On fully mixed and multidimensional extensions of the Caputo and Riemann-Liouville derivatives, related Markov processes and fractional differential equations

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Abstract

From the point of view of stochastic analysis the Caputo and Riemann-Liouville derivatives of order $\alpha \in (0,2)$ can be viewed as (regularized) generators of stable Lévy motions interrupted on crossing a boundary. This interpretation naturally suggests fully mixed, two-sided or even multidimensional generalizations of these derivatives, as well as a probabilistic approach to the analysis of the related equations. These extensions are introduced and some well-posedness results are obtained that generalize, simplify and unify lots of known facts. This probabilistic analysis leads one to study a class of Markov processes that can be constructed from any given Markov process in $\mathbf{R}^d$ by blocking (or interrupting) the jumps that attempt to cross certain closed set of 'check-points'.

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1 Introduction

For a general background in fractional calculus and fractional equations we refer to books [6], [14], [28], [29], see also survey [24], for the crucial link with CTRW to [32], [16] and [27], and for the numerous applications in natural science to [33] and [34].

The aim of this paper is to present a systematic treatment of a class of equations that include fractional derivatives as very particular cases. Unlike the mostly analytic studies of fractional differential equations (see the reference above), the present treatment and the corresponding far reaching extensions of fractional derivatives are based on a probabilistic point of view. This link with probability provides a powerful tool for the study of fractional equations. It is also worth mentioning the recent activity on proving probabilistic interpretation of solutions by analytic methods, see e.g. [11], while our approach provides such an interpretation as a starting point.

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From this point of view the basic Caputo and Riemann-Liouville (RL) derivatives of order $\alpha \in (0,2)$ can be viewed as (regularized) generators of stable Lévy motions interrupted on crossing a boundary. This interpretation naturally suggests fully mixed, two-sided or even multidimensional generalizations of these derivatives, as well as a probabilistic approach to the analysis of the related equations. These extensions are introduced leading to well-posedness results that generalize, simplify and unify lots of known facts. Some explicit solutions are also obtained. The corresponding probabilistic analysis leads one to study an interesting general class of Markov processes that can be constructed from any given Markov process in $\mathbf{R}^d$ by blocking (or interrupting) the jumps that attempt to cross certain closed set of check-points. This analysis is only initiated in the present work. Further development, as well as the application of this technique to the study of fractional in time and space diffusion equations and to the convergence of CTRW, will be discussed in separate publications.

The paper is organized as follows. In the next preliminary section we introduce in a convenient form the main objects of fractional calculus, the Caputo and RL derivatives.

In Section 3 we explain in detail the probabilistic meaning of these derivatives and their natural place in stochastic analysis leading, on the one hand side, to far reaching generalizations, and on the other hand, to a unified treatment of various equations by powerful tools of stochastic analysis. We distinguish the cases of $\beta \in (0,1)$ and $\beta \in (1,2)$, because in the first case the Caputo derivative has a direct probabilistic interpretation and in the second an additional regularization is needed. We also treat separately a multidimensional extension, as it includes one additional ingredient, the projection of a jump on a boundary.

The final Section 4 initiates the rigorous theory of the equations and processes introduced above by providing some basic examples (not at all exhaustive) of well-posedness results and explicit solutions that can be obtained by these tools, with main attention restricted to the analogs of the derivatives of order $\beta \leq 1$.

Appendix derives some equivalent versions of basic fractional derivatives mentioned in the next section without proof. These calculations should be obvious for specialists in fractional calculus and are given here for completeness.

We shall denote by $\mathbf{1}_M$ the indicator function of a set M.

2 Preliminaries: classical fractional derivatives

For convenience we recall here the basic definitions of fractional derivatives and their equivalent representations fitting our purposes.

We shall repeatedly use the Euler identity $\Gamma(x) = (x-1)\Gamma(x-1)$ for the Euler Gamma-function whenever appropriate without mentioning it.

Due to the formula for the iterated Riemann integral

$$I_a^n f(x) = \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} f(t) dt, \tag{1}$$

it is natural to extend it analytically, if $x > a$, to complex n with positive real part, leading to the following definition of the (right) fractional or Riemann-Liouville (RL) integral of order β (with positive real part):

$$I_a^\beta f(x) = I_{a+}^\beta f(x) = \frac{1}{\Gamma(\beta)} \int_a^x (x-t)^{\beta-1} f(t) dt. \tag{2}$$

Notice that for $x < a$, we have negative numbers $(x-t)$ in power $n-1$, so that the corresponding extension to complex (or even real) n leading to the so-called left RL integrals have some subtleties to be discussed later.

Noting that the derivation is the inverse operation to usual integration, the above definition suggests two notions of fractional derivative, the so-called RL (right) derivatives of order $\beta > 0, \beta \notin \mathbf{N}$:

$$D^{\beta}_{a+}f(x) = \frac{d^n}{dx^n} I^{n-\beta}_{a+} f(x) = \frac{1}{\Gamma(n-\beta)} \frac{d^n}{dx^n} \int_a^x (x-t)^{n-\beta-1} f(t)dt, \quad x > a, \tag{3}$$

and the so-called Caputo (right) derivative of order $\beta > 0, \beta \notin \mathbf{N}$:

$$D^{\beta}_{a+\star}f(x) = I^{n-\beta}_{a} \left[\frac{d^n}{dx^n} f\right](x) = \frac{1}{\Gamma(n-\beta)} \int_a^x (x-t)^{n-\beta-1} \left[\frac{d^n}{dt^n} f\right](t)dt, \quad x > a \tag{4}$$

where n is the maximal integer that is strictly less than $\beta + 1$.

Straightforward integration by parts (see Appendix) show that, for smooth enough f and $\beta \in (0,1)$, $x > a$,

$$D^{\beta}_{a+}f(x) = \frac{1}{\Gamma(-\beta)} \int_0^{x-a} \frac{f(x-z)-f(x)}{z^{1+\beta}} dz + \frac{f(x)}{\Gamma(1-\beta)(x-a)^{\beta}}, \tag{5}$$

$$D^{\beta}_{a+\star}f(x) = \frac{1}{\Gamma(-\beta)} \int_0^{x-a} \frac{f(x-z)-f(x)}{z^{1+\beta}} dz + \frac{f(x)-f(a)}{\Gamma(1-\beta)(x-a)^{\beta}}, \tag{6}$$

implying

$$D^{\beta}_{a+\star}f(x) = D^{\beta}_{a+}[f-f(a)](x) = D^{\beta}_{a+}f(x) - \frac{f(a)}{\Gamma(1-\beta)|x-a|^{\beta}}. \tag{7}$$

In particular it follows that for smooth bounded integrable functions, the right RL and Caputo derivatives coincide for $a = -\infty$, $\beta \in (0,1)$, and one defines the fractional derivative in generator form as their common value:

$$\frac{d^{\beta}}{dx^{\beta}} f(x) = D^{\beta}_{-\infty+}f(x) = D^{\beta}_{-\infty+\star}f(x) = \frac{1}{\Gamma(-\beta)} \int_0^{\infty} \frac{f(x-z)-f(x)}{z^{1+\beta}} dz. \tag{8}$$

Analogously (see Appendix for detail), for smooth enough f and $\beta \in (1,2)$, $x > a$, one finds that

$$D^{\beta}_{a+}f(x) = \frac{1}{\Gamma(-\beta)} \int_0^{x-a} \frac{f(x-z)-f(x)+f'(x)z}{z^{1+\beta}} dz + \frac{f(x)(x-a)^{-\beta}}{\Gamma(1-\beta)} + \frac{\beta f'(x)(x-a)^{1-\beta}}{\Gamma(2-\beta)}, \tag{9}$$

$$D^{\beta}_{a+\star}f(x) = \frac{1}{\Gamma(-\beta)} \int_0^{x-a} \frac{f(x-z)-f(x)+f'(x)z}{z^{1+\beta}} dz$$
$$+ \frac{(f(x)-f(a))(x-a)^{-\beta}}{\Gamma(1-\beta)} + \frac{(\beta f'(x)-f'(a))(x-a)^{1-\beta}}{\Gamma(2-\beta)}, \tag{10}$$

so that

$$D^{\beta}_{a+\star}f(x) = D^{\beta}_{a+}[f-f(a)-f'(a)(.-a)](x) = D^{\beta}_{a+}f(x) - \frac{f(a)(x-a)^{-\beta}}{\Gamma(1-\beta)} - \frac{f'(a)(x-a)^{1-\beta}}{\Gamma(2-\beta)}. \tag{11}$$

Again for smooth bounded integrable functions, the right RL and Caputo derivatives coincide for $a = -\infty$, $\beta \in (1,2)$, and one defines the fractional derivative in generator form as their common value:

$$\frac{d^\beta}{dx^\beta} f(x) = D^\beta_{-\infty+} f(x) = D^\beta_{-\infty+\star} f(x) = \frac{1}{\Gamma(-\beta)} \int_0^\infty \frac{f(x-z) - f(x) + f'(x)z}{z^{1+\beta}} dz. \quad (12)$$

Turning to the left derivative notice that for $x < a$ formula (1) rewrites as

$$I^n_a f(x) = \frac{(-1)^n}{(n-1)!} \int_x^a (t-x)^{n-1} f(t) dt, \quad (13)$$

suggesting several possible normalizations for the analytic continuation in n and the corresponding inversions (fractional derivatives). We are interested only in the derivatives of order less than 2. For a unified probabilistic interpretation of these derivatives it is convenient to choose the left versions of (3), (4) as follows. For $\beta \in (0,1)$:

$$D^\beta_{a-} f(x) = -\frac{1}{\Gamma(1-\beta)} \frac{d}{dx} \int_x^a (t-x)^{-\beta} f(t)\, dt, \quad x < a, \quad (14)$$

$$D^\beta_{a-\star} f(x) = -\frac{1}{\Gamma(1-\beta)} \int_x^a (t-x)^{-\beta} f'(t)\, dt, \quad x < a; \quad (15)$$

for $\beta \in (1,2)$:

$$D^\beta_{a-} f(x) = \frac{1}{\Gamma(2-\beta)} \frac{d^2}{dx^2} \int_x^a (t-x)^{1-\beta} f(t)\, dt, \quad x < a, \quad (16)$$

$$D^\beta_{a-\star} f(x) = \frac{1}{\Gamma(2-\beta)} \int_x^a (t-x)^{1-\beta} f''(t)\, dt, \quad x < a. \quad (17)$$

