model: convection-diffusion

For $q \geq 1$

\[
\begin{align*}
    u_t - \Delta u + (|u|^{q-1}u)_x &= 0 \quad \text{in} \quad (0, \infty) \times \mathbb{R} \\
    u(0) &= u_0
\end{align*}
\]

- Asymptotic Behaviour by using

\[
\frac{d}{dt} \int_{\mathbb{R}^d} |u|^p \, dx = -\frac{4(p-1)}{p} \int_{\mathbb{R}^d} |\nabla(|u|^{p/2})|^2 \, dx.
\]


The first term in the asymptotic behaviour

With $M = \int u_0$, Escobedo & Zuazua JFA ’91 proved

$q > 2,$

$$\lim_{t \to \infty} t^{1/2(1-1/p)} \|u(t) - MG_t\|_{L^p(\mathbb{R})} = 0$$

$q = 2,$

$$\lim_{t \to \infty} t^{1/2(1-1/p)} \|u(t) - f_M(x, t)\|_{L^p(\mathbb{R})} = 0$$

where $f_M(x, t) = t^{-1/2} f_M\left(\frac{x}{\sqrt{t}}, 1\right)$ is the unique solution of the viscous Bourgers equation

$$\begin{cases} 
U_t = U_{xx} - (U^2)_x \\
U(0) = M\delta_0
\end{cases}$$

$1 < q < 2$, Escobedo, Vazquez, Zuazua, ARMA ’93

$$\lim_{t \to \infty} t^{1/q(1-1/p)} \|u(t) - U_M(x, t)\|_{L^p(\mathbb{R})} = 0$$
Some ideas of the proof

- For $q > 2$

$$u(t) = S(t)u_0 + \int_0^t S(t - s)(u^q)_x(s)\,ds$$

and use that the nonlinear part decays faster than the linear one.

- $q = 2$ scaling: introduce $u_\lambda(x, t) = \lambda u(\lambda x, \lambda^2 t)$, write the equation for $u_\lambda$ and observe that the estimates for $u$ are equivalent to the fact that

$$u_\lambda(x, 1) \to f_M(x) \text{ in } L^1(\mathbb{R})$$

where $f_M$ is a solution of the viscous Bourgers equation with initial data $M\delta_0$

Main idea: in some moment you have to use compactness
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Main idea: in some moment you have to use compactness
Proof: the so-called ”four step method” :

- scaling - write the equation for $u_\lambda$
- estimates and compactness of $\{u_\lambda\}$
- passage to the limit
- identification of the limit

- $1 < q < 2$, read EVZ's paper, entropy solutions, etc...
Proof: the so-called ”four step method” :

- scaling - write the equation for $u_\lambda$
- estimates and compactness of $\{u_\lambda\}$
- passage to the limit
- identification of the limit

- $1 < q < 2$, read EVZ’s paper, entropy solutions, etc...

\[
\begin{cases}
  u_t(t, x) = (J_1 * u - u)(t, x) + (J_2 * (f(u)) - f(u))(t, x), & t > 0, x \in \mathbb{R}, \\
  u(0, x) = u_0(x), & x \in \mathbb{R}.
\end{cases}
\]

- $J_1$ and $J_2$ are nonnegatives and verify
  $$\int_{\mathbb{R}^d} J_1(x) \, dx = \int_{\mathbb{R}^d} J_2(x) \, dx = 1.$$
- $J_1$ even function
- $f(u) = |u|^{q-1}u$ with $q > 1$
- $q > 2$ similar estimates as in the local case
- the case $q = 2$ open until recently :)

Liviu Ignat (BCAM&IMAR)
P. Laurençot, Asymptotic Analysis ’05, considered the following model for radiating gases

\[
\begin{aligned}
\left\{
  & u_t + \left(\frac{u^2}{2}\right)_x = K \ast u - u \\
  & \text{in } (0, \infty) \times \mathbb{R}
\end{aligned}
\]

\[
  u(0) = u_0
\]

where \( K(x) = e^{-|x|}/2 \).

- take care in defining the solutions, entropy solutions, Schochet and Tadmor, ARMA 1992, Lattanzio & Marcati JDE 2003, D. Serre, Scalar conservation laws, etc...
- good news: asymptotic behaviour by scaling
- bad news: Oleinik estimate for \( u \): \( u_x \leq \frac{1}{t} \)
Linear problem revised

\[
\begin{cases}
  u_t(x, t) = \int_{\mathbb{R}} J(x - y)(u(y, t) - u(x, t)) \, dy, & x \in \mathbb{R}, \ t > 0, \\
  u(x, 0) = u_0(x), & x \in \mathbb{R}.
\end{cases}
\]

