Controllability of systems of Schrödinger equations.

Alberto Mercado Saucedo.

Universidad Técnica Federico Santa María, Valparaíso Chile.

Joint work with Marcos Lopez-García (UNAM) and Luz de Teresa (UNAM).
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Controllability of PDE's. Introduction.

The objective is **to drive the solution of a given evolution PDE** to some prescribed target, acting on the equation by means of a control, which could be a right hand side, boundary data, or a coefficient appearing in the equation.

Several mathematical tools:

- Unique continuation properties,
- Multiplier methods,
- Spectral properties,
- Carleman inequalities.

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We consider the control problem of **Schrödingier equation** in an open bounded set $\Omega \subset \mathbb{R}^n$ with a distributed control h: :

$$\begin{cases} i\partial_t u + \Delta u &= h1_\omega \quad \Omega \times (0,T) \\ u &= 0 \quad \Gamma \times (0,T) \\ u(0) &= u_0 \quad \Omega \end{cases}$$

Where

$$\omega \subset \Omega$$

is the control zone,
and

$$h \in L^2(\omega \times (0,T))$$

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We intend to impose that $u(T)$ would take a prescribed value.

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The main kinds of controllability properties of evolution partial differential equations are as follows:

1. **Approximate Controllability**. This property means that any state can be approximated (in a suitable norm) by a solution of the system.
2. **Exact Controllability**. This property means that any state can be reached by a solution of the system.
3. **Null Controllability**. This property states that any solution can be driven to rest.
4. **Controllability to the trajectories**. This property means that any non-controlled trajectory that is a solution of the system can be reached by a controlled trajectory. This property is equivalent to null controllability for linear equations, and it is especially important for non-linear equations.

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We can find the adjoint equation

$$(P^*)\begin{cases} i\partial_t \varphi + \Delta\varphi = 0 & \Omega \times (0,T) \\ \varphi = 0 & \Gamma \times (0,T) \\ \varphi(T) = \varphi_T & \Omega \end{cases}$$

Duality controllability / observability

We have:

Theorem

- *(P) is* **approximate controllable** *iff*

$$\varphi = 0 \text{ in } \omega \times (0,T) \quad \Longrightarrow \quad \varphi \equiv 0 \qquad \text{(UCP)}$$

- *(P) is* **exactly controllable** *iff*

$$\exists C > 0 \qquad \int_\Omega |\varphi_T|^2 \leq C \int_0^T \int_\omega |\varphi|^2 \qquad \forall \varphi_T \qquad \text{(Obs)}$$

- *(P) is null- controllable iff*

$$\exists C > 0 \qquad \int_\Omega |\varphi(0)|^2 \leq C \int_0^T \int_\omega |\varphi|^2 \qquad \forall \varphi_T \qquad \text{(Obs-0)}$$

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Dualidad controlabilidad / observabilidad

Boundary controllability:

$$\begin{cases} i\partial_t u + \Delta u = 0 & \Omega \times (0,T) \\ u = h\mathbf{1}_{\Gamma_0} & \Gamma \times (0,T) \\ u(0) = u_0 & \Omega \end{cases}$$

where

$$\Gamma_0 \subset \partial\Omega$$

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Observability properties.

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To prove observability inequalities:

- Multiplier methods
- Spectral properties
- Microlocal Analysis
- Carleman inequalities

Geometric control. Wave Equation.

Definition

We say $\Gamma_0 \subset \partial\Omega$ satisfies **GC** *hypothesis in Ω at time T, if every ray in Ω, with the laws of the Geometric Optics, intersects Γ_0 in some time $0 \leq t \leq T$.*

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(Bardos-Lebeau-Rauch 1992)
The **GC** *hypothesis for Γ_0 is equivalent to the controllability of Wave Equation with control acting in Γ_0.*

Controllability of Schrodinger equation.