When $\beta \in (0,1)$ and $x < a$, similar calculations as for the right derivative (see (109)) lead to the following analogs of (5), (6):

$$D^\beta_{a-} f(x) = \frac{1}{\Gamma(-\beta)} \int_0^{a-x} \frac{f(x+z) - f(x)}{z^{1+\beta}} dz + \frac{f(x)}{\Gamma(1-\beta)(a-x)^\beta}, \quad (18)$$

$$D^\beta_{a-\star} f(x) = \frac{1}{\Gamma(-\beta)} \int_0^{a-x} \frac{f(x+z) - f(x)}{z^{1+\beta}} dz + \frac{f(x) - f(a)}{\Gamma(1-\beta)(a-x)^\beta}, \quad (19)$$

implying

$$D^\beta_{a-\star} f(x) = D^\beta_{a-}[f - f(a)](x) = D^\beta_{a-} f(x) - \frac{f(a)}{\Gamma(1-\beta)(a-x)^\beta}. \quad (20)$$

When $\beta \in (1,2)$, $x < a$, one obtains

$$D^\beta_{a-} f(x) = \frac{1}{\Gamma(-\beta)} \int_0^{a-x} \frac{f(x+z) - f(x) - f'(x)z}{z^{1+\beta}} dz + \frac{f(x)(a-x)^{-\beta}}{\Gamma(1-\beta)} - \frac{\beta f'(x)(a-x)^{1-\beta}}{\Gamma(2-\beta)}, \quad (21)$$

$$D^\beta_{a-\star} f(x) = \frac{1}{\Gamma(-\beta)} \int_0^{a-x} \frac{f(x+z) - f(x) - f'(x)z}{z^{1+\beta}} dz$$

$$+\frac{(f(x)-f(a))(a-x)^{-\beta}}{\Gamma(1-\beta)}-\frac{(\beta f'(x)-f'(a))(a-x)^{1-\beta}}{\Gamma(2-\beta)}, \tag{22}$$

so that

$$D^\beta_{a-\star}f(x)=D^\beta_{a-}[f-f(a)-f'(a)(.-a)](x)=D^\beta_{a-}f(x)-\frac{f(a)(a-x)^{-\beta}}{\Gamma(1-\beta)}+\frac{f'(a)(a-x)^{1-\beta}}{\Gamma(2-\beta)}. \tag{23}$$

For smooth bounded integrable functions the left fractional derivatives in generator form become

$$\frac{d^\beta}{d(-x)^\beta}f(x)=D^\beta_{\infty-}f(x)=D^\beta_{\infty-\star}f(x)=\frac{1}{\Gamma(-\beta)}\int_0^\infty\frac{f(x+z)-f(x)}{z^{1+\beta}}dz, \tag{24}$$

$$\frac{d^\beta}{d(-x)^\beta}f(x)=D^\beta_{\infty-}f(x)=D^\beta_{\infty-\star}f(x)=\frac{1}{\Gamma(-\beta)}\int_0^\infty\frac{f(x+z)-f(x)-f'(x)z}{z^{1+\beta}}dz, \tag{25}$$

for $\beta\in(0,1)$ and $\beta\in(1,2)$ respectively.

It is straightforward to see that the pairs of operators (8), (24) and (12), (25) are dual in the sense that

$$\left(\frac{d^\beta}{dx^\beta}f,g\right)=\left(f,\frac{d^\beta}{d(-x)^\beta}g\right)$$

for $\beta\in(0,1)\cup(1,2)$ and sufficiently regular functions f,g, where the pairing (f,g) denotes of course the usual L^2-product: $(f,g)=\int f(x)g(x)dx$. This fact also justifies the notation $d^\beta/d(-x)^\beta$, since for $\beta=1$ the operators d/dx and $-d/dx=d/d(-x)$ are dual.

3 Probabilistic interpretation and extensions of RL and Caputo derivatives

3.1 The case $\beta\in(0,1), d=1$

It is well known that the operators $-d^\beta/d(-x)^\beta$ and $d^\beta/d(-x)^\beta$ from (24) and (25) respectively are the generators of stable Lévy motions without negative jumps, the (annoying) discrepancy in the sign reflects the fact that $\Gamma(-\beta)<0$ for $\beta\in(0,1)$ and $\Gamma(-\beta)>0$ for $\beta\in(1,2)$. In particular, for $\beta\in(0,1)$, the corresponding Lévy process is a stable subordinator (an increasing process). Similarly, the operators $-d^\beta/dx^\beta$ and d^β/dx^β for $\beta\in(0,1)$ and $\beta\in(1,2)$ respectively generate stable Lévy motions without positive jumps, which can be obtained by inversion of the processes generate by $-d^\beta/d(-x)^\beta$ and $d^\beta/d(-x)^\beta$.

Let us now look specifically at the decreasing process $X(t)$ generated by $A=-d^\beta/dx^\beta$, $\beta\in(0,1)$. Let us modify it by forbidding it (interrupting on an attempt) to cross a boundary $x=a$ with an $a\in\mathbf{R}$, that is, all jumps aimed to land to the left of the chosen barrier-point a are forced to land exactly at a. Analytically, this procedure means changing the generator A to

$$A_{a+\star}f(x)=-\frac{1}{\Gamma(-\beta)}\int_0^{x-a}\frac{f(x-z)-f(x)}{z^{1+\beta}}\,dz-\frac{1}{\Gamma(-\beta)}\int_{x-a}^\infty\frac{f(a)-f(x)}{z^{1+\beta}}\,dz,\quad x>a,$$

which rewrites as

$$A_{a+\star}f(x)=-\frac{1}{\Gamma(-\beta)}\int_0^{x-a}\frac{f(x-z)-f(x)}{z^{1+\beta}}\,dz+\frac{f(a)-f(x)}{\Gamma(1-\beta)(x-a)^\beta}\,dz,\quad x>a. \tag{26}$$

In this expression we recognize the Caputo derivative (4), with inverted sign. Killing the process at the boundary-point $x = a$ means analytically to set $f(a) = 0$, in which case (26) turns to RL fractional derivative (with inverted sign).

We conclude that the transition from the free derivative d^β/dx^β to the Caputo right derivative at a is a particular case of the procedure of interrupting a decreasing process on crossing a boundary. Namely, let an operator

$$Af(x) = \int_0^\infty (f(x-y) - f(x))\nu(x, dy)$$

with a kernel $\nu(x, .)$ on $(0, \infty)$ such that

$$\sup_x \int_0^\infty \min(1, |y|)\nu(x, dy) < \infty \tag{27}$$

generate a decreasing Feller process. Then the corresponding process interrupted (and hence stopped) at a boundary $x = a$ has the generator

$$A_{a+\star} f(x) = \int_0^{x-a} (f(x-y) - f(x))\nu(x, dy) + (f(a) - f(x)) \int_{x-a}^\infty \nu(x, dy), \quad x > a. \tag{28}$$

Similarly, the transition from the free derivative $d^\beta/d(-x)^\beta$ to the Caputo left derivative at a is a particular case of the procedure of interrupting an increasing process on crossing a boundary, that is, the transition from a process on $\mathbf{R}$ generated by

$$Af(x) = \int_0^\infty (f(x+y) - f(x))\nu(x, dy)$$

to the process on the interval $(-\infty, a]$ generated by

$$A_{a-\star} f(x) = \int_0^{a-x} (f(x+y) - f(x))\nu(x, dy) + (f(a) - f(x)) \int_{a-x}^\infty \nu(x, dy), \quad x < a. \tag{29}$$

From this point of view, the natural extension of this procedure to the general processes of bounded variation generated by the operators

$$Af(x) = \gamma(x) f'(x) + \int_{-\infty}^\infty (f(x+y) - f(x))\nu(x, dy) \tag{30}$$

with a kernel $\nu(x, .)$ on $\mathbf{R} \setminus \{0\}$ such that

$$\sup_x \int_{-\infty}^\infty \min(1, |y|)\nu(x, dy) < \infty \tag{31}$$

is a transition to the process in a given interval $[a, b]$ with arbitrary $-\infty \le a < b \le \infty$ with jumps interrupted on an attempt to cross the boundary (i.e. all jumps aiming to land outside $[a, b]$ are forced to land on its nearest point), that is the process generated by the operator

$$A_{[a,b]\star} f(x) = \gamma(x) f'(x) + \int_{a-x}^{b-x} (f(x+y) - f(x))\nu(x, dy)$$

$$+(f(b)-f(x))\int_{b-x}^{\infty}\nu(x,dy)+(f(a)-f(x))\int_{-\infty}^{a-x}\nu(x,dy), \quad x\in(a,b). \tag{32}$$

To deal with fractional derivatives we are mostly interested in the motion inside $[a,b]$. However, for the analysis it is convenient to be able to have the corresponding process extended to all $\mathbf{R}$. This can be done in various ways, though two natural approaches can be distinguished. In the first one all jumps are supposed to be restricted to jump in the direction of D only, for instance one can choose the extension of $A_{[a,b]\star}$ to all x is given by the expression

$$A_{[a,b]\star}f(x)=\gamma(x)f'(x)+\mathbf{1}_{x<b}\int_0^{\infty}(f[(x+y)\wedge b]-f(x))\nu(x,dy)$$

$$+\mathbf{1}_{x>a}\int_{-\infty}^{0}(f[(x+y)\vee a]-f(x))\nu(x,dy). \tag{33}$$

In the second approach we stick to the idea of interruption on crossing the boundary, so that D and its complement are treated symmetrically. To present this approach in a proper generality assume a finite set $B=\{b_1<\cdots<b_k\}$ or a countable set $B=\{b_i, i\in\mathbf{N}\}$, with $b_i<b_{i+1}$ for any i, is chosen. For instance, in the above setting B is a two-point set $B=\{a,b\}$. For any $x\in\mathbf{R}$ let us now define $b_+(x)$ and $b_-(x)$ as the nearest point of B to the right and to the left of x respectively (x excluded in both cases even if $x\in B$). Then the modification of the process on $\mathbf{R}$ generated by (30), with jumps interrupted on crossing B (think of B as a set of road blocks or check points placed to control the free motion given by A) can be specified by the generator

$$A_{B\star}f(x)=\gamma(x)f'(x)+\int_{-\infty}^{\infty}(f([(x+y)\wedge b_+(x)]\vee b_-(x))-f(x))\nu(x,dy). \tag{34}$$

Clearly for $x\in(b_i,b_{i+1})$ this generator coincides with (32), where $a=b_i, b=b_{i+1}$. In order to study the fractional differential equations in intervals we can choose to work with the processes given by (33) or (34), as their behavior until they reach the boundary ∂D is identical. For more general domains D these different extensions can lead of course to different problems.

In [18], [19] the author referred to generators of type (30) as to the generators of order at most one aiming to stress that they can be considered as fully mixed fractional derivatives of order not exceeding one.

If we kill the process generated by A at the boundary, the generator (32) turns to

$$A_{[a,b]}f(x)=\int_{a-x}^{b-x}(f(x+y)-f(x))\nu(x,dy)-f(x)\left[\int_{b-x}^{\infty}\nu(x,dy)+\int_{-\infty}^{a-x}\nu(x,dy)\right], \tag{35}$$

for $x\in(a,b)$, which represents the corresponding extension of RL fractional derivative.