Theorem

Let \( u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \). Then

\[
\lim_{t \to \infty} t^{1/2(1-1/p)} \| u(t) - MG_t \|_{L^p(\mathbb{R})} = 0
\]

where

\[
G_t(x) = \frac{1}{\sqrt{4\pi t}} \exp \left( -\frac{x^2}{4t} \right)
\]

is the heat kernel and \( M = \int_{\mathbb{R}} u_0(x) \, dx \).
• Main question: how to use the scaling here?
• Main difficulty: the lack of the smoothing effect present in the case of the heat equation

\[ u(t) = e^{-t} \varphi + K_t * \varphi = \text{smooth as initial data + smooth part} \]
\[ = \text{no smoothing effect} \]

• an idea: do the scaling for the smooth part instead of \( u \)

\[ v(x, t) = u(x, t) - e^{-t}u_0(x) \]
• Main question: how to use the scaling here?
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\[ u(t) = e^{-t}\varphi + K_t \ast \varphi = \text{smooth as initial data} + \text{smooth part} \]
\[ = \text{no smoothing effect} \]

• an idea: do the scaling for the smooth part instead of \( u \)

\[ v(x, t) = u(x, t) - e^{-t}u_0(x) \]
It follows that $v(x, t)$ verifies the equation:

\[
\begin{cases}
  v_t(x, t) = e^{-t}(J * u_0)(x) + (J * v - v)(x, t), & x \in \mathbb{R}, \ t > 0, \\
  v(x, 0) = 0, & x \in \mathbb{R}.
\end{cases}
\]

The ”four step method” can be applied and

\[
\lim_{t \to \infty} t^{1/2(1-1/p)} \|v(t) - MG_t\|_{L^p(\mathbb{R})} = 0
\]

Then, back to $u$ and obtain the asymptotic behaviour

This gives us the hope that the scaling could work in the nonlinear nonlocal models.
In a joint work with A. Pazoto we have considered the model

\[
\begin{aligned}
  u_t &= J * u - u + (|u|^{q-1}u)_x, \quad x \in \mathbb{R}, \ t > 0 \\
  u(0) &= \varphi.
\end{aligned}
\] (3)

Question: for \( q > 2 \) may we obtain similar results as in the case of the classical convection-diffusion: leading term given by the heat kernel? Answer: YES by scaling :)

**Theorem (L.I. & A. Pazoto, 2012)**

For any \( \varphi \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \) the solution \( u \) of system (3) satisfies

\[
\lim_{t \to \infty} t^{\frac{1}{2}(1-\frac{1}{p})} \|u(t) - MG_t\|_{L^p(\mathbb{R})} = 0
\] (4)
In a joint work with A. Pazoto we have considered the model

$$\begin{cases}
    u_t = J * u - u + (|u|^{q-1}u)_x, & x \in \mathbb{R}, t > 0 \\
    u(0) = \varphi.
\end{cases} \quad (3)$$

Question: for $q > 2$ may we obtain similar results as in the case of the classical convection-diffusion: leading term given by the heat kernel?

Answer: YES by scaling :)

**Theorem (L.I. & A. Pazoto, 2012)**

*For any $\varphi \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ the solution $u$ of system (3) satisfies*

$$\lim_{t \to \infty} t^{\frac{1}{2} \left(1 - \frac{1}{p}\right)} \|u(t) - MG_t\|_{L^p(\mathbb{R})} = 0 \quad (4)$$
We introduce the scaled functions

\[ u_\lambda(t, x) = \lambda u(\lambda^2 t, \lambda x) \quad \text{and} \quad J_\lambda(x) = \lambda J(\lambda x). \]

Then \( u_\lambda \) satisfies the system

\[
\begin{cases}
  u_{\lambda,t} = \lambda^2 (J_\lambda \ast u_\lambda - u_\lambda) + \lambda^{2-q} (u_\lambda^q)_x, & x \in \mathbb{R}, \ t > 0 \\
  u_\lambda(0, x) = \varphi_\lambda(x) = \lambda \varphi(\lambda x), & x \in \mathbb{R}.
\end{cases}
\] (5)

Question: four step method?
Estimates on $u_\lambda$

There exists $M = M(t_1,t_2,\|\varphi\|_{L^1(\mathbb{R})}, \|\varphi\|_{L^\infty(\mathbb{R})})$ such that

$$\|u_\lambda\|_{L^\infty(t_1,t_2,L^2(\mathbb{R}))} \leq M,$$

(6)

$$\lambda^2 \int_{t_1}^{t_2} \int_{\mathbb{R}} \int_{\mathbb{R}} J_\lambda(x-y)(u_\lambda(x) - u_\lambda(y))^2 \, dx \, dy \leq M.$$  