- **Jaffad (1990)** Interior controllability in a rectangle with observations in an open subdomain $\omega \subset \Omega \subset \mathbb{R}^2$.
- **Lebeau (1992)** If $\Gamma_0 \subset \partial\Omega$ satisfies **GC** hypothesis for some $T_0 > 0$, then Schrödinger equation is controllable with controls acting on Γ_0, for all $T > 0$.
- **Machtyngier (1994)**. Boundary controllability in Ω (boundary C^3), with observations in $\Gamma_0 = \{x \in \partial\Omega \,:\, n(x) \cdot (x - x_0) \geq 0\}$.
- **Komornik 1994**. Generalization of Jaffad for $\mathbb{R}^n$.
- **Phung (2001)** Controllability in L^2.
- **Rosier, Zhang (2009)** Controllability in L^2 of the nonlinear equation.
- **Ramdani, Takahashi, Tenenbaum, Tucsnak (2005)**, and **Tenenbaum, Tucsnak (2009)**, Boundary controllability in a rectangle, with observations in any open Γ_0 containing a corner.
- See survey **Zuazua (2003)**, *Remarks on the controllability of the Schrödinger equation*.

Recall that the GC hypothesis in Lebeau result are not necessary.

Carleman inequalities.

Carleman inequalities were introduced by Trosten Carleman in 1939 in the study of uniqueness for some PDE's.
Since then, this tool has been widely used in the study of unique continuation properties, controllability:

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Carleman inequalities are used to study:

- Control problems of equations with non-regular lower order terms.
- Control problems of semi-linear equations.
- Some inverse problems.

Carleman estimates. An example.

Consider $L = \Delta$ for functions $w \in C_c^\infty(\Omega)$.
We define

$$L_\phi w = e^{-\lambda\phi} L(e^{\lambda\phi} w)$$

$$\Delta(e^{\lambda\phi} w) = e^{\lambda\phi}\left(\lambda^2|\nabla\phi|^2 w + \lambda\Delta\phi w + 2\lambda\nabla\phi\cdot\nabla w + \Delta w\right)$$

If $\phi(x) = \alpha\cdot x$ with $\alpha \in \mathbb{R}^n \setminus \{0\}$ then:

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A is self-adjoint.
B is anti-adjoint.

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$$L_\phi w = \underbrace{\lambda^2|\alpha|^2 w + \Delta w}_{Aw} + \underbrace{2\lambda\alpha\cdot\nabla w}_{Bw}$$

We have

$$\|L_\phi w\|_{L^2}^2 = \|Aw\|_{L^2}^2 + \|Bw\|_{L^2}^2 + 2\langle Aw, Bw\rangle_{L^2} \tag{2}$$

A is self-adjoint and B is anti-adjoint (and both have constant coefficients), we get

$$2\langle Aw, Bw\rangle_{L^2} = \langle [A,B]w, w\rangle_{L^2} = 0, \quad \forall\, w \in C_c^\infty(\Omega)$$

Thus

$$\|L_\phi w\|_{L^2} \geq 2\lambda\|\alpha\cdot\nabla w\|_{L^2} \tag{3}$$

$$\geq \lambda\delta\|w\|_{L^2} \tag{4}$$

Which means that

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In other cases:

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We have

$$\|L_\phi w\|_{L^2}^2 = \|Aw\|_{L^2}^2 + \|Bw\|_{L^2}^2 + 2\langle Aw, Bw\rangle_{L^2} \tag{6}$$

$$2\langle Aw, Bw\rangle_{L^2} = \langle [A,B]w, w\rangle_{L^2} = \text{lower order + boundary terms}, \quad \forall\, w\in C_c^\infty(\Omega)$$

Thus

$$\begin{aligned} \|L_\phi w\|_{L^2} &\geq 2\lambda\|\alpha\cdot\nabla w\|_{L^2} \qquad (7)\\ &\geq \lambda\delta\|w\|_{L^2} \qquad (8)\end{aligned}$$

Which means that

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Carleman inequalities.

Given a differential operator P and a smooth function ϕ, we define

$$P_\phi = e^{\lambda\phi} P e^{-\lambda\phi}$$

Remark that $P_\phi = p(x, D + i\lambda\nabla\phi)$

For instance, ϕ is pseudoconvex if:

- For $P = \partial_t - \Delta$ if $|\nabla\phi| \neq 0$
- For $P = \partial_t^2 - \Delta$ if ϕ is convex.
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Boundary condition: Usually is required $\dfrac{\partial\phi}{\partial\nu} < 0$ in $\partial\Omega$.

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Theorem (Carleman inequalities)

If ϕ is pseudoconvex with respect to P then

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for λ large enough.