Therefore, from probabilistic point of view, the natural extension of the basic linear fractional equation

$$D^{\beta}_{a+\star}f(x)=-\lambda f(x), \quad x>a, \tag{36}$$

with initial condition $f_a=f(a)$ and $\lambda>0$ given, is the problem of finding f on $[a,\infty)$ satisfying

$$A_{a+\star}f(x)=\lambda f(x), \quad f(a)=f_a,$$

with $A_{a+\star}$ given by (28), that is the equation

$$\int_0^{x-a} (f(x-y)-f(x))\nu(x,dy) + (f(a)-f(x)) \int_{x-a}^{\infty} \nu(x,dy) = \lambda f(x), \quad f(a)=f_a. \tag{37}$$

This equation includes the extensions of (36) with various mixed fractional derivatives, that is, the problem

$$\sum_j \omega_j D^{\beta_j}_{a+\star} f(x) = -\lambda f(x), \quad f(a)=f_a, \tag{38}$$

with some finite collection of numbers $\omega_j > 0$ and $\beta_j \in (0,1)$ or even more exotic versions with ω_j or β_j being functions of x. Mixed derivatives and related fractional equations are actively studied recently by analytical methods, see e. g. [7], [29], [30], [12].

Moreover, it is now natural to formulate the two-sided version of (36) as the equation

$$D^{\beta}_{a+\star} f(x) + D^{\beta}_{b-\star} f(x) = -\lambda f(x), \quad f(a)=f_a, f(b)=f_b \tag{39}$$

on the interval $x \in [a,b]$, which represents the simplest case of a more general equation

$$A_{[a,b]\star} f(x) = \lambda f(x), \quad f(a)=f_a, f(b)=f_b \tag{40}$$

with $A_{[a,b]\star}$ given by (33). In particular, the mixed-derivatives extension of (39) is the problem

$$-\sum_{j=1}^k \omega_j D^{\beta_j}_{a+\star} f(x) - \sum_{j=1}^k \gamma_j D^{\beta_j}_{b-\star} f(x) = \lambda f(x), \quad f(a)=f_a, f(b)=f_b, \tag{41}$$

with some collection of numbers (or, more generally, functions) $\omega_j > 0, \gamma_j > 0, \beta_j \in (0,1)$ and $\lambda > 0$ extending for instance the problem considered in [10]. The well-posedness for this problem is covered by Theorem 4.2 below.

As solutions to the equation $D_{a+\star} f(x) = g(x)$ with a given g and initial condition $f(a) = a$ can be given in terms of fractional integrals, the two-sided analog of fractional integral becomes the solution to the problem

$$D^{\beta}_{a+\star} f(x) + D^{\beta}_{b-\star} f(x) = -g(x), \quad f(a)=f_a, f(b)=f_b, \tag{42}$$

with given f_a, f_b and g on $[a,b]$, which again represents the simplest case of a more general problem

$$A_{[a,b]\star} f(x) = g(x), \quad f(a)=f_a, f(b)=f_b, \tag{43}$$

or with the linear term included

$$A_{[a,b]\star} f(x) = \lambda f(x) + g(x), \quad f(a)=f_a, f(b)=f_b, \tag{44}$$

suggesting a new class of extensions of the notion of fractional integral, which is alternative to a more classical one, see [15], which is based on the variations of special functions used as the kernels (which is natural from an analytic perspective).

Remark 1. *Notice that equation* (44) *can be rewritten in terms of RL derivative. Namely, let* $v(x)$ *be a smooth function on* $[a,b]$ *with* $v(a)=f_a, v(b)=f_b$*. Then for the function* $\phi = f - v$ *problem* (44) *rewrites as*

$$A_{[a,b]\star}\phi = \lambda\phi + g - A_{[a,b]\star}v + \lambda v, \quad \phi(a)=\phi(b)=0, \tag{45}$$

which is equivalent to

$$A_{[a,b]}\phi = \lambda\phi + \tilde{g}, \quad \phi(a)=\phi(b)=0, \quad \tilde{g} = g + (\lambda - A_{[a,b]\star})v. \tag{46}$$

Particular examples of boundary value problems with fractional derivatives are actively studied, mostly by analytical techniques, see e. g. [13], [14] or [26] and references therein, for distributed or mixed derivatives see also [25], [11] and [35].

3.2 The case $\beta \in (0,1), d > 1$

Let us now turn to the multidimensional extension of this interruption procedure. The analog of RL derivative arising from a process in $\mathbf{R}^d$ and a domain $D \subset \mathbf{R}^d$ is the generator of the process killed on leaving D. For Caputo version this is more subtle, as we have to specify a point where a jump crosses the boundary. Below we choose the most natural model assuming that a trajectory of a jump follows shortest path (a straight line in Euclidean case). Suppose A is a generator of a Feller process $X_t(x)$ in $\mathbf{R}^d$ with the generator of type

$$Af(x) = (\gamma(x), \nabla)f(x) + \int_{\mathbf{R}^d}(f(x+y)-f(x))\nu(x,dy) \tag{47}$$

with a kernel $\nu(x,.)$ on $\mathbf{R}^d \setminus \{0\}$ such that

$$\sup_x \int_{\mathbf{R}^d} \min(1,|y|)\nu(x,dy) < \infty, \tag{48}$$

that is, in the terminology of [18], [19], a generator or order at most one.

Let D be an open convex subset of $\mathbf{R}^d$ with boundary ∂D and closure $\bar{D}$. For $x \in \mathbf{R}^d$ and a unit vector e let $L_{x,e} = \{x+\lambda e, \lambda \geq 0\}$ be the ray drawn from x in the direction e. For $x \in \mathbf{R}^d$, let

$$D(x) = \{y \in \mathbf{R}^d : L_{x,y/|y|} \cap \bar{D} \neq \emptyset\}.$$

In particular, $D(x) = \bar{D}$ for all $x \in D$. Furthermore, for $y \in D(x)$, let

$$\lambda(x, y/|y|) = \max\{R > 0 : x + Ry/|y| \in \bar{D}\}.$$

Let us introduce the restriction function $R_D(x,y)$, $x \in \mathbf{R}^d$, $y \in D(x)$, by the formula

$$R_D(x,y) = \begin{cases} x+y, & \text{if } |y| \leq \lambda(x,y/|y|) \\ x + \lambda(x,y/|y|)y/|y|, & \text{if } |y| \geq \lambda(x,y/|y|) \end{cases} \tag{49}$$

The process $X_t(x)$ with jumps interrupted on crossing ∂D when jumping from inside D and forced to jump in the direction of $\bar{D}$ when jumping from outside of D can be defined by the generator

$$A_{D\star}f(x) = (\gamma(x), \nabla)f(x) + \int_{D(x)}[f(R_D(x,y)) - f(x)]\nu(x,dy), \tag{50}$$

which represents a multidimensional extension of the Caputo boundary operator in D arising from (33).

To define a multidimensional analog of (34) let us assume more generally that $\tilde{D}$ is an arbitrary open subset of $\mathbf{R}^d$, so that $B = \mathbf{R}^d \setminus \tilde{D}$ is closed. Let us define

$$\tilde{\lambda}(x, y/|y|) = \min\{R > 0 : x + Ry/|y| \in B\}$$

for any y, with the convention that $\tilde{\lambda}(x, y/|y|) = \infty$ if the ray $L_{x,y/|y|}$ does not intersect B at all, and let the projection-on-the-boundary function $\tilde{R}_B(x, y)$ be defined for all $x, y \in \mathbf{R}^d$ by the same formula (49), but with $\tilde{\lambda}$ instead of λ. Then the analog of (34), that is the modification of the process on $\mathbf{R}^d$ generated by (47), obtained by interrupting jumps on an attempt to cross B, is specified by the generator

$$A_{\tilde{D}\star} f(x) = (\gamma(x), \nabla) f(x) + \int_{\mathbf{R}^d} [f(\tilde{R}_D(x, y)) - f(x)] \nu(x, dy). \tag{51}$$

For a convex domain D the operators (50) and (51) with $B = \partial D$ and $\tilde{D} = \mathbf{R}^d \setminus B$ coincide for $x \in D$, so that when one is interested in the random motion inside $\bar{D}$ one can work with either of the processes generated by (50) or (51) and stopped on the boundary of D.

As pointed out above, the process $X_t(x)$ killed on the boundary, which represents a multidimensional extension of the RL boundary operator in D arising from (46), is specified by the generator

$$A_D f(x) = \int_{\mathbf{R}^d} [f(x + y)\mathbf{1}_{x+y \in \bar{D}} - f(x)] \nu(x, dy), \quad x \in D. \tag{52}$$

Remark 2. *In dimension $d = 1$ the projection $z \to R_{(a,b)}(x, z) = [z \wedge a] \vee b$ of the real numbers to the interval $[a, b]$ (used in (33)) does not depend on x and is clearly the most natural one. In higher dimensions one can imagine several reasonable extensions. Our choice used above was meant to preserve the direction of a jump. Another reasonable choice could be the definition of the projection $R_D(x, z)$ as the point on $\bar{D}$ nearest to z (both choices coincide in $d = 1$). This would lead to a different multi-dimensional extension of the Caputo derivative.*

Assuming for simplicity that the kernel ν has a density, $\nu(x; y)$, with respect to Lebesgue measure, consider the following three basic examples.