(7)

and

$$\|u_\lambda,t\|_{L^2(t_1,t_2,H^{-1}(\mathbb{R}))} \leq M.$$  

(8)

Q: Aubin-Lions Lemma? Not exactly ... since we have no gradients :( but an integral term in the second estimate
Theorem

Let $1 \leq p < \infty$ and $\Omega \subset \mathbb{R}$ open. Let $\rho : \mathbb{R} \to \mathbb{R}$ be a nonnegative smooth continuous radial functions with compact support, non identically zero, and $\rho_n(x) = n^d \rho(nx)$. Let $\{f_n\}$ be a sequence of functions in $L^p(\mathbb{R})$ such that

$$\int_{\Omega} \int_{\Omega} \rho_n(x-y)|f_n(x) - f_n(y)|^p \, dx \, dy \leq \frac{M}{n^p}. \quad (9)$$

The following hold:

1. If $\{f_n\}$ is weakly convergent in $L^p(\Omega)$ to $f$ then $f \in W^{1,p}(\Omega)$ for $p > 1$ and $f \in BV(\Omega)$ for $p = 1$.

2. Assuming that $\Omega$ is a smooth bounded domain in $\mathbb{R}$ and $\rho(x) \geq \rho(y)$ if $|x| \leq |y|$ then $\{f_n\}$ is relatively compact in $L^p(\Omega)$.\[\]
Compact sets in $L^p(0, T, B)$

Theorem (Simon ’87)

Let $\mathcal{F} \subset L^p(0, T, B)$. $\mathcal{F}$ is relatively compact in $L^p(0, T, B)$ for $1 \leq p < \infty$, or $C(0, T, B)$ for $p = \infty$ if and only if

1. $\{ \int_{t_1}^{t_2} f(t) dt, \ f \in \mathcal{F} \}$ is relatively compact in $B$ for all $0 < t_1 < t_2 < T$.

2. $\| \tau_h f - f \|_{L^p(0, T-h, B)} \to 0$ as $h \to 0$ uniformly for $f \in \mathcal{F}$.

Main idea: put together the previous results to obtain compactness for $u_\lambda$
Theorem

Let \( u_n \) be a sequence in \( L^2(0, T, L^2(\mathbb{R})) \) such that

\[
\| u_n \|_{L^\infty(0,T,L^2(\mathbb{R}))} \leq M, \quad (10)
\]

\[
n^2 \int_0^T \int_\mathbb{R} \int_\mathbb{R} J_n(x - y)(u_n(x) - u_n(y))^2 \, dx \, dy \leq M \quad (11)
\]

and

\[
\| \partial_t u_n \|_{L^2(0,T,H^{-1}(\mathbb{R}))} \leq M. \quad (12)
\]

Then there exists a function \( u \in L^2((0, T), H^1(\mathbb{R})) \) such that, up to a subsequence,

\[
u_n \rightarrow u \quad \text{in} \quad L^2_{loc}((0, T) \times \mathbb{R}). \quad (13)
\]
Proof:

Follow carefully the steps in Simon’s paper + BBM&R static criterium + tricky inequalities + Lufthansa flight

Consequence: we can apply the ”four step method”
Proof:

Figure: Caipirinha

Follow carefully the steps in Simon’s paper + BBM&R static criterium + tricky inequalities + Lufthansa flight

Consequence: we can apply the "four step method"
Conclusion: the scaling method works for some nonlocal problems but you have to take care

Related work

- \( u_t = J * u - u + G * u^2 - u^2, \int G = 1 \)


- \( u_t = u_{xx} + \int_{\mathbb{R}} K(x - y) \left( \frac{u(t,x) + u(t,y)}{2} \right)^2 dy, \int K = 0 \)
- \( u_t = J * u - u + \int_{\mathbb{R}} K(x - y) \left( \frac{u(t,x) + u(t,y)}{2} \right)^2 dy \)
- \( u_t = u_{xx} + K * u^2 = u_{xx} + \partial_x (G * u^2) \)
There are connections with Degasperis-Procesi (Coclite 2006, 2009)

\[ u_t + \partial_x \left[ \frac{u^2}{2} + G * \left( \frac{3}{2} u^2 \right) \right] = 0 \]

or Camassa-Holm

\[ u_t + \partial_x \left[ \frac{u^2}{2} + G * \left( u^2 + \frac{1}{2} (\partial_x u)^2 \right) \right] = 0 \]
Second term for the above models

\[ u_t = u_{xx} + G \ast u^q - u^q, \quad 1 < q < 2 \]

\[ u_t = J \ast u - u + G \ast u^q - u^q, \quad 1 < q < 2 \]

\[ u_t = u_{xx} + (u^{q-1}(K \ast u))_x, \quad 1 < q < 2 \]
THANKS for your attention !!!