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If the previous properties are not satisfied in a set $\omega \subset \Omega$ or $\omega \subset \partial\Omega$, then

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Theorem (Carleman inequalities)

If ϕ is pseudoconvex with respect to P in $\Omega \setminus \omega$ then

$$\|v\|_{H^m_\lambda} \leq C\|P_\phi v\|_{L^2} + \|v\|_{H^m(\omega)}$$

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In the original variable, we get:

$$\|e^{-\lambda\phi} w\|_{H^m} \leq C\|e^{-\lambda\phi} P w\|_{L^2} + \underbrace{\|e^{-\lambda\phi} w\|_{H^m(\omega)}}_{observation}$$

Carleman for Schrödinger Equation.

Theorem (Baudouin, Puel 2002.)

If exists an $x_0 \in \mathbb{R}^n$ *such that*

$$\Gamma_0 \supset \{x \in \partial\Omega \,:\, (x - x_0) \cdot n(x) > 0\}.$$

Then

$$\int_0^T \int_\Omega \left(\rho|\nabla w|^2 + \rho^3 |w|^2\right) dxdt \leq C \int_0^T \int_{\Gamma_0} \rho \left|\frac{\partial w}{\partial \nu}\right|^2 dxdt \tag{10}$$

The weight function is

$$\psi(x) = |x - x_0|^2.$$

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Schrödinger Equation. Carleman inequality.

Remark: The precise hypothesis:

1. $\lambda|\nabla\psi.\,\xi|^2 + D^2\psi\,(\xi,\overline{\xi}) \geq \varepsilon\,|\xi|^2$ for all $\xi \in \mathbb{C}^n$
2. $\Gamma_0 \supset \{x \in \partial\Omega\ :\ \nabla\psi \cdot n(x) > 0\}$

1. From the Carleman estimate, we can deduce an observability estimate in H^1.
2. This implies a controllability result in H^{-1}
3. Used to prove stability of some inverse problems.

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Schrödinger Equation. More general hypothesis.

In the search of stability without the GC hypothesis:

Theorem (M, Osses, Rosier 2008)

If $\psi > 0$, $\nabla\psi \neq 0$ *is such that*

$$\lambda|\nabla\psi.\,\xi|^2 + D^2\psi\left(\xi,\bar{\xi}\right) \geq 0$$

then we have the Carleman estimate

$$\int_0^T\int_\Omega \left(\rho|\nabla w\cdot\nabla\psi|^2 + \rho^3|w|^2\right)dxdt \leq C\int_0^T\int_{\Gamma_0}\rho\left|\frac{\partial w}{\partial\nu}\right|^2 dxdt \tag{11}$$

Schrödinger equation.

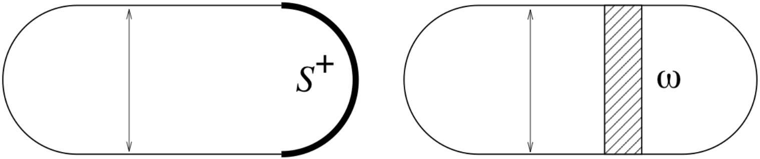

Figure : Observation regions in a stadium for the Schrödinger equation using degenerate Carleman weights with spatial dependence of the type $x \cdot e_1$.

For $\psi = \xi \cdot x$ we get

Theorem (M, Osses, Rosier 2008)

Interior observability:

$$\int_0^T \int_\Omega \rho^3 |w|^2 dxdt \leq C \int_0^T \int_\omega \rho(|w|^2 + |\nabla w|^2) dxdt \tag{12}$$

where $\omega \subset \Omega$ is such that is intersected by any right line in direction ξ.

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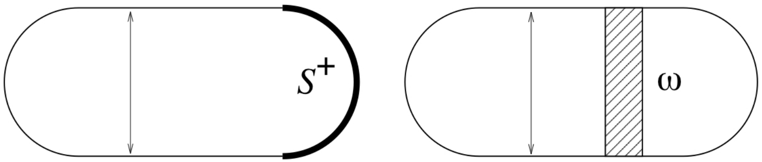

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Controllability of Schrödinger equation.

This implies a result for the controllability of a Schrodinger equation:

$$\begin{cases} i\partial_t u + \Delta u &= h1_\omega & \Omega \times (0,T) \\ u &= 0 & \Gamma \times (0,T) \\ u(0) &= u_0 & \Omega \end{cases}$$

Corolary (from the Carleman of M - Osses - Rosier (2008))

For each $u_0 \in L^2(\Omega)$ *exists*

$$h \in L^2(0,T;H^{-1}(\Omega))$$

supported in ω *such that*

$$u(T) = 0$$

Remark: In the Carleman estimate, we have only observation of $\nabla_\xi w := \nabla w \cdot \xi$.