If D is a half space

$$D = D_b = \{(x_1, x_2) \in \mathbf{R}^{d+1} : x_1 < b, x_2 \in \mathbf{R}^d\}, \tag{53}$$

then

$$R_D(x, y) = \tilde{R}_D(x, y) = \left(b, x_2 + \frac{b - x_1}{y_1} y_2\right), \quad x_1 < b \le x_1 + y_1,$$

$$A_{D\star} f(x) = \int_{\mathbf{R}^d} dy_2 \int_{-\infty}^{b-x_1} dy_1 \nu(x; y) [f(x + y) - f(x)]$$

$$+ \int_{\mathbf{R}^d} dy_2 \int_{b-x_1}^{\infty} dy_1 \nu(x; y) [f\left(b, x_2 + \frac{b - x_1}{y_1} y_2\right) - f(x)]. \tag{54}$$

If D is a band

$$D = D_{(a,b)} = \{(x_1, x_2) \in \mathbf{R}^{d+1} : a < x_1 < b, x_2 \in \mathbf{R}^d\}, \tag{55}$$

then, for $x_1 \in (a, b)$,

$$\begin{aligned} A_{D\star} f(x) &= \int_{\mathbf{R}^d} dy_2 \int_{a-x_1}^{b-x_1} dy_1 \nu(x; y)[f(x+y) - f(x)] \\ &+ \int_{\mathbf{R}^d} dy_2 \int_{b-x_1}^{\infty} dy_1 \nu(x; y)[f\left(b, x_2 + \frac{b-x_1}{y_1} y_2\right) - f(x)] \\ &+ \int_{\mathbf{R}^d} dy_2 \int_{-\infty}^{a-x_1} dy_1 \nu(x; y)[f\left(a, x_2 + \frac{a-x_1}{y_1} y_2\right) - f(x)]. \end{aligned} \tag{56}$$

If D is the unit ball $D = \{x : |x| < 1\}$ in $\mathbf{R}^d$, then, for $x \in D$,

$$R_D(x, y) = \tilde{R}_D(x, y) = x + \lambda(x, y) y, \quad \lambda(x, y) = \frac{1}{|y|} \left[\sqrt{\left(x, \frac{y}{|y|}\right)^2 + 1 - |x|^2} - \left(x, \frac{y}{|y|}\right) \right],$$

$$A_{D\star} f(x) = \int_{\mathbf{R}^d} dy \, \nu(x; y) \left(\mathbf{1}_{\lambda(x,.)\geq 1}[f(x+y) - f(x)] + \mathbf{1}_{\lambda(x,.)<1}[f(x + \lambda(x, y) y) - f(x)] \right). \tag{57}$$

3.3 The case $\beta \in (1, 2)$

Let us now look at the derivatives of order $\beta \in (1, 2)$. Interrupting the Lévy process with only negative jumps generated by (12) on crossing the boundary $\{x = a\}$ means changing its generator to the operator

$$\tilde{D}^\beta_{a+} f(x) = \frac{1}{\Gamma(-\beta)} \int_0^{x-a} \frac{f(x-z) - f(x) + f'(x) z}{z^{1+\beta}} dz + \frac{1}{\Gamma(-\beta)} \int_0^{x-a} \frac{f(a) - f(x) + f'(x) z}{z^{1+\beta}} dz, \tag{58}$$

which rewrites as

$$\begin{aligned} \tilde{D}^\beta_{a+} f(x) &= \frac{1}{\Gamma(-\beta)} \int_0^{x-a} \frac{f(x-z) - f(x) + f'(x) z}{z^{1+\beta}} dz \\ &+ \frac{(f(x) - f(a))(x-a)^{-\beta}}{\Gamma(1-\beta)} + \frac{\beta f'(x)(x-a)^{1-\beta}}{\Gamma(2-\beta)}. \end{aligned} \tag{59}$$

This expression differs from the Caputo derivative (10) by the term containing $f'(a)$:

$$D^\beta_{a+\star} f(x) = \tilde{D}^\beta_{a+} f(x) - \frac{f'(a)(x-a)^{1-\beta}}{\Gamma(2-\beta)}, \tag{60}$$

which one could expect as operator (10) does not have a structure that allows one to interpret it as a generator for a Markov process precisely because of this term containing $f'(a)$ (recall that $\Gamma(-\beta) > 0$ and $\Gamma(1-\beta) < 0$ here).

In order to see the meaning of this correcting term let us observe that if $f \in C^2[a,\infty)$, then, up to terms tending to zero, $\tilde{D}^{\beta}_{a+} f(x)$ behaves like

$$f'(x)(x-a)^{1-\beta}\left[\frac{1}{\Gamma(1-\beta)}+\frac{\beta}{\Gamma(2-\beta)}\right]=\frac{f'(x)(x-a)^{1-\beta}}{\Gamma(2-\beta)},$$

as $x \to a$, and thus tends to $\pm\infty$ if $f'(a)$ is positive or negative respectively. Thus subtraction of the term containing $f'(a)$ in (60) is a regularization of $\tilde{D}^{\beta}_{a+}$ that makes it finite on smooth functions.

On the other hand, killing the process generated by (60) at the boundary point $x = a$ means setting $f(a) = 0$ and then (60) turns exactly into the Riemann-Liouville derivative

$$D^{\beta}_{a+} f(x)=\frac{1}{\Gamma(-\beta)}\int_0^{x-a}\frac{f(x-z)-f(x)+f'(x)z}{z^{1+\beta}}dz+\frac{f(x)(x-a)^{-\beta}}{\Gamma(1-\beta)}+\frac{\beta f'(x)(x-a)^{1-\beta}}{\Gamma(2-\beta)}, \tag{61}$$

precisely as in the case $\beta \in (0,1)$.

Extension of this procedure is thus clear. Namely, starting with a Feller process with negative jumps, say with the generator

$$Af(x)=\int_0^{\infty}[f(x-z)-f(x)+f'(x)z]\nu(x,dz) \tag{62}$$

such that

$$\sup_x \int_0^{\infty} z^2\nu(x,dz)<\infty, \qquad \int_0^{\infty} z\nu(x,dz)=\infty \tag{63}$$

one can form the corresponding process with jumps interrupted at a as the process generated by

$$\begin{aligned}\tilde{A}_{a+}f(x)&=\int_0^{x-a}[f(x-z)-f(x)+f'(x)z]\nu(x,dz)\\&+(f(a)-f(x))\int_{x-a}^{\infty}\nu(x,dz)+f'(x)\int_{x-a}^{\infty}z\nu(x,dz).\end{aligned} \tag{64}$$

Its main term of asymptotics as $x \to a$ is

$$f'(x)\int_{x-a}^{\infty}(z-(x-a))\nu(x,dz),$$

which is unbounded unless $f'(a) = 0$. Thus the analog of the Caputo fractional derivative is obtained by subtracting this 'infinity':

$$A_{a+\star}f(x)=\tilde{A}_{a+}f(x)-f'(a)\int_{x-a}^{\infty}(z-(x-a))\nu(a,dz). \tag{65}$$

Moreover, it implies the following limiting behavior for $f \in C^2[a,\infty)$:

$$\begin{aligned}&\lim_{x\to a_+} A_{a+\star}f(x)=0,\\&\lim_{x\to a_+}\tilde{A}_{a+}f(x)=0, \text{ if } f'(a)=0.\end{aligned} \tag{66}$$

Remark 3. *Notice that generator* (64) *has variable coefficients even if underlying process was a Lévy process. Hence it is not directly clear whether* (64) *generates a well defined process. This issue will be addressed in the next sections.*

Similarly one defines $A_{a-\star}$ for Feller processes with positive jumps. More generally, for any Feller process on $\mathbf{R}$ generated by the operator

$$Af(x) = G(x)f''(x) + \gamma(x)f'(x) + \int_{\mathbf{R}} [f(x+z) - f(x) - f'(x)z\chi(z)]\nu(x, dz), \tag{67}$$

where χ is some mollifier (that traditionally is taken as either $\mathbf{1}_{|z|\le 1}$ or as $1/(1+z^2)$), one defines the corresponding process with jumps interrupted on crossing an interval $[a, b]$ as the process generated by the operator

$$\tilde{A}_{[a,b]}f(x) = G(x)f''(x) + \gamma(x)f'(x) + \int_{a-x}^{b-x} [f(x+z) - f(x) - f'(x)z\chi(z)]\nu(x, dz)$$

$$+(f(b)-f(x)) \int_{b-x}^{\infty} \nu(x, dz) + (f(a)-f(x)) \int_{-\infty}^{a-x} \nu(x, dz) - f'(x) \int_{\mathbf{R}\setminus(a-x,b-x)} z\chi(z)\nu(x, dz) \tag{68}$$

for $x \in (a, b)$ (when such process is well defined), or for arbitrary x

$$\tilde{A}_{[a,b]}f(x) = G(x)f''(x) + \gamma(x)f'(x) + \mathbf{1}_{x<b} \int_0^{\infty} (f[(x+z) \wedge b] - f(x) - f'(x)z\chi(z))\, \nu(x, dz)$$

$$+\mathbf{1}_{x>a} \int_{-\infty}^0 (f([(x+z) \vee a] - f(x) - f'(x)z\chi(z))\nu(x, dz). \tag{69}$$

The corresponding analog of the Caputo derivative is obtained by subtracting the singularity at the boundary points, that is, if

$$\int_{-1}^0 |z|\nu(x, dz) = \infty, \quad \int_0^1 z\nu(x, dz) = \infty, \tag{70}$$

then, for $x \in (a, b)$,

$$A_{[a,b]\star}f(x) = \tilde{A}_{[a,b]}f(x) - f'(b) \int_{b-x}^{\infty} [(b-x) - z\chi(z)]\nu(x, dz) - f'(a) \int_{-\infty}^{a-x} [(a-x) - z\chi(z)]\nu(x, dz). \tag{71}$$

The analog of the Riemann-Liouville derivative is obtained from $\tilde{A}_{[a,b]}f(x)$ by setting the boundary values of f to zero yielding the operator of the processes generated by $\tilde{A}_{[a,b]}f(x)$, but killed at the boundary:

$$A_{[a,b]}f(x) = G(x)f''(x) + \gamma(x)f'(x) + \int_{a-x}^{b-x} [f(x+z) - f(x) - f'(x)z\chi(z)]\nu(x, dz)$$

$$-f(x) \int_{\mathbf{R}\setminus(a-x,b-x)} \nu(x, dz) - f'(x) \int_{\mathbf{R}\setminus(a-x,b-x)} z\chi(z)\nu(x, dz). \tag{72}$$

But what is the relation between equations involving $A_{[a,b]\star}$ and $\tilde{A}_{[a,b]}$? The point is that if $\tilde{A}_{[a,b]}f = g$ on $[a, b]$ for some bounded g, then necessarily $f'(a) = f'(b) = 0$ (as otherwise $\tilde{A}_{[a,b]}$ would be unbounded for $x \to a$ or $x \to b$) and hence $\tilde{A}_{[a,b]}f = A_{[a,b]\star}$, so that at least classical solutions for the problems

$$\tilde{A}_{[a,b]}f(x) = \lambda f(x) + g(x), \quad f(a) = f_a, f(b) = f_b \tag{73}$$

also solve the problem

$$A_{[a,b]\star}f(x) = \lambda f(x) + g(x), \quad f(a) = f_a, f(b) = f_b. \tag{74}$$

The problem (73) can be naturally settled probabilistically.

