Numerical bifurcation analysis of PSPM.

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Motivation

A General Physiologically Structured Population Model (PSPM)

Equilibria and bifurcations

- Equilibria
- Bifurcations

Numerical computation of equilibria and bifurcations

- One parameter variation
- Two parameter variation
- Examples
- Present and future work
Biological motivation:

- Many biological and ecological processes can be modeled with physiologically structured population models.
- Using dynamical systems and bifurcation theory in biological problems, it is possible to obtain biological conclusions from the mathematical results.
- Applications to cell biology, ecology, epidemiology, fishery industry...

Mathematical motivation:

- Formulate a general deterministic model VFE-DDE
- Analyze equilibria and bifurcations under parameter variation. ODE → VFE-DDE.
- Hopf bifurcation.

Computational motivation:

- Existing software can not compute equilibria and bifurcation curves for PSPM.
- Develop codes to compute equilibria and bifurcations under parameter variation, including Hopf bifurcation.
The model: 1 PSP interacts with an unstructured environment.

The environment:

- $\mathcal{E} = (I, E)$ unstructured environmental variables. $I$ interactions or measures, $E$ concentrations.
- Environmental history $\mathcal{E}_t(\theta) = \mathcal{E}(t + \theta)$, $\theta \leq 0$ → Incorporate history dependence

The structured population:

- i-state space $\mathcal{X}$.
- $X_0$ unique state at birth → All the individuals that at time $t$ have age $a$ have the same state $X(a, \mathcal{E}_t) \in \mathcal{X}$.
- Several life stages in which the processes at the i-level differ.

Dynamics at the individual level: The behavior of an individual with state $x := X(a, \mathcal{E}_t)$ is determined by

- Reproduction $\beta(x, \mathcal{E}(t))$
- Interaction $\gamma(x, \mathcal{E}(t))$
- Development $g(x, \mathcal{E}(t))$
- Mortality $\mu(x, \mathcal{E}(t))$
Dynamics at the population level:

- $b(t)$ population birth rate.
- $\mathcal{F}(a, \mathcal{E}_t)$ survival probability.

\[
\begin{align*}
    b(t) &= \int_0^h \beta(x, \mathcal{E}(t))\mathcal{F}(a, \mathcal{E}_t)b(t - a)\,da \\
    I(t) &= \int_0^h \gamma(x, \mathcal{E}(t))\mathcal{F}(a, \mathcal{E}_t)b(t - a)\,da \\
    \frac{d}{dt}E(t) &= F(\mathcal{E}(t))
\end{align*}
\] (1-3)

Particular case  \( F^i(\mathcal{E}(t)) = G^i(\mathcal{E}(t))E^i(t), \quad i \in \mathcal{I} \) (4)

- $x := X(a, \mathcal{E}_t)$ and $\mathcal{F}(a, \mathcal{E}_t)$ by solving the ODE system:

\[
\begin{align*}
    \frac{d}{d\alpha}x(\alpha) &= g(x(\alpha), \mathcal{E}_t(-a + \alpha)) \quad 0 < \alpha \leq a \\
    x(0) &= X_0 \\
    \frac{d}{d\alpha}\bar{\mathcal{F}}(\alpha) &= -\mu(x(\alpha), \mathcal{E}_t(-a + \alpha))\bar{\mathcal{F}}(\alpha) \quad 0 < \alpha \leq a \\
    \bar{\mathcal{F}}(0) &= 1
\end{align*}
\] (5-6)
**Equilibria:** \((b, I, E)\) such that \(b(t) = b, I(t) = I, E(t) = E\ \forall t.

- **Equilibrium conditions:**
  \[
  \bar{b} \left(1 - \int_0^h \beta(x, E)F(a, E) da\right) = 0 \quad (7)
  
  I - \bar{b} \int_0^h \gamma(x, E)F(a, E) da = 0 \quad (8)
  
  F(E) = 0 \quad (9)
  
- **Types of equilibrium states:**
  - Trivial \((b, I, E) = (0, 0, 0)\)
  - Nontrivial \(b \neq 0, I \neq 0, E \neq 0\)
  - \(b\)-trivial \(b = 0, I = 0, E \neq 0\)
  - \(E\)-trivial \(b \neq 0, I \neq 0, E = 0\)

- **Particular case:** \(F^i(E) = G^i(E)E^i, \quad i \in \mathcal{I}, \quad \mathcal{K} \subset \mathcal{I}\)
  - \(\mathcal{K}\)-trivial \(b \neq 0, I \neq 0, E^i = 0\) for \(i \in \mathcal{K}, E^j \neq 0\) for \(j \notin \mathcal{K}\)
  - \((b, \mathcal{K})\)-trivial \(b = 0, I = 0, E^i = 0\) for \(i \in \mathcal{K}, E^j \neq 0\) for \(j \notin \mathcal{K}\)
Bifurcations: under one parameter variation

- Transcritical
- Saddle-node

One parameter variation:

- Compute equilibrium curve
  \[ F(x, \alpha) = 0, \quad F : \mathbb{R}^{n+1} \to \mathbb{R}^n \quad \rightarrow \text{numerical continuation (predictor-corrector)} . \]
- Detect bifurcation points \( \rightarrow \text{test functions} \).
  - Transcritical \text{not regular point} \( \rightarrow \text{decompose the model: regular curve + test function} \)
  - Saddle node \( \det(F_x(x, \alpha)) = 0 \) (same as ODE)
  - Hopf
- Decompose model

\[ \begin{align*}
(b, l, E, \alpha)_0 & \quad \rightarrow \quad \text{Decompose model} \\
F(E) & \quad \rightarrow \quad \text{Equilibrium conditions} \\
g, \mu, \beta, \gamma & \quad \rightarrow \quad \text{Compute curve: apply numerical continuation} \\
\text{parameters} & \quad \rightarrow \quad \text{At each step, detect bifurcation} \\
\text{other ingredients} & \quad \rightarrow \quad \text{Compute bifurcation. Newton}
\end{align*} \]
Two parameter variation:

- Compute bifurcation curve
  \[ F(x, \alpha) = 0, \quad F : \mathbb{R}^{n+1} \to \mathbb{R}^n \] defined by Equilibrium conditions + test function
  \[ \rightarrow \text{numerical continuation (predictor-corrector)}. \]

- Transcritical \[ \rightarrow \text{decomposed model: regular equilibrium + test function} \]
- Saddle node (bordering technique, same as ODE)
- Hopf

\[ (b, I, E, \alpha)_0 \rightarrow \text{Decompose model} \]
\[ \rightarrow \text{Equilibrium conditions} \]
\[ \rightarrow \text{Compute curve: apply numerical continuation} \]
\[ \rightarrow \text{Bifurcation conditions} \]
Examples:

- Consumer-Resource model (Daphnia).


Sánchez, de Roos, Diekmann, Getto. Numerical bifurcation analysis of PSPM.
Examples:

- Predator-Prey-Resource model.

Examples:

- Cannibalistic model with two resources.
- Non cannibalistic model with two resources.


Present work:

1. Hopf bifurcation.
   ▶ Test functions.
   ▶ Characteristic equation.


Future plans:

1. Extend the model to several PSP.
2. Analyze stability properties of PSPM using numerical spectral methods.
THANK YOU FOR YOUR ATTENTION