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System of Schrödinger equations.

Let $\Omega \subset \mathbb{R}^N$ be a bounded, open set with C^2 boundary, $N \geq 1$. Let ω and $\mathcal{O}$ be two nonempty open subsets of Ω.
We consider the following cascade system of Schrödinger equations:

$$\begin{cases} ip_t + \Delta p = h\mathbf{1}_\omega & \text{in } Q, \\ p = 0 & \text{on } \Sigma, \\ p(x,0) = p^0(x) & \text{in } \Omega, \end{cases} \tag{13}$$

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where p^0, u^0 are given, h is a control with support in $\omega \times (0,T)$.

In this system, the problem is to prove controllability with:

1. Less controls than equations.
2. No geometric hypothesis on ω.
3. General case $\omega \cap \mathcal{O} = \emptyset$.

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Some references.

- **Rosier, de Teresa (2011)**. Periodic boundary conditions, $\omega \cap \mathcal{O} = \emptyset$.
- **Alabau, Leautaud (2013)**. Both ω and $\mathcal{O}$ satisfies GC hypothesis
- **Alabau, (2013)**. Systems of N equations.
- **Dehman, Le-Rousseau, Leatud (2014)**. Equations in manifolds.

We are able to deal with 2 of 3 stated objectives.

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System of Schrödinger equations.

For the system, our main controllability result is:

Theorem (Lopez-García - M - de Teresa (preprint 2014))

Let $X := H_2 \cap H_0^1(\Omega)$. *Suppose there exists an open set*

$$\widetilde{\omega} \subset \omega \cap \mathcal{O}$$

a observation set for the degenerate Carleman inequality, and such that

$$\xi \cdot n(x) = 0 \ \text{ for each } x \in \partial\widetilde{\omega} \cap \partial\Omega.$$

Then for each $(p^0, u^0) \in L^2(\Omega)^2$ *there exists a control* $h \in L^2(0, T; X')$ *supported in* $\omega \times (0, T)$ *such that*

$$(p(T), u(T)) = (0, 0).$$

Idea of the proof

We consider the adjoint system

$$\begin{cases} iz_t + \Delta z = 0 & \text{in } Q, \\ z = 0 & \text{on } \Sigma, \\ z(x,T) = z^0(x) & \text{in } \Omega, \end{cases} \tag{15}$$

$$\begin{cases} iq_t + \Delta q = z\rho_{\mathcal{O}} & \text{in } Q, \\ q = 0 & \text{on } \Sigma, \\ q(x,T) = q^0(x) & \text{in } \Omega, \end{cases} \tag{16}$$

Adding the Carleman inequalities for the adjoint system, we get

$$\int_0^T \int_\Omega \rho^3 (|z|^2 + |q|^2) dx dt \leq C \int_0^T \int_\omega \rho (|z|^2 + |\nabla_\xi z|^2 + |q|^2 + |\nabla_\xi q|^2) dx dt \tag{17}$$

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Idea of the proof

We can eliminate one observation:

$$\begin{aligned}\int_0^T\int_\omega \rho|z|^2 &= \int_0^T\int_\omega \rho\overline{z}(Pq)\\ &= \int_0^T\int_\omega \rho\overline{Pz}q + \text{Lower order terms}\\ &\le \varepsilon\int_0^T\int_\omega \rho|z|^2 + \int_0^T\int_\omega(|q|^2+|\nabla q|^2)\end{aligned} \tag{18}$$

and

$$\begin{aligned}\int_0^T\int_\omega \rho|\nabla_\xi z|^2 &= \int_0^T\int_\omega \rho\overline{\nabla}_\xi z\nabla_\xi(Pq)\\ &= \int_0^T\int_\omega \rho\overline{\nabla_\xi Pz}\nabla_\xi q + \text{Lower order terms}\\ &\le \varepsilon\int_0^T\int_\omega \rho(|\nabla_\xi z|^2) + C\|q\|_{H^2(\omega)}\end{aligned} \tag{19}$$

We have used the hypothesis on the boundary in order to avoid boundary terms. □

Ongoing work

1. Control of H^1 initial conditions with L^2 controls.
2. Remove hypothesis on $\partial(\omega \cap \mathcal{O})$.
3. The case $\omega \cap \mathcal{O} = \emptyset$.

Thanks!

Eskerrak!

¡Muchas gracias!