Remark 4. *The subtraction of singularity in definition* (71) *makes it dependent on representation* (67). *This is however not very essential for solving the corresponding fractional differential equations, because operator $\tilde{A}$ is not representation dependent.*

Similarly in multidimensional case the analog of the Caputo derivative can be obtained by subtracting the singularity from the generator of the process obtained from an initial one by restricting the jumps to a chosen domain $\bar{D}$. For instance, for a Feller process generated by the operator

$$Af(x) = \int_{\mathbf{R}^d} [f(x+z) - f(x) - (f'(x), z)\chi(z)]\nu(x, dz), \tag{75}$$

the process with jumps restricted to land on $\bar{D}$ (stopped-on-crossing-the-boundary process) for D the half-space (53) has the generator

$$\tilde{A}_D f(x) = \int_{\mathbf{R}^d} dy_2 \int_{-\infty}^{b-x_1} dy_1 \nu(x; y)[f(x+y) - f(x) - (f'(x), y)\chi(y)]$$
$$+ \int_{\mathbf{R}^d} dy_2 \int_{b-x_1}^{\infty} dy_1 \nu(x; y)[f\left(b, x_2 + \frac{b - x_1}{y_1} y_2\right) - f(x) - (f'(x), y)\chi(y)], \tag{76}$$

for $x \in D$, which is a direct extension of (54). However, unlike (54), $\tilde{A}_D f(x)$ now diverges as x tends to a boundary point if $\partial f / \partial x_1(x) \neq 0$ there. In fact, as one sees directly, the main term of $\tilde{A}_D f(x)$ as $(x_1, x_2) \to (b, x_2)$ is

$$\frac{\partial f}{\partial x_1}(x_1, x_2) \int_{b-x_1}^{\infty} dy_1 \int_{\mathbf{R}^d} dy_2 \, (b - x_1 - y_1)\nu(b, x_2; y_1, y_2).$$

Hence the analog of the Caputo derivative is

$$A_{D\star} f(x) = \tilde{A}_D f(x) - \frac{\partial f}{\partial x_1}(b, x_2) \int_{b-x_1}^{\infty} dy_1 \int_{\mathbf{R}^d} dy_2 \, (b - x_1 - y_1)\nu(b, x_2; y_1, y_2). \tag{77}$$

4 Basic well-posedness results and examples

4.1 The case $\beta \in (0, 1), d = 1$

Let us start with the operator (32) assuming for simplicity that the kernels ν have densities with respect to Lebesgue measure. Therefore let

$$Af(x) = \gamma(x) f'(x) + \int_{\mathbf{R}} (f(x+y) - f(x))\nu(x, y) \, dy. \tag{78}$$

Choosing for definiteness extension (33) (of (32) defined for $x \in (a,b)$) let

$$A_{[a,b]\star}f(x)=\gamma(x)f'(x)+\mathbf{1}_{x<b}\int_0^\infty (f[(x+y)\wedge b]-f(x))\nu(x,dy)$$

$$+\mathbf{1}_{x>a}\int_{-\infty}^0 (f[(x+y)\vee a]-f(x))\nu(x,dy), \qquad (79)$$

which can be represented as the sum of the generators of a decreasing process, an increasing process and a drift:

$$A_{[a,b]\star}=A_{a+\star}+A_{b-\star}+\gamma(.)\frac{d}{dx}$$

with

$$A_{a+\star}f(x)=\mathbf{1}_{x>a}\int_{-\infty}^0 (f[(x+y)\vee a]-f(x))\nu(x,dy),$$

$$A_{b-\star}f(x)=\mathbf{1}_{x<b}\int_0^\infty (f[(x+y)\wedge b]-f(x))\nu(x,dy).$$

Let $-\infty<a<b<\infty$. Let us denote, as usual, by $C[a,b]$ (resp. $C_\infty(\mathbf{R})$) the Banach space of continuous functions on $[a,b]$ (resp. on $\mathbf{R}$ vanishing at infinity), by $C_\infty(-\infty,a]$ (resp. $C_\infty[a,\infty)$) the Banach space of continuous functions on $(-\infty,a]$ (resp. $[a,\infty)$) vanishing at infinity, by $C^k_\infty(\mathbf{R})$, $C^k_\infty(-\infty,a]$, $C^k_\infty[a,\infty)$, $C^k[a,b]$ the subspaces of functions of the corresponding spaces $C_\infty(\mathbf{R})$, $C_\infty(-\infty,a]$, $C_\infty[a,\infty)$, $C[a,b]$, having derivatives up to order k from $C_\infty(\mathbf{R})$, $C_\infty(-\infty,a]$, $C_\infty[a,\infty)$, $C[a,b]$, respectively.

Recall also that, for a domain $D\in\mathbf{R}^d$ with boundary ∂D and a Markov process $X_x(t)$ in $\mathbf{R}^d$ or just in D, a point $x\in\partial D$ is called regular if $\tau_D(x)\to 0$ in probability for $x\to x_0$, $x\in D$, where τ_D is the exit time from D. We say that it is regular in expectation if $\mathbf{E}\tau_D(x)\to 0$ for $x\to x_0$, $x\in D$.

Remark 5. *1. A point zero for $D=(0,\infty)$ is regular for a Brownian motion, but not regular in expectation, as $\mathbf{E}\tau_D(x)=\infty$ for all $x\in D$. 2. We are using the notion of a regular point arising from the theory of parabolic PDEs (see e. g. [9] or [19]), which is different from the corresponding notion used in the theory of Lévy processes (see [22]).*

Theorem 4.1. *Assume that $\gamma(x)$ is continuously differentiable with a bounded derivative and that $\nu(x,y)$ is a continuous function of two variables, which is continuously differentiable with respect to the first variable and has the following uniform bounds and tightness property*

$$\sup_x\int |y|\nu(x,y)\,dy<\infty, \quad \sup_x\int |y|\left|\frac{\partial}{\partial x}\nu(x,y)\right|dy<\infty, \qquad (80)$$

and

$$\lim_{\delta\to 0}\sup_x\int_{|y|\le\delta}|y|\nu(x,y)\,dy=0. \qquad (81)$$

(i) Then the operator $A_{a+\star}$ generates a Feller process on $[a,\infty)$ and a Feller semigroup on $C_\infty[a,\infty)$ with the invariant core $C^1_\infty[a,\infty)$, the operator $A_{b-\star}$ generates a Feller process on $(-\infty,b]$ and a Feller semigroup on $C_\infty(-\infty,b]$ with the invariant core $C^1_\infty(-\infty,b]$.

(ii) Moreover, the operator $A_{[a,b]\star}$ generates a Feller process on $\mathbf{R}$ and a Feller semigroup on $C_\infty(\mathbf{R})$ and, if $\gamma(x)=0$, then $A_{[a,b]\star}$ generates also a Feller process on $[a,b]$ and a Feller semigroup on $C[a,b]$ with the invariant core $C^1[a,b]$.

(iii) If

$$\int_{-\infty}^{0} \min(|y|,\epsilon)\nu(a,y)\,dy > C\epsilon^r \tag{82}$$

or

$$\int_{0}^{\infty} \min(y,\epsilon)\nu(b,y)\,dy > C\epsilon^r \tag{83}$$

for some $C>0$, $r\in(0,1)$, then the point a (resp. b) is regular in expectation for the first (resp. second) process in (i).

(iv) If $\gamma(a)<0$ (resp. $\gamma(b)>0$), then then the point a (resp. b) is a regular in expectation boundary point of the interval (a,b) for the process generated by $A_{[a,b]\star}$.

(v) If $A_{[a,b]\star}$ is the operator on the l.h.s. of (41) and $\omega_{j0}>0, \gamma_{j0}>0$, where β_{j0} is the unique maximum of all β_j, then the points a and b are regular in expectation boundary points of (a,b) for the process generated by $A_{[a,b]\star}$.

Proof. (i) Notice that the operator $A_{[a,b]\star}$ is a bounded operator $C^1[a,b]\to C[a,b]$ such that

$$\lim_{x\to a_+} A_{a+\star}f(x)=0, \quad \lim_{x\to b_-} A_{b-\star}f(x)=0 \tag{84}$$

for $f\in C^1[a,b]$, as follows from

$$\lim_{\delta\to 0}\delta\int_{\mathbf{R}\setminus[-\delta,\delta]}\nu(x,y)\,dy=0. \tag{85}$$

Remark 6. *Equation* (85) *is a consequence of the first bound in* (80). *In fact, since*

$$\int_\delta^1 y\nu(x,y)\,dy=\delta F_x(\delta)+\int_\delta^1 F_x(y)\,dy,$$

where $F_x(y)=\int_y^1\nu(x,z)\,dz$, it follows that the both terms on the r.h.s. of this equation are uniformly bounded. Hence, the l.h.s. of this equation and the second term on the r.h.s. converge to $\int_0^1 y\nu(x,u)\,dy$ implying $\delta F_x(\delta)\to 0$ as $\delta\to 0$.

We shall follow the strategy of proof from Theorem 5.1.1 of [19]. Let us now work for definiteness with $A_{b-\star}$ (other cases are dealt with analogously). Differentiating (and using straightforward cancelations) yields

$$\frac{d}{dx}A_{b-\star}f(x)=\int_0^{b-x}[f'(x+y)-f'(x)]\,\nu(x,y)\,dy-f'(x)\int_{b-x}^{\infty}\nu(x,y)\,dy$$

$$+\int_0^{b-x}(f(x+y)-f(x))\frac{\partial\nu}{\partial x}(x,y)\,dy+(f(b)-f(x))\int_{b-x}^{\infty}\frac{\partial\nu}{\partial x}(x,y)\,dy.$$

Thus if f solves the equation

$$\dot f=A_{b-\star}f,$$

then $g=f'$ (if exists) solves the equation

$$\dot g=\int_0^{b-x}[g(x+y)-g(x)]\,\nu(x,y)\,dy-g(x)\int_{b-x}^{\infty}\nu(x,y)\,dy$$

$$+\int_0^{b-x} dy \int_0^y g(x+z)dz \frac{\partial \nu}{\partial x}(x,y) + \int_x^b g(z)dz \int_{b-x}^\infty \frac{\partial \nu}{\partial x}(x,y)\, dy. \tag{86}$$

Let us introduce the approximation $A_{b-\star h}$ for our operator obtained by changing $\nu(x,y)$ to $\nu_h(x,y) = \mathbf{1}_{|y|>h}\nu(x,y)$. For any h the operator $A_{b-\star h}$ is bounded in $C(-\infty,b]$ and hence generates a conservative Feller semigroup T_t^h there. Moreover, the operator on the r.h.s. of (86) becomes also bounded when ν is replaced by ν_h, so that this equation becomes well-posed in $C(-\infty,b]$, so that T_t^h is also a strongly continuous semigroup in $C^1(-\infty,b]$.

The key observation now is that T_t^h are bounded in $C^1(-\infty,b]$ uniformly in h, because the first two terms on the r.h.s. of (86) represent a conditionally positive operator with a negative coefficients at g, which therefore generates a positivity preserving contraction in $C(-\infty,b]$, and the last two terms are uniformly (in h) bounded operators. Hence $(T_t^h f)'(x)$ (by $'$ we denote the derivative with respect to x) are uniformly bounded for all $h \in (0,1]$ and t from any compact interval whenever $f \in C^1_\infty(-\infty,b]$. Therefore, writing

$$(T_t^{h_1} - T_t^{h_2})f = \int_0^t T_{t-s}^{h_2}(A_{b-\star h_1} - A_{b-\star h_2})T_s^{h_1}\, ds$$

for arbitrary $h_1 > h_2$ and estimating

$$|(A_{b-\star h_1} - A_{b-\star h_2})T_s^{h_1} f(x)| \le \int_{h_2 \le |y| \le h_1} |(T_s^{h_1} f)(x+y) - (T_s^{h_1} f)(x)|\nu(x,y)\, dy$$

$$\le \int_0^{h_1} \|(T_s^{h_1} f)'\| \nu(x,y)|y|\, dy = o(1)\|f\|_{C(-\infty,b]}, \quad h_1 \to 0,$$

yields

$$\|(T_t^{h_1} - T_t^{h_2})f\| = o(1)t\|f\|_{C^1(-\infty,b]}, \quad h_1 \to 0. \tag{87}$$

Therefore the family $T_t^h f$ converges to a family $T_t f$, as $h \to 0$, which also forms a strongly continuous semigroup in $C(-\infty,b]$. Writing

$$\frac{T_t f - f}{t} = \frac{T_t f - T_t^h f}{t} + \frac{T_t^h f - f}{t}$$

and noting that by (87) the first term is of order $o(1)\|f\|_{C^1(-\infty,b]}$ as $h \to 0$ allows one to conclude that $C^1(-\infty,b]$ belongs to the domain of the generator of the semigroup T_t in $C(-\infty,b]$.

Applying to T_t the procedure applied above to T_t^h (differentiating the evolution equation with respect to x) shows that T_t defines also a strongly continuous semigroup in $C^1(-\infty,b]$, and hence $C^1(-\infty,b]$ is an invariant core for T_t.

Remark 7. *Notice that the semigroup T_t extends also to the strongly continuous semigroup on $C_\infty(\mathbf{R})$ generated by $A_{b-\star}$ with an invariant domain C^1_∞, because the right condition of (84) ensures a smooth gluing with the value $A_{b-\star} f x) = 0$ for $x > b$ for any $f \in C^1(\mathbf{R})$.*

(ii) This is quite similar and is omitted.

(iii) To prove regularity of the boundary, we shall use the method of Lyapunov functions. Namely, to show that, say, b is regular for the last process in (i), it is sufficient to find a continuous function f in a neighborhood of $[a,b]$ such that f is differentiable for $x \in (a,b)$, $f(b) = 0$, and for $x \in (c,b)$ with some $c \in (a,b)$ one has $f(x) > 0$, $A_{[a,b]\star} f(x) < 0$

(see Proposition 6.3.2 of [19]). As such function one can take $f_\omega(x)=(b-x)^\omega$ with some $\omega\in(0,1)$. In fact, clearly $f_\omega(b)=0$, $f_\omega(x)>0$ for $x<b$ and for x approaching b from the left, $A_{[a,b]\star}f(x)$ is of order

$$-(b-x)^{\omega-1}\int_0^\infty \min(y,b-x)\nu(b,y)\,dy$$

which tends to $-\infty$ as $x\to b$ for sufficiently small ω under the assumptions of (iii).

(iv) Using the same Lyapunov function $f_\omega(x)=(b-x)^\omega$ one sees that the drift term always dominates the jump part part of the generator.

(v) Proving regularity in a general case can be subtle. However, for the process from (v) the same Lyapunov function $f_\omega(x)=(b-x)^\omega$ yields the required result.

□

Theorem 4.1 allows one to solve equations involving $A_{[a,b]\star}$ or $A_{a\pm\star}$ by the standard techniques of stochastic analysis. Namely, let us denote by $X_x(t)$ the Markov process generated by A of (78) and by $X_x^*(t)[a,b]$ the Markov process on $[a,b]$ generated by $A_{[a,b]\star}$.

Let us stress that the process $X_x^*(t)[a,b]$ (which is interrupted but not stopped at the boundary) is generated by $A_{[a,b]\star}$ defined on the domain $C^1_\infty(\mathbf{R})$ (or else on the domain $C^1[a,b]$ if $\gamma(b)\le 0$ and $\gamma(a)\ge 0$); the process $X_x^*(t)[a,b;stop]$ on $[a,b]$ obtained from $X_x^*(t)[a,b]$ by stopping at the boundary is generated by $A_{[a,b]\star}$ defined on the domain, which is a subspace of $C^1[a,b]$ of functions f such that $A_{[a,b]\star}f(a)=A_{[a,b]\star}f(b)=0$; the process $X_x^*(t)[a,b;kill]$ killed at the boundary is generated by $A_{[a,b]\star}$ defined on the domain, which is a subspace of $C^1[a,b]$ of functions f such that $A_{[a,b]\star}f(a)=A_{[a,b]\star}f(b)=0$ and $f(a)=f(b)=0$. It is easily seen that if $\int_{-\infty}^a \nu(a,y)dy=\infty$ and $\int_b^\infty \nu(b,y)dy=\infty$, the operator $A_{[a,b]\star}$ generates a strongly continuous semigroup (of the process $X_x^*(t)[a,b]$ killed on the boundary) on the subspace $C_0[a,b]$ of $C[a,b]$ consisting of functions vanishing at a and b.

Let τ denote the first exit time for $X_x^*(t)[a,b]$ or $X_x(t)$ from (a,b):

$$\tau=\inf\{t\ge 0: X_x^*(t)[a,b]\notin(a,b)\}=\inf\{t\ge 0: X_x(t)\notin(a,b)\}.$$

Applying Dynkin's martingale to a function $f\in C^1[a,b]$ and Doob's optional sampling theorem for the stopping time τ (see e. g. [8] for the presentation of these two basic tools of stochastic analysis) yields

$$f(x)=\mathbf{E}\left[f(X_x^*(\tau)[a,b])-\int_0^\tau (A_{[a,b]\star}f)(X_x^*(s)[a,b])\,ds\right]$$

Hence if f is a solution to problem (43), then

$$f(x)=f(a)\mathbf{P}(X_x^*(\tau)[a,b]=a)+f(b)\mathbf{P}(X_x^*(\tau)[a,b]=b)-\mathbf{E}\int_0^\tau g(X_x^*(s)[a,b])\,ds. \quad (88)$$

Moreover, since the trajectories of $X_x(t)$ and $X_x^*(t)[a,b]$ coincide till time τ this can be expressed entirely in terms of the process $X_x(t)$ as

$$f(x)=f(a)\mathbf{P}(X_x(\tau)\le a)+f(b)\mathbf{P}(X_x(\tau)\ge b)-\mathbf{E}\int_0^\tau g(X_x(s))\,ds. \quad (89)$$

Introducing, as is common in the theory of Lévy processes, the occupation-till-exit measure $H(x, dy)$ on (a, b) by

$$H(x, B) = \mathbf{E} \int_0^\tau \mathbf{1}_B(X_x(s))\, ds, \tag{90}$$

allows one to rewrite (89) as

$$f(x) = f(a)\mathbf{P}(X_x(\tau) \le a) + f(b)\mathbf{P}(X_x(\tau) \ge b) - \int_a^b g(y)H(x, dy). \tag{91}$$

Thus we obtain the following result.

Theorem 4.2. *(i) Under the assumptions of Theorem 4.1 problem* (43) *can have at most one classical solution. If* $\mathbf{E}\tau_{[a,b]}(x) < \infty$ *for all* $x \in D$*, the probabilities* $\mathbf{P}(X_x(\tau) \le a)$*,* $\mathbf{P}(X_x(\tau) \ge b)$ *are functions from* $C^1[a, b]$ *and the measure* $H(x, dy)$ *is continuously weakly differentiable in* x *(so that the integral on the r.h.s. of* (91) *belongs to* $C^1[a, b]$ *for any* $g \in C[a, b]$*), then formula* (91) *supplies the unique classical solution to problem* (43)*.*

Remark 8. *In the classical analysis of boundary-value problems for partial differential equations, problems like the one in* (43) *are usually understood to mean that the main equation there holds for all* x *excluding the boundary. The necessity of this attitude is easily seen here. Namely, if* f *belongs to the domain of the generator of a stopped or killed process, then the value of this generator on* f *at a boundary point should vanish. Thus only for* g *vanishing on the boundary the solution to* (43) *can belong to the domain of the generator and satisfy the main equation up to the boundary.*

Furthermore, as is known from stochastic analysis, formula (91) makes sense as a generalized solution under more general assumptions. Not going into much detail, let us only menton one particular situation. Namely, one says that a continuous function $f(x)$ on $[a, b]$ is a generalized solution to problem (43) with $g = 0$, if f belongs to the domain of the generator of the semigroup $T_t^{stop}[a, b]$ of the stopped process $X_x^*(t)[a, b; stop]$ on $[a, b]$ (obtained by the closure of the operator $A_{[a,b]\star}$ defined on the domain, which is a subspace of $C^1[a, b]$ of functions f such that $A_{[a,b]\star}f(a) = A_{[a,b]\star}f(b) = 0$) and satisfies $A_{[a,b]\star}f = 0$, or equivalently $T_t^{stop}[a, b]f = f$. The following fact is a consequence of a general theory of boundary points (here the regularity of the boundary is crucial), see e. g. Theorem 6.2.3 of [19] for detail.

Theorem 4.3. *Under the assumptions of Theorem 4.1, formula* (91) *supplies a unique generalized solution to problem* (43)*.*

Furthermore, to solve problem (44) one utilizes the process

$$f(X_x^*(\tau)[a, b])e^{-\lambda t} + \int_0^t e^{-\lambda s}(\lambda - A_{[a,b]\star})f(X_x^*(s)[a, b])\, ds,$$

which is known (see e. g. [8]) to be a martingale for any $f \in C^1[a, b]$ (under the conclusions of Theorem 4.1). Again by the optional sampling theorem it follows that if f solves (43), then

$$f(x) = \mathbf{E}[f(X_x^*(\tau)[a, b])e^{-\lambda\tau}] = f(a)\mathbf{E}[e^{-\lambda\tau}\mathbf{1}_{X_x(\tau)\le a}] + f(b)\mathbf{E}[e^{-\lambda\tau}\mathbf{1}_{X_x(\tau)\ge b}]. \tag{92}$$

Formula (92) again enures uniqueness of classical solution to (43) and yields its integral representation in case the expectations involved in the r.h.s. of (92) are sufficiently regular functions of x.

As an example, let us consider problem (42) on the interval $[a,b] = [-1,1]$. The corresponding process $X_x(t)$ of Theorem 4.2 is a symmetric Lévy motion on $\mathbf{R}$ with index $\beta \in (0,1)$. For this process all ingredients of formula (91) are known, see e. g. [3]:

$$\mathbf{P}(X_x(\tau) \geq 1) = 2^{1-\beta} \frac{\Gamma(\beta)}{(\Gamma(\beta/2))^2} \int_{-1}^{x} (1-u^2)^{-1+\beta/2}\, du, \tag{93}$$

and $H(x,dy)$ has the density

$$H(x,y) = 2^{-\beta} \pi^{-1/2} \frac{\Gamma(1/2)}{(\Gamma(\beta/2))^2} \int_0^z (u+1)^{-1/2} u^{\beta/2-1} |x-y|^{\beta-1}\, du, \tag{94}$$

where

$$z = (1-x^2)(1-y^2)/(x-y)^2.$$

Thus (91) yields a solution to problem (42) in closed form. Some formulas for exit probabilities are also available for nonsymmetric Lévy motions, see [31] and [23], thus yielding explicit solutions to a slightly more general (compared to (42)) problem

$$\alpha_1 D^{\beta}_{a+\star} f(x) + \alpha_2 D^{\beta}_{b-\star} f(x) = -g(x), \quad f(a) = f_a, f(b) = f_b, \tag{95}$$

with arbitrary positive constants α_1, α_2.

Similar results hold for the equations on a half-line involving the operators $A_{a\pm\star}$.

4.2 The case $\beta \in (0,1), d > 1$

Let us now consider operator (46) assuming again for simplicity that the kernel ν has a density, $\nu(x,y)$, with respect to Lebesgue measure.

The killed processes generated by (52) (which are well studied for Lévy processes, see e. g. [4]) are generally easier for analysis than stopped processes. Therefore we concentrate on the analysis of operator (50), which is more involved. Let us consider only the case when D is the semi-space D_b or the band $D_{(a,b)}$, see (53), (55), where $A_{D\star}$ is given by (54) and (56).

In what follows we have to use a rather ugly additional condition

$$\Omega(\epsilon,x) = \epsilon \int_{\epsilon}^{\infty} dy_1 \int_{\mathbf{R}^d} dy_2 \mathbf{1}_{|y|\leq 1} \nu(x;y_1,y_2) \frac{y_2^2}{y_1^2} \leq C\omega(\epsilon,x) \tag{96}$$

with a constant C, where

$$\omega(\epsilon,x) = \int_{\epsilon}^{\infty} dy_1 \int_{\mathbf{R}^d} dy_2 \nu(x;y_1,y_2).$$

Its reasonability relies on the fact that it holds for stable-like processes in $\mathbf{R}^{d+1}$ with

$$\nu(x;y) = \frac{a(x)}{|y|^{d+1+\beta}}, \quad \beta \in (0,1), \tag{97}$$

and hence for a variety of standard examples.

In fact, since

$$\int_\epsilon^\infty dy_1 \int_{\mathbf{R}^d} dy_2 \mathbf{1}_{|y|\le 1} = \int_\epsilon^1 r^d \, dr \int_0^{\arccos(r)} \sin^{d-1}\phi \, d\phi \, dn,$$

where dn is Lebesgue measure on the unit sphere in $\mathbf{R}^d$ (or just a coefficient 2 in case $d=1$) with the total area $|S^{d-1}|$, one has, assuming (97), that

$$\Omega(\epsilon,x) \le a(x)|S^{d-1}|\epsilon \int_\epsilon^1 r^{-1-\beta} \, dr \int_0^{\arccos(r)} d\phi \frac{\sin\phi}{\cos^2\phi}$$

$$= a(x)|S^{d-1}|\epsilon \int_\epsilon^1 r^{-1-\beta} \, dr \int_r^1 \frac{dz}{z^2} \le \frac{a(x)}{1+\beta}\epsilon^{-\beta}$$

and

$$\omega(\epsilon,x) \ge a(x)|S^{d-1}| \int_\epsilon^1 r^{-1-\beta} \, dr \int_0^{\arccos(r)} \sin\phi \, d\phi$$

$$= a(x)|S^{d-1}|\epsilon \int_\epsilon^1 r^{-1-\beta} \, dr \int_r^1 dz = a(x)\left[\frac{\epsilon^{-\beta}-1}{\beta} - \frac{1-\epsilon^{1-\beta}}{1-\beta}\right],$$

implying (96).

Extending one-dimensional notations for function spaces used above we shall denote by $C_\infty[\bar{D}]$ the Banach space of continuous functions on $\bar{D}$ vanishing at infinity and by $C^1_\infty[\bar{D}]$ its subspace of functions with first order partial derivatives belonging to $C_\infty[\bar{D}]$.

Theorem 4.4. *Assume that $\nu(x;y) = \nu(x_1,x_2;y_1,y_2)$ is a continuous function, which is continuously differentiable with respect to x and has uniform bounds* (80) *and tightness property* (81)*, where the integrals are over $\mathbf{R}^{d+1}$, and additionally the bound on the second derivative with respect to x_2:*

$$\sup_x \int |y| \left|\frac{\partial^2}{\partial x_2^2}\nu(x_1,x_2;y_1,y_2)\right| dy < \infty. \tag{98}$$

Moreover, assume that (96) *holds with a constant C and that $\nu(x;y_1,y_2)$ is an even function of y_2.*

(i) Then, for a semi-space $D = D_b$, the operator $A_{D\star}$ generates a Feller process $X_t^*(x)$ on $\bar{D}$ and a Feller semigroup on $C_\infty[\bar{D}]$ with invariant core $C^1_\infty[\bar{D}]$.

(ii) If additionally (98) *holds also with the integral in y_1 taken over $(-\infty,\epsilon)$, then also for a band $D = D_{(a,b)}$ the operator $A_{D\star}$ generates a Feller process $X_t^*(x)$ on $\bar{D}$ and a Feller semigroup on $C_\infty[\bar{D}]$ with invariant core $C^1_\infty[\bar{D}]$.*

Proof. The proof follows the same lines as in Theorem 4.1 above. Let us deal only with domain D_b. Approximating ν by $\nu_h(x;y) = \mathbf{1}_{y_2>h}\nu(x;y)$ we get semigroups T_h on $C_\infty(\bar{D})$, which are uniformly bounded and preserves twice continuous differentiability with respect to the second variable x_2 and bounds to these derivatives. This holds because differentiation of the equation $\dot{f} = A_{\mathbf{D}\star}f$ with respect to x_2 does not feel the boundary so-to-say, that is we get

$$\frac{d}{dt}\frac{\partial f}{\partial x_2} = \int_{\mathbf{R}^d} dy_2 \int_{-\infty}^{b-x_1} dy_1 \nu(x;y)[\frac{\partial f}{\partial x_2}(x+y) - \frac{\partial f}{\partial x_2}(x)]$$

$$+\int_{\mathbf{R}^d} dy_2 \int_{b-x_1}^{\infty} dy_1 \nu(x;y)[\frac{\partial f}{\partial x_2}\left(b, x_2+\frac{b-x_1}{y_1}y_2\right)-\frac{\partial f}{\partial x_2}(x)]$$

$$+\int_{\mathbf{R}^d} dy_2 \int_{-\infty}^{b-x_1} dy_1 \frac{\partial \nu}{\partial x_2}(x;y)[f(x+y)-f(x)]$$

$$+\int_{\mathbf{R}^d} dy_2 \int_{b-x_1}^{\infty} dy_1 \frac{\partial \nu}{\partial x_2}(x;y)[f\left(b, x_2+\frac{b-x_1}{y_1}y_2\right)-f(x)],$$

and similarly for the second derivative in x_2 (with h and without it). The problem arises when differentiating the equation $\dot{f} = A_{\mathbf{D}\star} f$ with respect to x_1 yielding the equation

$$\frac{d}{dt}g = \int_{\mathbf{R}^d} dy_2 \int_{-\infty}^{b-x_1} dy_1 \nu(x;y)[g(x+y)-g(x)] - g(x)\int_{\mathbf{R}^d} dy_2 \int_{b-x_1}^{\infty} dy_1 \nu(x;y)$$

$$-\int_{\mathbf{R}^d} dy_2 \int_{b-x_1}^{\infty} dy_1 \nu(x;y)\frac{y_2}{y_1}\frac{\partial f}{\partial x_2}\left(b, x_2+\frac{b-x_1}{y_1}y_2\right) \tag{99}$$

(other terms cancel as in one-dimensional case) for $g = \partial f/\partial x_1$. Similar equation hods for ν_h instead of ν. Because of assumed symmetry of ν this rewrites as

$$\frac{d}{dt}g(x) = \int_{\mathbf{R}^d} dy_2 \int_{-\infty}^{b-x_1} dy_1 \nu(x;y)[g(x+y)-g(x)] - g(x)\omega(b-x_1, x) + \phi(x;f) \tag{100}$$

with

$$\omega(\epsilon, x) = \int_{\mathbf{R}^d} dy_2 \int_{\epsilon}^{\infty} dy_1 \nu(x;y),$$

$$\phi(x;f) = -\int_{\mathbf{R}^d} dy_2 \int_{b-x_1}^{\infty} dy_1 \nu(x;y)\frac{(b-x_1)y_2^2}{y_1^2}\frac{\partial^2 f}{\partial x_2^2}\left(b, x_2+\theta\frac{b-x_1}{y_1}y_2\right),$$

where $\theta \in (0,1)$ so that, for $\partial^2 f/\partial x^2$ bounded by a constant c, ϕ is bounded:

$$|\phi(x;f)| \le c(b-x_1)\int_{\mathbf{R}^d} dy_2 \int_{b-x_1}^{\infty} dy_1 \nu(x;y)\frac{y_2^2}{y_1^2}.$$

Equation (100) rewrites in the mild form as

$$g_t(x) = g_0(x) + \int_0^t e^{-\omega(b-x_1,x)(t-s)} A g_s(x)\, ds + \int_0^t e^{-\omega(b-x_1,x)(t-s)} \phi(x;f)\, ds, \tag{101}$$

where

$$Ag(x) = \int_{\mathbf{R}^d} dy_2 \int_{-\infty}^{b-x_1} dy_1 \nu(x;y)[g(x+y)-g(x)].$$

By (96), the last term in (101) is uniformly bounded, and hence equation (101) and its versions with ν_h instead of ν have uniformly bounded solutions for bounded g_0. Hence we can now complete the proof as in Theorem 4.1. □

One can now get a direct multi-dimensional version of Theorem 4.2 for the boundary value problems

$$A_{D\star} f = g, \quad f|_{\partial D} = \phi, \tag{102}$$

with g in D and ϕ on ∂D given, which represent the simplest multidimensional analogs of linear equations with the Caputo derivatives. Alternatively, one can also analyze such problems via the reduction to killed processes (that is, to the analogs of RL derivatives), see Remark 1.

We shall not go into detail of general domains D here, but note that the problem

$$A_{D\star} f = \lambda f + g, \quad f|_{\partial D} = \phi, \tag{103}$$

for A generating a stable Lévy process in $\mathbf{R}^d$ and D the ball $D = \{y \in \mathbf{R}^d : |y| < r\}$ can be solved explicitly, using multidimensional extensions of formulas (93) and (94) given also in [3].

4.3 The case $\beta \in (1,2)$

For the case $\beta \in (0,1)$ above we constructed the interrupted process on its own and then look at its stopping, which is quite natural. However, as we noted, the boundary-value problems (at least in one-dimensional case) for corresponding operators can be expressed in terms of the initial process stopped at the boundary. We shall follow this approach here, as the study of interrupted process becomes rather subtle.

Let us reduce our attention to one-dimensional processes only generated by the operators

$$Af(x) = \int [f(x+y - f(x) - yf'(x)]\nu(x,y)\, dy$$

with the density ν satisfying

$$\sup_x \int_{\mathbf{R}} (|y| \wedge |y|^2)\nu(x,y)\, dy < \infty. \tag{104}$$

The question of whether such an operator generates a uniquely defined process is non-trivial already in the case of this simple A, which can be looked at as the fully mixed-order fractional derivative. To go ahead, we shall use additional assumptions of regularity and monotonicity. The following statement is a particular case of Theorem 4.1 of [17]:

Proposition 4.1. *Assume that ν is twice continuously differentiable with respect to the first variable satisfying*

$$\sup_x \int (|y| \wedge |y|^2)|\frac{\partial}{\partial x}\nu(x,y)|\, dy < \infty, \quad \sup_x \int (|y| \wedge |y|^2)|\frac{\partial^2}{\partial x^2}\nu(x,y)|\, dy < \infty, \tag{105}$$

and that the functions

$$\int_a^\infty \nu(x,y),\, dy \qquad \int_{-\infty}^{-a} \nu(x,y)\, dy \tag{106}$$

are non-decreasing and non-increasing respectively for any $a > 0$. Then the operator A generates a Feller process $X_t(x)$ on $\mathbf{R}$ and a Feller semigroup with the space $C^2_\infty(\mathbf{R})$ being an invariant core. The process $X_t(x)$ is stochastically monotone (but we will not use this latter fact).

Next let $-\infty < a < x < b < \infty$ and let

$$\tilde{A}_{[a,b]} f(x) = \int_{a-x}^{b-x} [f(a \vee [(x+y) \wedge b]) - f(x) - yf'(x)]\nu(x,y)\, dy \tag{107}$$

be the operator representing the corresponding process $X_x^*(t)[a,b]$ interrupted on an attempt to cross the boundary of $[a,b]$ (the processes on $(-\infty,b]$ or $[a,\infty)$ with a one-sided boundary are considered analogously and will not be looked at) and

$$A_{[a,b]\star}f(x) = \tilde{A}_{[a,b]}f(x) - f'(b)\int_{b-x}^{\infty}[(b-x)-z]\nu(x,z)\,dz - f'(a)\int_{-\infty}^{a-x}[(a-x)-z]\nu(x,z)\,dz \tag{108}$$

the corresponding analog of Caputo's derivative (see (71)).

As above, Proposition 4.1 allows us to apply the standard tools of stochastic calculus. Namely, let $f \in C^2[a,b]$ such that $f'(a) = f'(b) = 0$. Then we can continue it to all $\mathbf{R}$ by setting $f(x) = f(b)$ for $x > b$ and $f(x) = f(a)$ for $x < a$ and it will become a bounded continuously differentiable function on $\mathbf{R}$. Denoting as above by τ the exit time from (a,b), that is $\tau = \inf\{t : X_t(x) \notin (a,b)\}$ and applying to f Dynkin's martingale we get again (89), or else (91), using the kernel $H(x,dy)$ defined by (90) and assuming f solves problem (73) with $\lambda = 0$.

Remark 9. *Actually we can use Dynkin's martingale only for twice continuously differentiable functions, and our (extended) f may have discontinuities of the second derivatives on the boundary, but this can be settled via approximation, as the final expression* (91) *does not involve the second derivative of f on the boundary.*

Therefore we get the following version of Theorem 4.2 for the present case $\beta \in (1,2)$:

Theorem 4.5. *(i) Under the assumptions of Proposition 4.1 problem* (73) *with $\lambda = 0$ can have at most one classical solution. If the probabilities $\mathbf{P}(X_x(\tau) \le a)$, $\mathbf{P}(X_x(\tau) \ge b)$ are functions from $C^2[a,b]$ and the measure $H(x,dy)$ is continuously weakly differentiable in x (so that the integral on the r.h.s. of* (91) *belongs to $C^2[a,b]$ for any $g \in C[a,b]$), formula* (91) *supplies the unique classical solution to* (73) *with $\lambda = 0$.*

(ii) Generally under the assumptions of Proposition 4.1, formula (91) *supplies a unique generalized solution to* (73) *with $\lambda = 0$.*

As was noted solutions to (73) solve also (74) under the additional assumption that the first derivative of the solution vanishes on the boundary, which yields a rather tamed (and expected) non-uniqueness for (74).

5 Appendix

For completeness we deduce here the expressions (5), (6) and (9), (10) from the original definitions.

For $\beta \in (0,1)$ and $x > a$ integration by parts yields

$$I_{a+}^{1-\beta}f(x) = \frac{1}{\Gamma(1-\beta)}\int_a^x (x-t)^{-\beta}f(t)\,dt = -\frac{1}{\Gamma(1-\beta)}\int_a^x \frac{d}{dt}\left[\frac{(x-t)^{1-\beta}}{1-\beta}\right]f(t)\,dt$$

$$= \frac{1}{\Gamma(1-\beta)}\left[\frac{(x-a)^{1-\beta}}{1-\beta}\right]f(a) + \frac{1}{\Gamma(1-\beta)}\int_a^x \left[\frac{(x-t)^{1-\beta}}{1-\beta}\right]f'(t)\,dt,$$

so that

$$D_{a+}^{\beta}f(x) = \frac{d}{dx}I_{a+}^{1-\beta}f(x) = \frac{f(a)}{\Gamma(1-\beta)(x-a)^{\beta}} + \frac{1}{\Gamma(1-\beta)}\int_a^x (x-t)^{-\beta}f'(t)\,dt. \tag{109}$$

Another integration by parts using

$$f'(t) = \frac{d}{dt}(f(t) - f(x)),$$

yields

$$D^{\beta}_{a+}f(x) = \frac{f(a)}{\Gamma(1-\beta)(x-a)^{\beta}} - \frac{f(a)-f(x)}{\Gamma(1-\beta)(x-a)^{\beta}} - \frac{\beta}{\Gamma(1-\beta)}\int_a^x \frac{f(t)-f(x)}{(x-t)^{1+\beta}}\,dt,$$

which equals to the r.h.s. of (5). On the other hand,

$$D^{\beta}_{a+\star}f(x) = I^{1-\beta}_{a+}f'(x) = \frac{1}{\Gamma(1-\beta)}\int_a^x (x-t)^{-\beta}f'(t)\,dt,$$

which differs from (109) by $f(a)(x-a)^{-\beta}/\Gamma(1-\beta)$ yielding (6), (7).

For $\beta \in (1,2)$ and $x > a$ one has

$$D^{\beta}_{a+\star}f(x) = \frac{1}{\Gamma(2-\beta)}\int_a^x (x-t)^{1-\beta}f''(t)\,dt, \tag{110}$$

which rewrites as

$$= \frac{1}{\Gamma(2-\beta)}(x-a)^{1-\beta}(f'(x)-f'(a)) + \frac{1-\beta}{\Gamma(2-\beta)}\int_a^x (x-t)^{-\beta}(f'(t)-f'(x))\,dt.$$

Another integration by parts using

$$f'(t) - f'(x) = \frac{d}{dt}(f(t) - f(x) - (t-x)f'(x))$$

yields

$$D^{\beta}_{a+\star}f(x) = \frac{1}{\Gamma(2-\beta)}(x-a)^{1-\beta}(f'(x)-f'(a))$$

$$-\frac{1}{\Gamma(1-\beta)}(x-a)^{-\beta}(f(a)-f(x)-(a-x)f'(x)) + \frac{1}{\Gamma(-\beta)}\int_a^x \frac{f(t)-f(x)-(t-x)f'(x)}{(x-t)^{1+\beta}}\,dt,$$

which equals the r.h.s. of (10).

On the other hand, again for $\beta \in (1,2)$ and $x > a$,

$$I^{2-\beta}_{a+}f(x) = \frac{1}{\Gamma(2-\beta)}\int_a^x (x-t)^{1-\beta}f(t)\,dt,$$

which rewrites by integration by parts as

$$I^{2-\beta}_{a+}f(x) = \frac{f(a)}{\Gamma(2-\beta)}\frac{(x-a)^{2-\beta}}{2-\beta} + \frac{1}{\Gamma(2-\beta)}\int_a^x \frac{(x-t)^{2-\beta}}{2-\beta}f'(t)\,dt,$$

and by yet another integration by parts as

$$I^{2-\beta}_{a+}f(x) = \frac{f(a)}{\Gamma(2-\beta)}\frac{(x-a)^{2-\beta}}{2-\beta} + \frac{f'(a)(x-a)^{3-\beta}}{\Gamma(2-\beta)(2-\beta)(3-\beta)} + \int_a^x \frac{(x-t)^{3-\beta}f''(t)}{\Gamma(2-\beta)(2-\beta)(3-\beta)}\,dt.$$

Consequently,

$$D_{a+}^{\beta} f(x) = \frac{d^2}{dx^2} I_{a+}^{2-\beta} f(x) = \frac{1-\beta}{\Gamma(2-\beta)}(x-a)^{-\beta} f(a) + \frac{f'(a)(x-a)^{1-\beta}}{\Gamma(2-\beta)} + \int_a^x \frac{(x-t)^{1-\beta} f''(t)}{\Gamma(2-\beta)}\, dt.$$

Comparing this with (110) yields (9) and (11).

Similarly, for $\beta \in (1,2)$ and $x < a$ one has by definition (17) that

$$D_{a-\star}^{\beta} f(x) = \frac{1}{\Gamma(2-\beta)} \int_x^a (t-x)^{1-\beta} f''(t)\, dt$$

$$= -\frac{1}{\Gamma(2-\beta)}(a-x)^{1-\beta}(f'(x)-f'(a)) - \frac{1-\beta}{\Gamma(2-\beta)} \int_x^a (t-x)^{-\beta}(f'(t)-f'(x))\, dt.$$

By integration by parts this rewrites as

$$D_{a-\star}^{\beta} f(x) = -\frac{1}{\Gamma(2-\beta)}(x-a)^{1-\beta}(f'(x)-f'(a))$$

$$-\frac{1}{\Gamma(1-\beta)}(a-x)^{-\beta}(f(a)-f(x)-(a-x)f'(x)) + \frac{1}{\Gamma(-\beta)} \int_x^a \frac{f(t)-f(x)-(t-x)f'(x)}{(t-x)^{1+\beta}}\, dt,$$

which equals the r.h.s. of (22). Similarly, (21) is obtained.

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